REAL ANALYSIS PRELIM EXAM
AUGUST 20, 2014

(1) (a) Suppose $\sum a_n$ and $\sum b_n$ are convergent. Prove or disprove that $\sum a_n b_n$ is necessarily convergent.

(b) Suppose $\sum a_n$ and $\sum b_n$ are absolutely convergent. Prove or disprove that $\sum a_n b_n$ is necessarily convergent.

(2) Evaluate $\displaystyle\int_{\mathbb{R}^3} \exp\left(-\|x\|^3\right)\, dx$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\|x\|$ denotes the usual norm, and $dx = dx_1 dx_2 dx_3$ is the usual volume element.

(3) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 0$ for $x \in [-\pi, 0]$, $g(x) = 1$ for $x \in (0, \pi)$ and $g(t + 2\pi) = g(t)$ for all $t \in \mathbb{R}$.

(a) Prove that the Fourier series of g is

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin((2k-1)x).$$

(b) Describe in what way(s) the series converges or does not converge to g.

(4) State and prove the ratio test for real series.

(5) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x^2 - y^2, -2xy + 1)$. Prove that there exists an $\varepsilon > 0$ such that, for all fixed (u, v) with $|(u + 1, v - 1)| < \varepsilon$, there exists exactly two distinct points (x_1, y_1) and (x_2, y_2) so that $F(x_j, y_j) = (u, v)$ for $j = 1, 2$.

(6) Suppose that $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Let v be a fixed vector in $\mathbb{R}^n$, and define the function $H : \mathbb{R}^n \to \mathbb{R}^n$ by $H(x) = g(x) + 3\|x\|\, v$, for all $x \in \mathbb{R}^n$. Prove that H is continuous, using the ε-δ definition.

(7) For a fixed set $S \subset \mathbb{R}^n$, let $\phi(x) = 1$ for $x \in S$ and 0 otherwise. Prove that ϕ is continuous at c if and only if c is not in $\partial S = \overline{S} \cap \overline{(\mathbb{R}^n - S)}$.

(8) Let f be a bounded, Riemann integrable function on $[0, 1]$ such that $\lim_{x \to 1^-} f(x) = 3$. Using the definition, prove that the function $g(x) = \frac{f(x)}{\sqrt{1-x}}$ is improper Riemann integrable on the interval $(0, 1)$.