Name: ______________________________

Time: 4 hours

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

(10pts.) 1. Evaluate

$$\oint_C (y^2 + \arctan(x^3))\,dx + (3xy - e^{y^4})\,dy,$$

where C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, traversed counterclockwise.

(10pts.) 2. Find the maximum and minimum values of $x^2 + y^2 - 2x + 4y + 3$ on the closed disk $x^2 + y^2 \leq 9$, and state where they occur.

(10pts.) 3. Suppose $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ if n is a positive integer greater than 1. Prove that the sequence (x_n) has limit 0, and that $\lim_{n\to\infty} x_{n+1}/x_n = 1/2$.

(10pts.) 4. Let (x_n) be a sequence of real numbers. Prove that $\lim_{n\to\infty} x_n = L$ if and only if every subsequence of (x_n) has in turn a subsequence that converges to L.

(10pts.) 5. Suppose f is continuous on $[0,1]$. Set $f_0 = f$, and inductively define a sequence $\{f_n\}$ by the formula

$$f_{n+1}(x) = \int_0^x f_n(t)dt.$$

Prove that $\{f_n\}$ converges uniformly to the zero function on $[0,1]$.

(10pts.) 6. Suppose (a_n) is a sequence of positive real numbers and that

$$p = \lim_{n\to\infty} \frac{\log(1/a_n)}{\log n}$$

is a real number.

(a) Prove that $\sum_n a_n$ converges if $p > 1$.

(b) Prove that $\sum_n a_n$ diverges if $p < 1$.

(10pts.) 7. Let E be a bounded subset of of $\mathbb{R}^n$ with nonempty interior. Prove that there is a strictly positive number r such that E contains some open ball B of radius r, but E contains no open ball of radius strictly greater than r.

(10pts.) 8. Let $f\colon I \to \mathbb{R}$ be differentiable, where I is an open interval containing $[0,1]$. Prove that if f' is continuous on $[0,1]$, then for all $\epsilon > 0$ there is an $h_0 > 0$ such that whenever $0 < |h| < h_0$ and $x, x+h \in [0,1]$, it follows that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \epsilon.$$