REAL ANALYSIS PRELIM EXAM

AUGUST 23, 2016

(1) Suppose (a_n) is a sequence of real numbers such that $0 < a_n < 1$ for all n. Show that $\sum a_n$ is convergent if and only if $\sum \arcsin a_n$ is convergent.

(2) Let $p\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim_{x\to\infty} p(x)$ and $\lim_{x\to\infty} p'(x)$ exist. Prove that $\lim_{x\to\infty} p'(x) = 0$.

(3) Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function that is not the zero function.

  (a) Prove that there exists a ball $B_\varepsilon(x_0)$ of radius $\varepsilon > 0$ centered at some $x_0 \in \mathbb{R}^n$ such that $\int_{B_\varepsilon(x_0)} g(x)\ dx \neq 0$.

  (b) Suppose that the condition that g is continuous is replaced with g being Riemann integrable. Determine whether or not the conclusion of (a) is still valid.

(4) Let $f\colon [-1,1] \to \mathbb{R}$ be any continuous function, and define $h\colon [0,1] \to \mathbb{R}$ by $h(0) = f(0)$ and

$$h(x) = \inf_{-x<t<x} f(t).$$

  (a) Prove that h is decreasing.

  (b) Prove or disprove that h is necessarily continuous.

(5) Let f and g be two Riemann integrable functions on $[a,b]$.

  (a) Prove or disprove that $h(x) = \max\{f(x), g(x)\}$ is Riemann integrable.

  (b) Answer the same question as (a) when f and g are two functions on $[0,\infty)$ that are improper Riemann integrable.

(6) Justify your answers to the following questions with proof.

  (a) Does there exist a continuous function $g : \mathbb{R} \to (0,1)$ that is 1-1 and onto?

  (b) Does there exist a continuous function $g\colon \mathbb{R} \to [0,1]$ that is 1-1 and onto?

(7) A real-valued function $f(x)$ is *convex* on an interval I if, for all $a, b \in I$ and $t \in [0,1]$,

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b).$$

Prove that if $f(x)$ is convex on I, then $e^{f(x)}$ is also convex on I.

(8) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x,y) = (y + yx^4, x^2 + y^2 - 16)$.

  (a) Prove that there exists $\varepsilon > 0$ such that for every (u,v) in the ball $B_\varepsilon(0,0)$ of radius ε centered at the origin, the set $F^{-1}\{(u,v)\}$ consists of two points.

  (b) Show that the following statement is false: "For every $(u,v) \in \mathbb{R}^2$, the set $F^{-1}\{(u,v)\}$ consists of two points."