Real Analysis Preliminary Exam
September 12, 2020
4 hours

(1) Suppose $a > 0$ and that $f : \mathbb{R} \to [a, \infty)$ is uniformly continuous. Prove that $\ln f$ is uniformly continuous on $\mathbb{R}$.

(2) How many times is the function $f(x) = \sin|x| - |x|$ differentiable?

(3) Let C be the curve formed by intersecting the cylinder $x^2 + y^2 = R^2$ with the plane $z = ax + by$ oriented counterclockwise when viewed from above and let

$$F(x, y, z) = (y^2, 5z, 2x).$$

Compute the line integral of F along C.

(4) Find a finite sum that estimates

$$\int_0^1 \frac{x}{2 + x^5}\, dx$$

to within 0.01. You do not need to calculate the value of your sum, but must prove that it is within 0.01 of the actual value of the integral.

(5) (a) Find the Fourier series for the function $f : \mathbb{R} \to \mathbb{R}$ that has period 2 and satisfies $f(x) = \pi x + \pi$ for $-1 \le x \le 1$.
(b) Find all $x \in \mathbb{R}$ for which the Fourier series does not converge to $f(x)$, and determine what it converges to at those points.

(6) For the infinite series $\sum_{n \ge 0} a_n$, suppose that $\sum_{n \ge 0} a_n^4$ converges.
(a) Prove or disprove that $\sum_{n \ge 0} a_n^3$ necessarily converges.
(b) Prove or disprove that $\sum_{n \ge 0} a_n^5$ necessarily converges.

(7) For a set $S \subset \mathbb{R}^n$, let $\chi_S(x) = 1$ for $x \in S$ and 0 otherwise. Prove that χ_S is continuous at c if and only if c is not in $\partial S = (\overline{S}) \cap (\overline{\mathbb{R}^n - S})$.

(8) Let $S(n) = \sum_{j=1}^{n} \frac{1}{n + j}$.
(a) Prove that

$$\lim_{n \to \infty} S(n) = \ln 2 \ .$$

(b) Prove that

$$\ln 2 - \frac{1}{4n} \le S(n) \le \ln 2 \ .$$