# REAL ANALYSIS PRELIMINARY EXAMINATION

AUGUST 2022

(1) Let $f(x) = x$.

  (a) Find the Fourier series of $f(x)$ on $[-\pi, \pi]$.

  (b) Graph the function $g : \mathbb{R} \to \mathbb{R}$ to which the Fourier series converges pointwise.

  (c) Does the Fourier series converge to $g$ uniformly?

(2) Let $f(x, y) = \sqrt{|xy|}$.

  (a) Prove that $f$ is continuous at $(x, y) = (0, 0)$.

  (b) Prove that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(x, y) = (0, 0)$.

  (c) Prove that $f$ is not differentiable at $(x, y) = (0, 0)$.

(3) For $\alpha \neq 0$, evaluate $\displaystyle\lim_{x \to 0} \left( \frac{\alpha^2}{\sin^2 \alpha x} - \frac{1}{\sin^2 x} \right)$.

(4) Find $\displaystyle\int\int_S \vec{F} \cdot \vec{N} dS$, where $\vec{F}(x, y, z) = (x + 2yz, -y + z^2, 3z^3 - xy)$, $S$ is the surface of the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$, and $\vec{N}$ is the outward unit normal to the surface.

(5) Let $(a_n)$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n$ converges.

  (a) Prove that the series $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ also converges.

  (b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ such that the series $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges.

(6) (a) Prove that the set of the limit points of a sequence of real numbers is a closed set.

(b) Construct a sequence of real numbers so that the set of the limit points of the sequence is precisely the set of natural numbers $\mathbb{N}$.

(7) Suppose that $f$ is a twice continuously differentiable function with positive second derivative on the closed interval $[0, 2]$. Suppose in addition that $f(0) = 1$, $f(1) = -1$, and $f(2) = 2$. Prove that $f$ has exactly two zeros on $(0, 2)$.

(8) Suppose that functions $f$ and $g$ are Riemann integrable on the interval $[a, b]$ and that

$$\int_a^b f(x)dx > \int_a^b g(x)dx.$$

Use the definition of Riemann integrability to prove that there exists $[c, d] \subset [a, b]$ with $c < d$ such that $f(x) > g(x)$ on $[c, d]$.