# Real Analysis Preliminary Exam
# January, 2011

## Directions: All problems require justification. The time limit is 4 hours.

1. Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

2. Find the maximum and minimum values of $x^2 + 4y - 6z$ on the half ellipsoid $x^2 + 2y^2 + 3z^2 = 125$, $z \geq 0$.

3. Let $g$ be a real-valued function that is differentiable and has a bounded derivative on $\mathbb{R}$. Prove that $g$ is uniformly continuous on $\mathbb{R}$.

4. Assume that $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function, and suppose that the graph of $F(x,y,z) = 0$ in $\mathbb{R}^3$ is a surface that is tangent to the plane $z = 2x - y + 3$ at the point $(1,-1,6)$.

   a. Prove that there exists an open disk $D$ of some positive radius centered at the point $(-1,6) \in \mathbb{R}^2$ and a function $g : D \to \mathbb{R}$ such that $F(g(u,v),u,v) = 0$ for all $u,v \in D$.

   b. Find all possible values of $\nabla g(-1,6)$.

5. For $f$ bounded on $[a,b]$ with $a < b$, show that f is Riemann-integrable on $[a,b]$ if and only if $f$ is Riemann-integrable on $[c,d]$ for all $c,d \in \mathbb{R}$ such that $a < c < d < b$.

6. State and prove the theorem about the existence of a radius of convergence for a general real power series in one variable.

7. Compute the volume of the solid unit ball in $\mathbb{R}^n$.

8. Prove that a monotone function on $\mathbb{R}$ has at most countably many points of discontinuity.

9. Let $M(f,a,b,n)$ denote the midpoint estimate for $\int_a^b f(x)\,dx$ using $n$ equal-width subdivisions of $[a,b]$. If $f$ is twice continuously differentiable, prove that

$$\int_a^b f(x)\,dx = M(f,a,b,n) + \frac{f''(c)(b-a)^3}{24n^2}$$

   for some $c \in [a,b]$.