PRELIMINARY REAL ANALYSIS EXAM

AUGUST, 2010

Work all problems, justifying everything, clearly identifying any major theorems used. In the unlikely event that you find an error in one of the problems, clearly state what the error is. If the statement of a problem is false, provide a counterexample. If you believe there is a typographical error, notify the proctor.

(1) State and prove the product rule for differentiation.

(2) (a) Prove the convergence properties of the geometric series. That is, find the conditions on a and r such that the series $\sum_{n=0}^{\infty} ar^n$ converges.

(b) State and prove the ratio test.

(3) Let f and g be uniformly continuous functions on $\mathbb{R}$.

(a) Must $f+g$ be uniformly continuous on $\mathbb{R}$?

(b) Must fg be uniformly continuous on $\mathbb{R}$?

(4) Consider a sequence of real numbers $(a_n)_{n\geq 0}$. Suppose that for every positive integer $k>1$ and every nonnegative integer r, the subsequence $(a_{nk+r})_{n\geq 0}$ converges. Must $(a_n)_{n\geq 0}$ converge?

(5) Let $(I_n = [a_n, b_n))_{n\geq 1}$ be a nested sequence of half-open intervals, closed at the left endpoint, open at the right.

(a) Give an example of such a sequence with $\bigcap_{n=1}^{\infty} [a_n, b_n) = \emptyset$.

(b) Prove that if (b_n) is **strictly** decreasing, then $\bigcap_{n=1}^{\infty} [a_n, b_n)$ is a point or a nonempty closed interval.

(6) Suppose that a real-valued, twice continuously differentiable function h satisfies $h''(x) \geq 0$ on $\mathbb{R}$. Prove that for every $x, y \in \mathbb{R}$ with $x<y$,

$$h\left(\frac{x+2y}{3}\right) \leq \frac{h(x)+2h(y)}{3}.$$

No "proof by picture" will be accepted.

(7) Prove that a monotone function on $[0,1]$ is Riemann integrable.

(8) Let T be a triangle in $\mathbb{R}^2$ with side lengths $a \geq b \geq c > 0$ and opposite angle measures $A,\ B,\ C$, in radians. Suppose that the triangle has its long side (with length a) on the positive x axis and that it has one vertex at $(0,0)$.

(a) Express the area of T as a double integral.

(b) Express the area of T as a line integral over the boundary of the triangle.

(c) Compute the integral in (b), and thereby express the area of T in terms of a, b, c, A, B, C.

(9) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function satisfying $\partial f/\partial x = \partial f/\partial y$ for all $(x,y) \in \mathbb{R}^2$. If $f(x,0) \geq 0$ for all $x \in \mathbb{R}$, show that $f(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$.

(10) (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is any continuous function and that $f(x) = a_0 + \sum_{n\geq 1} (a_n \cos(nx) + b_n \sin(nx))$ is its Fourier series on $[0, 2\pi]$. Prove that

$$b_1 \leq \frac{1}{4\pi} + \int_0^{2\pi} [f(x)]^2 \ dx.$$

(b) Find all functions f as above so that $b_1 = \frac{1}{4\pi} + \int_0^{2\pi} [f(x)]^2 \ dx$.