Preliminary Real Analysis Exam

(1) Let $f(x) = \sum_{n=0}^{\infty} \frac{x^{\frac{5n}{3}}}{(n!)^2}$ for $x \in [0,1]$. Prove that $f(x)$ is continuous.

(2) Evaluate the integral $\iint_S (y\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n} \; dS$, where S is the surface $\{(x,y,z) : x^2 + y^2 = z, z \leq 4\}$ and $\mathbf{n}$ is the unit normal to S that points away from the z axis.

(3) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$.

(a) State the definition of differentiability of f at the point (a,b).

(b) Prove that if f has continuous first order partial derivatives in a neighborhood of (a,b), then f is differentiable at (a,b).

(4) (a) Let f be a nonnegative, continuous function on the nonnegative reals. For a positive integer n, let $I_n = \int_0^n f(x)\; dx$. Prove that $\int_0^\infty f(x)\; dx$ converges if and only if the sequence (I_n) converges.

(b) Show that the conclusion in (a) may be false if the hypothesis that f is nonnegative is dropped.

(5) Let f be twice continuously differentiable. Prove that, given x and h, there exists θ such that

$$f(x+h) - 2f(x) + f(x-h) = f''(\theta)h^2.$$

(6) Let $f(x)$ be infinitely differentiable and odd. Suppose the Fourier series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx)$$

converges to $f(x)$ on $(-1,0)$ and the Fourier series

$$c_0 + \sum_{n=1}^{\infty} (c_n \cos 2\pi nx + d_n \sin 2\pi nx)$$

converges to $f(x)$ on $(0,1)$. Express the Fourier series of period 2 that converges to $f(x)$ on $(-1,1)$ in terms of $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$.

(7) Let S_0 be the surface $x^2 + y^2 + z^2 = 1$. Let S_1 be the surface $x^2 + y^2 - z^2 = 1$. Let S_2 be the surface $x^2 - y^2 - z^2 = 1$. For what $i \neq j$ is there a continuous surjection $S_i \to S_j$? A short justification is sufficient in each case.

(8) Suppose that g is a twice differentiable function on $\mathbb{R}$ such that $|g''(x)| \leq K$. Prove that there exists a constant $C \in \mathbb{R}$ such that

$$e^{g(x)} \leq C\, e^{Cx^2}.$$