REAL ANALYSIS PRELIMINARY EXAMINATION

AUGUST 28, 2017

(1) State and prove the Ratio Test for convergence of an infinite series $\sum a_n$.

(2) (a) State the theorem for changing variables in the integral

$$\iiint_Q f(x, y, z) dx\, dy\, dz,$$

where Q is a region in $\mathbb{R}^3$.

(b) Find the volume of the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

(3) Let $g(x) = \sin(x^5 + x^7)$. Find $\dfrac{\partial^{17} g}{\partial x^{17}}(0)$.

(4) Let $f(x) = |x|^\alpha$, where $\alpha \in \mathbb{R}$. For which $\alpha > 0$ is f differentiable at 0?

(5) Consider the function $f : [0, 1] \to [0, 1]$ that assigns to x the value 0 if x is irrational, and the value $\frac{1}{q^2}$ if x is a rational number of the form $\frac{p}{q}$, where p, q are positive integers without common factors.

(a) Show that f is continuous at $x_0 \in [0, 1]$ if and only if x_0 is irrational.

(b) Prove or disprove that f is Riemann integrable.

(6) Let $(f_n : \mathbb{R} \to \mathbb{R})_{n \geq 1}$ be a sequence of differentiable functions that converge uniformly to f on $\mathbb{R}$.

(a) Must the sequence of derivatives $(f_n')_{n\geq 1}$ also converge uniformly on $\mathbb{R}$?

(b) Must $\left(\int_0^x f_n(t)\, dt\right)_{n\geq 1}$ converge to $\int_0^x f(t)\, dt$ for $x \in [0, 1]$? If so, must the convergence be uniform on $[0, 1]$?

(7) Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}$ and $f : \Omega \to \mathbb{R}$ given by

$$f(x, y, z) = yx^z + zx^y.$$

(a) Explain why f is differentiable.

(b) Find the gradient of f at $(1, 1, 1)$.

(c) Near $(1, 1, 1)$, can one write z as a function of x and y in the relation $f(x, y, z) - 2 = 0$?

(8) Let the function $h : \mathbb{R} \to \mathbb{R}$ be periodic with period 2π and have a continuous second derivative everywhere. Prove there exists a constant C independent of n such that the Fourier coefficients of h satisfy

$$|a_n| \leq \frac{C}{n^2}, \qquad |b_n| \leq \frac{C}{n^2}, \qquad n \geq 1.$$