1. Consider a linearly ordered set X with the order topology.
   (a) Show X is Hausdorff.
   (b) If X is infinite and well-ordered, show there are infinitely many $x \in X$ such that $\{x\}$ is open.
   (c) Give an example of X infinite and well-ordered where the topology is not discrete.
2. Let A' denote the set of limit points of a subset of a topological space. Let $A \subset X$ and $B \subset Y$. In $X \times Y$, prove that $(A \times B)' \supset A' \times B'$ and give an example that shows equality may not hold.
3. Prove that a metric space X is 2nd countable if and only if it is separable.
4. Let X and Y be topological spaces and $f : X \to Y$ a function.
   (a) Suppose $f : X \to Y$ be continuous. Let the sequence (x_n) converge to x. Prove $(f(x_n))$ converges to $f(x)$.
   (b) Now assume X is 1st countable. Suppose whenever (x_n) converges to x, we also have $(f(x_n))$ converges to $f(x)$. Prove f is continuous.
5. Let X be a countable metric space. Show that X has a basis consisting of sets that are both open and closed.
6. Let X be a metric space with no isolated points and let S be a discrete subspace of X. Show that $\bar{S}$ contains no open set.
7. Suppose that a compact metric space has at most countably many points. Find such a space with infinitely many isolated points and infinitely many non-isolated points.
8. Provide a proof or counterexample: The intersection of a decreasing sequence of compact, connected sets in a Hausdorff space is connected.
9. Suppose A and B are subsets of a space X, and that $A \cup B$ and $A \cap B$ are connected.
   (a) If A and B are closed, prove they are also connected.
   (b) Give an example where A and B are not connected.
10. Prove that a compact metric space has a countable dense subset.
11. A space is Lindelöf if every cover by open sets has a countable subcover. Prove that a closed subspace of a Lindelöf space is Lindelöf.
12. Let X be a metric space. Prove that:
    (a) For any $A \subset X$, $d(x;A) = inf\{d(x;a) : a \in A\}$ defines a continuous function from X to $\mathbb{R}$.
    (b) $\bar{A} = \{x \in X : d(x;A) = 0\}$.
    (c) X is a normal topological space.
13. Let A be a subspace of a regular space X. Show that X/A is Hausdorff if and only if A is closed.
14. For subsets A and B of a metric space with distance function d, define $d(A,B) = inf\{d(a,b) : a \in A, b \in B\}$.
    (a) If A and B are compact, prove there exist $a \in A$ and $b \in B$ such that $d(A,B) = d(a,b)$.
    (b) Give a counterexample if “compact” is replaced by “closed.”
15. Give an example of a subset of $\mathbb{R}^n$ having uncountably many connected components. Can such a subset be open? Closed?
16. Show that $\{0,1\}^{[2,3]}$ is separable but not second countable. (Note that $\{0,1\}^{[2,3]} = \left\{\text{functions } f : [2,3] \to \{0,1\}\right\}$, the Cartesian product of an uncountable number of copies of the two point Hausdorff space $\{0,1\}$, indexed by the interval $[2,3]$.)
17. Let $X = C([0,1])$ be the space of continuous, real-valued functions on $[0,1]$, with the topology generated by sets of the form

$$U(f,\varepsilon) = \{g \in C([0,1]) : \sup_{t \in I} |f(t) - g(t)| < \varepsilon\},$$

    for $f \in X$, $\varepsilon > 0$. Prove or disprove that
    a. X is Hausdorff.
    b. X is locally connected.
    c. X is locally compact.
18. For $\alpha \in \mathbb{R}$, let X_α be the quotient $\mathbb{R}^2/\sim_\alpha$, where the equivalence relation is defined by

$$(x,y) \sim_\alpha (x, \alpha x + y)$$

for all $x, y \in \mathbb{R}^2$. Find necessary and sufficient conditions that the quotient topology on X_α is Hausdorff.

19. Show that a compact metric space cannot be isometric to a proper subset of itself.
20. Let $E_1, E_2, \ldots$ be nonempty closed subsets of a complete metric space (X, d) with $E_{n+1} \subset E_n$ for all positive integers n, and such that $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$, where $\operatorname{diam}(E)$ is defined to be

$$\sup\{d(x,y) \mid x, y \in E\}.$$

Prove that $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$.
21. A standard theorem states that a continuous real valued function on a compact set is bounded. Prove the converse: If K is a subset of $\mathbb{R}^n$ and if every continuous real valued function on K is bounded, then K is compact.
22. Let K be a nonempty compact set in a metric space with distance function d. Suppose that $\varphi: K \to K$ satisfies

$$d(\varphi(x), \varphi(y)) < d(x,y)$$

for all $x \neq y$ in K. Show there exists precisely one point $x \in K$ such that $x = \varphi(x)$.
23. Prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.
24. (a) Prove that a continuous image of a compact space is compact.
 (b) Prove that a continuous image of a connected space is connected.
25. Prove that a compact Hausdorff space is both regular and normal.
26. Prove that a second-countable, compact topological space is necessarily sequentially compact (i.e. every sequence contains a convergent subsequence).
27. Let $X = \prod_{i=1}^{\infty}[0,1]$ and let $Y = \{x \in X | \pi_i(x) = 0 \text{ except for finitely many } i\}$.
 (a) Determine whether Y is compact when X is given the product topology.
 (b) Determine whether Y is compact when X is given the box topology.
28. Let X be a Hausdorff space.
 (a) If $S \subset X$ is a finite set, prove there is a collection $\{U_s | s \in U_s\}$ of pairwise disjoint open sets in X for which $s \in U_s$.
 (b) If S is infinite, prove the conclusion of part (a) may or may not be true.
29. Let $X \subset \mathbb{R}^2$ be the set of vertical lines with integer x-intercepts.
 (a) Describe the construction of $\hat{X}$, the one-point compactification of X, and explain its topology.
 (b) Determine whether $\hat{X}$ is homeomorphic to the Hawaiian earring, that is, the union of all circles in $\mathbb{R}^2$ with centers $(1/n; 0)$ and radius $1/n$ where n is a positive integer.
 (c) Determine whether $\hat{X}$ is homeomorphic to the quotient space of $\mathbb{R}$ that has $\mathbb{Z}$ identified to a point.
30. Let X be a topological space. Which of the following four properties hold for a subspace Y whenever they hold for X: compactness, second countability, local connectedness, regularity? (Prove or describe a counterexample.)
31. Show that if X and Y are connected spaces then so is $X \times Y$.
32. State the Unique Path Lifting Lemma from covering space theory and give a brief outline of its proof.
33. (a) State the Seifert-van Kampen theorem.
 (b) Use (a) to calculate the fundamental group of wedge of n circles (use induction).
34. Let X be homotopy equivalent to a singleton space.
 (a) Prove that X is path connected.
 (b) Prove that X is simply connected.
35. Prove that if $f_0, f_1 : X \to Y$ are homotopic maps, and $g_0, g_1 : Y \to Z$ are homotopic maps, then $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic maps. Prove that if X is a space and Y is a contractible space, then any two maps from X to Y are homotopic.

36. Prove the following weak version of the Seifert-van Kampen theorem: If $X = U \cup V$ where U, V are open, $U \cap V$ is path connected and x is in $U \cap V$ then $\pi_1(X;x)$ is generated by the images of $\pi_1(U;x)$ and $\pi_1(V;x)$ in $\pi_1(X;x)$.
37. Prove or disprove:
(a) If $f : X \to Y$ is continuous and injective, then $f\# : \pi_1(X;x) \to \pi_1(Y;f(x))$ is injective.
(b) If $f : X \to Y$ is continuous and surjective, then $f\# : \pi_1(X;x) \to \pi_1(Y;f(x))$ is surjective.
(c) If $c : A \subset X$ is the inclusion and $r : X \to A$ is a retraction then $c\#$ is injective and $r\#$ is surjective.
38. Let $C(X;\mathbb{R}) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}$. Let open balls under the sup norm be a basis for a topology on $C(X;\mathbb{R})$. Prove that $C(X;\mathbb{R})$ is contractible.
39. Prove the Zig-Zag lemma: let $0 \to C \to D \to E \to 0$ be a short exact sequence of chain complexes with the above maps being $f : C \to D, g : D \to E$. Show that there is a long exact sequence of homology groups that arises from this situation.
40. Consider the figure-eight X with base point x. Let $G = \langle a \rangle$ be the subgroup of $\pi_1(X,x)$ generated by the loop a which circles the right hand loop in X exactly once. Draw an explicit picture of the covering space $\tilde{X}$ whose projection $p : (\tilde{X},\tilde{x}) \to (X,x)$ takes $\pi_1(\tilde{X},\tilde{x})$ isomorphically onto G.
41. For each of the following, either give an example or explain why none exists.
(a) A connected space X such that $\pi_1(X)$ is a non-trivial finite group.
(b) A space X such that $H_2(X)$ is a non-trivial finite group.
(c) A retraction of the 2-sphere to the circle on the equator.
(d) A continuous function from the 2-sphere to itself with no fixed points.
42. Compute the relative homology groups $H_n(S^3,A)$, for all $n \geq 0$, where A is a finite set of points in S^3.
43. Prove that if $m \neq n$, $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.
44. Construct a map from S^2 to S^2 of degree two.
45. Suppose that M and N are closed surfaces. If $M\#N$ denotes the connected sum of M and N, derive a formula for the Euler characteristic of $M\#N$ in terms of the Euler characteristics of M and N.
46. Find the fundamental group of
    a. the torus.
    b. the genus two surface.
    c. real projective n-space $\mathbb{RP}^n$.
    d. the Klein bottle.
    e. the wedge $\mathbb{RP}^2 \vee \mathbb{RP}^2$.
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48. Construct a space whose fundamental group is the free product of $\mathbb{Z}_2$ with $\mathbb{Z}$.
49. Compute the homology groups (with integer coefficients) of
    a. The wedge $S^1 \vee S^2$ of a circle and a 2-sphere.
    b. $S^1 \times S^2$.
    c. The Klein bottle.
    d. The four-sphere with an embedded torus collapsed to a point.
50. Let X be 2-dimensional complex defined as an equilateral triangle with edges oriented clockwise and which are all identified.
    a. Find $H_*(X,\mathbb{Z})$.
    b. Find $H_*(X,\mathbb{Z}_3)$.

c. Prove or disprove that X is a surface.