Preliminary Real Analysis Exam

January 20, 2012

(1) For $k \geq 1$, suppose that $\mathbb{R}^k \setminus A$ contains no closed balls of positive radius. Prove that A is dense in $\mathbb{R}^k$.

(2) (a) Compute the Taylor series at 0 for $\cos x$ and prove it converges to $\cos x$ for all real x.
(b) Prove the convergence is uniform on closed, finite intervals $[a, b]$.
(c) Prove the convergence is not uniform on $(-\infty, \infty)$.

(3) (a) Prove that if f is differentiable at c, then $\displaystyle\lim_{h \to 0^+} \frac{f(c+2h) - f(c-h)}{h}$ exists.
(b) Show that the converse of (a) is false.

(4) (a) Find the Fourier series of

$$f(x) = x, \ 0 \leq x < 2,$$

and assume that f is periodic of period 2.
(b) Use this Fourier series to evaluate $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \ldots.$

(5) (a) Let C be the line segment from the point (x_1, y_1) to (x_2, y_2). Prove that

$$\int_C (-y dx + x dy) = x_1 y_2 - x_2 y_1.$$

(b) Suppose a polygon P has vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, listed in order. Prove that the area of P is

$$\frac{1}{2} \left| (x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n) \right|.$$

(6) Suppose there is a sequence of bounded functions $f_n : [a, b] \to \mathbb{R}$.
(a) Suppose that f_n converges uniformly to a function $f : [a, b] \to \mathbb{R}$. Prove that f is bounded on $[a, b]$.
(b) Suppose that each f_n is continuous on $[a, b]$, and suppose that $f_n \to f$ pointwise on $[a, b]$. Prove or disprove that f is bounded.

(7) For which $k \in \mathbb{R}$ is it true that $\int_{\mathbb{R}^n} |x|^k e^{-|x|} \, dx < \infty$?

(8) Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $h'(0) = 0$ and $h''(0) = 1$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by $F(x) = h(|x|)$. Prove that F is differentiable on $\mathbb{R}^n$.