Preliminary Real Analysis Exam

August 24, 2012

1. Let S be the surface which bounds the half of the solid ball of radius 5 centered at the origin with $y \geq 0$. Let N be the outward unit normal to S. Over this surface, find the flux

$$\int_S F \cdot N \, dA$$

of the vector field $F(x, y, z) = \left(6z, y^2, 2x\right)$.

2. Prove that f is differentiable at a if and only if

$$\lim_{h \to 0} \frac{f(a + h + h^2) - f(a)}{h}$$

exists.

3. Let $f(x) = x^2$ for $-\pi \leq x < \pi$, and assume that f is periodic of period 2π.

   (a) Find the Fourier series of f.

   (b) Use your series to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

4. Consider the series $S(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$.

   (a) Find the interval of convergence of this series.

   (b) Is the convergence on this interval uniform?

   (c) Find $S(1/2)$.

5. Prove that every sequence of real numbers has a monotone subsequence.

6. Let f be continuous on $[0, 1]$. For $x \in [0, 1]$, let

$$g_n(x) = \int_0^x f(y)(x - y)^n \, dy.$$

   (a) Find the pointwise limit of g_n as $n \to \infty$.

   (b) Is the convergence uniform?

7. Prove the Heine-Borel Theorem for $\mathbb{R}$: A subset $A \subset \mathbb{R}$ is closed and bounded if and only if every open cover of A has a finite subcover.

8. Consider the function $f(x, y) = \sqrt{|xy|}$.

   (a) Prove or disprove that f is continuous at $(x, y) = (0, 0)$.

   (b) Prove or disprove that $\partial f / \partial x$ and $\partial f / \partial y$ exist at $(x, y) = (0, 0)$.

   (c) Prove or disprove that f is differentiable at $(x, y) = (0, 0)$.