Real Analysis Preliminary Exam
August 21, 2015
4 hours

(1) Consider the sequence (a_k) defined inductively by $a_1 = \sqrt{6}$ and $a_{k+1} = \sqrt{6 + a_k}$. Prove that it converges, and find its limit.

(2) State and prove the chain rule for differentiable functions of one variable.

(3) Prove the following or give a counterexample: If $a_n \geq 0$ for all n and

$$\sum_{n=1}^{\infty} a_n \text{ converges, then } \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \text{ converges as well.}$$

(4) Prove that a monotone bounded function on $[a, b]$ is Riemann integrable on $[a, b]$.

(5) Put $f(x, y) = \dfrac{x^3 y}{x^4 + y^2}$ for all $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.
(a) Prove that f is continuous at $(0, 0)$.
(b) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0, 0)$.
(c) Is $f(x, y)$ differentiable at $(0, 0)$?

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be the function of period 2π such that $f(x) = x^3$ for $-\pi \leq x < \pi$.
(a) Prove that the Fourier series for f has the form $\sum_{n=1}^{\infty} b_n \sin nx$ (no cosine terms). Write, but do not evaluate, an integral formula for b_n in terms of n.
(b) Discuss the pointwise convergence of the Fourier series above on $\mathbb{R}$.
(c) Prove that

$$\sum_{n=1}^{\infty} b_n^2 = \frac{2\pi^6}{7}.$$

(7) Describe, with proof, the possible values of $\displaystyle\oint \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$, where C is a simple closed, smooth curve, traversed counterclockwise, that does not pass through $(0, 0)$.

(8) Let A be a compact subset of $\mathbb{R}^n$, $x \in A$. Let (x_i) be a sequence of points in A such that every convergent subsequence of (x_i) converges to x.
(a) Prove that the sequence (x_i) itself converges to x.
(b) Give an example showing (a) can fail if A is not compact.