(10pts.) 1. Suppose $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuous and that $\lim_{x\to\infty} f(x) = 0$. Prove that f is uniformly continuous.

(10pts.) 2. Compute

$$\int_0^2 \int_{\sqrt{3}x}^{\sqrt{4-x^2}} (x^2 + y^2)^{3/2} \, dy \, dx.$$

(10pts.) 3. Let s_n be the nth partial sum of a sequence (a_n), and suppose that

$$\lim_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right| = \frac{1}{2}.$$

Prove that $\sum a_n$ converges to zero.

(10pts.) 4. Let K be a closed and bounded nonempty subset of $\mathbb{R}^n$, and let x be a point in $\mathbb{R}^n$. Prove that there exists a point z in K such that $\|x - z\| = \inf\{\|x - y\| : y \in K\}$. Is the point z necessarily unique?

(10pts.) 5. Let $f\colon [0, 1] \to \mathbb{R}$ be a Riemann integrable function and write $\|f\|$ for the number

$$\left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2}.$$

a. Show that it is possible that $\|f\| = 0$ even if f is non-zero.

b. Show that if $\|f\| = 0$, then $f(x) = 0$ whenever f is continuous at x.

c. Show that if $f(x) = 0$ for all $x \in [0, 1]$ at which f is continuous, then $\|f\| = 0$.

(10pts.) 6. Suppose that f and g are smooth functions from $\mathbb{R}^3$ to $\mathbb{R}$ and let D be a solid whose boundary is a closed smooth surface Σ, oriented outward. Prove that

$$\iint_\Sigma (f\nabla g - g\nabla f) \cdot \vec{n} \, dS = \iiint_D \left(f\nabla^2 g - g\nabla^2 f \right) \, dV,$$

where ∇ is the gradient, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, and $\vec{n}$ is the outward unit normal vector field.

(10pts.) 7. Let (f_n) be a sequence of continuous functions on an interval $[a, b]$, and suppose that $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function f. Prove that

$$\int_a^b f(x) \, dx = \sum_{n=1}^{\infty} \left(\int_a^b f_n(x) \, dx \right).$$

(10pts.) 8. Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose $f(0) = 0$ and that $f(1) = 1$.

a. Suppose that in addition $f(a) = a$ for some $a \in (0,1)$. Prove that there exist at least two values of $x \in (0,1)$ such that $f'(x) = 1$.

b. Prove that there exists a real number $c \in (0,1)$ such that $f'(c) = 2c$.