Real Analysis Preliminary Exam
August 17, 2011

Work all eight problems, justifying your work. Each is worth 10 points. You have four hours.

(1) (a) Let a_n be a sequence of real numbers. Give the ϵ - N definition for convergence $\lim_{n\to\infty} a_n = a$.

(b) Use the definition to prove $\displaystyle\lim_{n\to\infty} \frac{5n^3+n^2+1}{2n^3} = \frac{5}{2}$.

(2) Let S be the piece of the plane $z = 4x + 6y$ for $0 \le x \le 1$, $0 \le y \le 2$. Let C be the boundary of the surface S, oriented so that it will look counterclockwise from above. Use Stokes's Theorem to compute $\int_C (2xz\ dx - 2yz\ dy)$.

(3) Let $f : [a, b] \to \mathbb{R}$ be bounded.
(a) Prove or disprove: If f is Riemann integrable on $[a, b]$, then so is f^2.
(b) Prove or disprove: If f^2 is Riemann integrable on $[a, b]$, then so is f.

(4) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge for $|x| < R$.
(a) Prove that $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} a_n x^{n+1}/(n+1)$ converge for $|x| < R$.
(b) Prove that, on $|x| < R$, f has derivative $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and antiderivative $\sum_{n=0}^{\infty} a_n x^{n+1}/(n+1)$.

(5) Let $(x_k)_{k\ge1}$ be a sequence in $\mathbb{R}^n$. Let L be the set of limits of convergent subsequences of (x_k). Prove that L is a closed set.

(6) For this problem, you may use any techniques you know, but may only use the following two facts about the number e: (i) $e > 0$ and (ii) $\lim_{h\to0}(e^h - 1)/h = 1$ and nothing about e^x other than general facts about exponentials b^x.
(a) Prove that $\dfrac{d}{dx} e^x = e^x$.
(b) Estimate e to within 0.2.

(7) Consider the function $f : [0, 1/e] \to \mathbb{R}$ defined by $f(x) = x^{1+x}$.
(a) Prove that the image of f is contained in $[0, 1/e]$.
(b) Prove that $|f(x) - f(y)| < |x - y|$ for all pairs of distinct x and y in $[0, 1/e]$.
(c) Prove that there is no $r < 1$ such that $|f(x) - f(y)| \le r|x - y|$ for all x and y in $[0, 1/e]$.

(8) Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable and satisfy $|f(x)| \leq \dfrac{C}{1+|x|^2}$ and $|f'(x)| \leq \dfrac{C}{1+|x|^2}$ for some constant C. Let $g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$. Let

$$\sum_{k=0}^{\infty} a_k \cos(2\pi kx) + \sum_{k=1}^{\infty} b_k \sin(2\pi kx)$$

be the Fourier series of g.

(a) Prove that $a_0 = \hat{f}(0)$ and, for $k \geq 1$, $a_k = \hat{f}(k) + \hat{f}(-k)$ and $b_k = (\hat{f}(k) - \hat{f}(-k))i$, where $\hat{f}$ is the Fourier transform of f: $\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\, dx$.

(b) Use the above to prove the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$