Real Analysis Preliminary Exam
August, 2023

Justification is required for all statements.

1. Let $f, g : [0,1] \to [0,\infty)$ be continuous functions such that $\sup_{0\le x\le 1} f(x) = \sup_{0\le x\le 1} g(x)$. Prove that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.

2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^2 function satisfying $\lim_{x\to 0} \frac{\phi(x)}{x} = a$. [Note that C^2 means that ϕ'' exists everywhere on $\mathbb{R}$ and is continuous.] For which $a \in \mathbb{R}$ does the series $\sum_{n=1}^{\infty} \phi\left(\frac{1}{n}\right)$ converge?

3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that
$$\lim_{n\to\infty} \int_0^1 nx(1-x^2)^{n-1} f(x)\, dx = \frac{f(0)}{2}.$$
[Hint: First show that $\int_0^1 nx(1-x^2)^{n-1}\, dx = \frac{1}{2}$.]

4. Is
$$g(x) = \sum_{n=2}^{\infty} \left(\frac{x}{\ln n}\right)^n$$
continuous on $\mathbb{R}$? Is g uniformly continuous on $\mathbb{R}$? Explain.

5. Let $G \subset \mathbb{R}^2$ be an open set, and suppose that $[0,1]\times[0,1] \subset G$. Show that there exists $\varepsilon > 0$ such that
$$[0, 1+\varepsilon] \times [0,1] \subset G.$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous positive function, and let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
$$F(x,y,z) = \left(\int_0^x f(t)\ dt, \int_0^{x+y} f(t)\ dt, \int_0^{x+y+z} f(t)\ dt\right).$$
Prove that in a small neighborhood of an arbitrary point of $\mathbb{R}^3$, F is invertible and the inverse is continuously differentiable.

7. Let $U \subset \mathbb{R}^2$ be the region bounded by parabolas $y = x^2$ and $x = y^2$. Evaluate
$$\int_{\partial U} \left(y + x^{2023}\right) dx + \left(2023x - e^{2023y^2}\right) dy,$$
assuming that ∂U is oriented counterclockwise.

8. Find the Fourier series for $p(x) = x^2$ on $[-\pi, \pi]$. Sketch a graph of the function defined on all of $\mathbb{R}$ to which the Fourier series converges pointwise. Is the convergence uniform? Use the Fourier series to show that
$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + ...$$