SAMPLE QUESTIONS FOR PRELIMINARY REAL ANALYSIS EXAM

VERSION 2.0

Contents

1. Undergraduate Calculus

(1.1) Find

$$\int_0^{2\pi} x^2 \sin(x)\ dx.$$

(1.2) Find

$$\lim_{x\to\frac{\pi}{2}} (\pi - 2x)\sec(x).$$

(1.3) Find

$$\int_1^2 \frac{\exp(-1/x)}{x^2} dx.$$

(1.4) Find $p, q \in \mathbb{Z}_{>0}$ such that

$$\frac{p}{q} = 0.01565656\overline{56},\ (p,q) = 1.$$

(1.5) Find the interval of convergence of the series

$$\sum_{n\geq 1} \frac{2^{-n}(x+1)^n}{n}.$$

(1.6) Evaluate

$$\int\int_A xy\ dA,$$

where $A = \{(x, y) : x \geq 0, y \geq 0, 2x + y \leq 4\}$.

(1.7) Find the critical points and inflection points of the function $\left(\sqrt{x}\right)^x$.

(1.8) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial function such that

$$p(0) = p(2) = 3,\ p'(0) = p'(2) = -1.$$

Find

$$\int_0^2 x\ p''(x)\ dx.$$

2. Limits and Continuity

(2.1) Let (x_n) and (y_n) be sequences of real numbers converging to x and y respectively. Prove that $(x_n y_n)$ converges to xy.

(2.2) Let f be continuous on $[a, b]$ and (x_n) be a Cauchy sequence with values in $[a, b]$. Prove that $(f(x_n))$ is a Cauchy sequence.

(2.3) Let (x_n) be a real sequence which satisfies

$$|x_n - x_{n+1}| < \frac{1}{n}$$

for all $n \geq 1$.

(a) If (x_n) is bounded, must it converge?

(b) If the subsequence (x_{2n}) converges, must (x_n) converge?

(2.4) Let a be a positive real number. Define a sequence (x_n) by

$$x_0 = 0, \quad x_{n+1} = a + x_n^2, \quad n \geq 0\,.$$

Find a necessary and sufficient condition on a in order that a finite limit $\lim\limits_{n\to\infty} x_n$ should exist.

(2.5) Prove that every sequence of real numbers contains a monotone subsequence.

(2.6) In each case, give an example of a sequence (a_n) that satisfies the inequality, or prove that no such sequence exists:

(a) $\limsup a_n \leq \lim a_n$

(b) $\limsup a_n \geq \lim a_n$

(c) $\limsup a_n \leq \liminf a_n$

(d) $\limsup a_n > \lim a_n > \liminf a_n$

(2.7) Prove that

$$\limsup n^{-2/n} = 1.$$

(2.8) Let

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ e^x & \text{otherwise} \end{cases}$$

Find the set $\{x \in \mathbb{R} : g \text{ is continuous at } x\}$.

3. Derivatives and the Mean Value Theorem

(3.1) For a finite number a, suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. State and prove the applicable version of L'Hôpital's rule.

(3.2) State and prove the Mean Value Theorem.

(3.3) Let

$$f(x) = \begin{cases} \dfrac{x}{2} + x^2 \sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) Show that f is differentiable, and compute $f'(0)$.

(b) Show that there exists $\delta > 0$ such that

$$\begin{aligned} f(x) < 0 \quad &\text{if} \quad -\delta < x < 0, \\ f(x) > 0 \quad &\text{if} \quad 0 < x < \delta. \end{aligned}$$

(c) Is f increasing on any open interval containing 0?

(3.4) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\frac{df}{dx} > 0$, then f is injective.

(3.5) Prove or disprove: if f and g are continuously differentiable functions on $(0,1)$, and if they satisfy

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = L < \infty, \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = 0,$$

then

$$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = L\ .$$

(3.6) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{x\to 0} \frac{f(x)}{x} = L, \ \ f(0) = 0,$$

for some $L \in \mathbb{R}$. Determine whether or not each statement below is necessarily true:

(a) f is differentiable at 0.

(b) $L = 0$.

(c) $\lim_{x\to 0} f(x) = 0$.

4. Infinite Series

(4.1) State and prove the ratio test for series.

(4.2) Suppose f has a continuous third derivative on $[-1,1]$. Prove that the series

$$\sum_{n=1}^{\infty} [nf(1/n) - nf(-1/n) - 2f'(0)]$$

converges.

(4.3) Let $x \in (0,1)$. Let p be a positive integer > 2. Show that there exists a sequence $(a_n)_{n\geq 1}$ of integers such that $0 \leq a_n < p$ and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

Also, show that if $p^n x \notin \mathbb{Z}$ for all $n \in \{0, 1, 2, ...\}$, then the sequence $(a_n)_{n\geq 1}$ is unique.

5. The Riemann Integral and the Mean Value Theorem for Integrals

(5.1) Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable over $[b,1]$ for all $0 < b < 1$.
(a) If f is bounded, prove f is Riemann integrable over $[0,1]$.
(b) What if f is not bounded?

(5.2) Suppose f is a continuous real-valued function. Show that

$$\int_0^1 f(x)\, x^2\, dx = \frac{1}{3}\, f(\xi)$$

for some $\xi \in [0,1]$.

(5.3) Show that there exist a constant a such that, for all integers $N \geq 2$,

$$\sum_{n=1}^{N} \frac{1}{\sqrt{n}} < 2\sqrt{N} - \sqrt{2} + \frac{a}{\sqrt{N}}.$$

(5.4) Consider the function defined by $f(x) = \sin(1/x)$, $x > 0$, $f(0) = 0$. Using the definition of Riemann integral, show that f is Riemann integrable on the interval $[0,1]$.

(5.5) Let f be differentiable on $[a,b]$ with $|f'(x)| \leq \beta$. Let P be a partition of $[a,b]$, and let $U(f,P)$ and $L(f,P)$ be the upper and lower Riemann sums. Prove that

$$U(f,P) - L(f,P) \leq \beta |P| (b-a),$$

where $|P|$ is the mesh of the partition.

(5.6) Prove the Mean-Value Theorem for Integrals: Let u and v be continuous real-valued functions on an interval $[a,b]$, and suppose that $v \geq 0$ on $[a,b]$. Then there exists a point ξ in $[a,b]$ such that

$$\int_a^b u(x)\, v(x)\, dx = u(\xi) \int_a^b v(x)\, dx.$$

(5.7) Let $f_k \to f$ uniformly on $[a,b]$ and f_k be Riemann integrable on $[a,b]$. Using the definition of Riemann integral, show that f is Riemann integrable on $[a,b]$ and

$$\int_a^b f(x)\, dx = \lim_{k\to\infty} \int_a^b f_k(x)\, dx.$$

(5.8) Suppose f is non-negative, bounded, and Riemann integrable on $[a,b]$. Prove that f^2 (the square of f) is also Riemann integrable on $[a,b]$.

(5.9) Prove or disprove that for f a continuous function defined on the interval (a,b),
(a) $\int_a^b |f|^p < \infty$ implies $\int_a^b |f|^q < \infty$ if $p > q > 1$.
(b) $\int_a^b |f|^p < \infty$ implies $\int_a^b |f|^q < \infty$ if $p > q > 0$.
(c) $\int_a^b |f|^p < \infty$ implies $\int_a^b |f|^q < \infty$ if $p > q$.
(d) $\int_a^b |f|^p < \infty$ implies $\int_a^b |f|^q < \infty$ if $q > p > 1$.

(5.10) Let f and g be continuous functions on $\mathbb{R}$ such that $f(x+1) = f(x)$ and $g(x+1) = g(x)$ for all $x \in \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)\ g(nx)\ dx = \int_0^1 f(x)\ dx \int_0^1 g(x)\ dx.$$

(5.11) Find some reasonable sufficient conditions on a function f so that

$$\frac{d}{dt}\int_a^b f(t,x)\ dx = \int_a^b \frac{\partial f}{\partial t}(t,x)\ dx.$$

(5.12) Let $f(x,y)$ be a positive function of x and y in $\mathbb{R}$.

(a) Suppose that for each $y \in \mathbb{R}$, $f(x,y)$ is Riemann integrable as a function of x over the interval $[a,b]$. Suppose for a fixed $y_0 \in \mathbb{R}$, $\lim_{y\to y_0} f(x,y)$ exists. Prove that

$$\lim_{y\to y_0}\int_a^b f(x,y)\,dx = \int_a^b \left(\lim_{y\to y_0} f(x,y)\right)\,dx.$$

(This includes the possibility that both sides are $+\infty$.)

(b) Suppose that for each $y \in [c,d]$, $f(x,y)$ is Riemann integrable as a function of x over the interval $[a,b]$. Similarly, suppose that for each $x \in [a,b]$, $f(x,y)$ is Riemann integrable as a function of y over the interval $[c,d]$. Prove that

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

(This includes the possibility that both sides are $+\infty$.)

(5.13) Let $f(x,y)$ be a function of x and y in $\mathbb{R}$.

(a) Suppose that for each $y \in \mathbb{R}$, $f(x,y)$ is Riemann integrable as a function of x over the interval $[a,b]$. Suppose for a fixed $y_0 \in \mathbb{R}$, $\lim_{y\to y_0} f(x,y)$ exists and is a Riemann integrable function. Prove that

$$\lim_{y\to y_0}\int_a^b f(x,y)\,dx = \int_a^b \left(\lim_{y\to y_0} f(x,y)\right)\,dx,$$

and both sides are finite.

(b) In the previous formula, show that it is possible that one side exists and the other does not, if the hypotheses are weakened.

(5.14) Suppose that for each $y \in [c,d]$, $f(x,y)$ is Riemann integrable as a function of x over the interval $[a,b]$. Similarly, suppose that for each $x \in [a,b]$, $f(x,y)$ is Riemann integrable as a function of y over the interval $[c,d]$.

(a) Prove that if $\int_a^b\int_c^d |f(x,y)|\,dy\,dx$ exists as a finite double Riemann integral, then

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

(b) In the previous formula, show that it is possible that one side exists but the other does not, if the hypothesis is weakened.

6. Improper Integrals

(6.1) Consider the integral

$$\int_0^\infty \frac{\sin(x)}{x}\ dx.$$

(a) Does it converge?

(b) Does it converge absolutely?

(6.2) Let the convolution $a * b$ of functions on $\mathbb{R}$ be defined by
$$(a * b)(x) = \int_{-\infty}^{\infty} a(y)\, b(y - x)\ dy.$$
Prove that if f and g are piecewise continuous functions on $\mathbb{R}$, then
$$\int_{-\infty}^{\infty} |f * g| \leq \int_{-\infty}^{\infty} |f| \int_{-\infty}^{\infty} |g|,$$
if we assume that all the integrals converge.

(6.3) Suppose that f is uniformly continuous and absolutely integrable on $\mathbb{R}$. Prove that
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos(nx)\ dx = 0.$$

(6.4) Find
$$\lim_{m\to\infty} \sqrt{m} \int_{-\infty}^{\infty} \left(1 + x^2\right)^{-m} dx,$$
or show the limit does not exist.

(6.5) Prove or disprove that

(a) Every bounded function f on $[0, 2]$ is Riemann integrable on $[0, 2]$.

(b) Every Riemann integrable function f on $[0, 2]$ is bounded.

(c) If
$$\lim_{b\to 2^-} \int_0^b f(x)dx$$
exists and is finite, then f is Riemann integrable on $[0, 2]$.

(6.6) Let
$$F(g)(t) := \int_0^{\infty} \exp(-tx) g(x)\, dx.$$

(a) Prove that if g is continuous and bounded on $(0, \infty)$, then $F(g)$ is continuous and bounded on $(1, \infty)$.

(b) Prove that if g is continuous and bounded on $(0, \infty)$, then it is not necessarily true that $F(g)$ is continuous and bounded on $(0, \infty)$.

(c) Prove that if g is continuous, bounded and improper Riemann integrable on $[0, \infty)$, then $F(g)$ is continuous, bounded and improper Riemann integrable on $[1, \infty)$.

(d) Prove or disprove the converses of (a) and (c).

(e) Show that even if g is continuous, bounded and improper Riemann integrable on $[0, \infty)$, then $F(g)$ is not necessarily Riemann integrable on $[0, \infty)$.

7. Uniform Continuity; Sequences and Series of Functions

(7.1) Prove that a uniform limit of continuous functions is continuous.

(7.2) Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for $n = 1, 2, 3, \ldots$, and satisfy $|f_n'(x)| \leq 1$. Assume $g(x) = \lim_{n\to\infty} f_n(x)$ exists for all x. Prove g is continuous.

(7.3) Let $f_n(x) = \dfrac{nx^n}{1 + nx^n}$.

(a) Find the pointwise limit of (f_n).

(b) Prove that (f_n) does not converge uniformly on $\mathbb{R}$.

(c) Prove that (f_n) converges uniformly on $[1, 2]$.

(7.4) Prove that a continuous function on a finite closed interval is uniformly continuous.

(7.5) Let f be a continuous, real-valued function defined on $\mathbb{R}$. Suppose f is uniformly continuous on $(-\infty, 0)$ and on $(0, \infty)$. Prove f is uniformly continuous on $\mathbb{R}$.

(7.6) Define h by

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 2 - x & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the sum $f(x) = \sum_{n=0}^{\infty} h(2^n x)$ converges for every $x \in [0, 1]$.
(b) Is $f(x)$ continuous on $[0, 1]$?
(c) Does the sum for f converge uniformly on $[0, 1]$?
(d) Does the sum $\sum_{n=0}^{\infty} h(2^n x)/n$ converge uniformly on $[0, 1]$?

(7.7) Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist $a > 0$ and $b > 0$ such $|f(x)| \leq a|x| + b$ for all x.

(7.8) Suppose that (g_j) is a sequence of continuous functions on $[0, 1]$ such that $g_1 \geq g_2 \geq \ldots$ and $\lim_{n\to\infty} g_n(x) = 0$ for all $x \in [0, 1]$. Prove that (g_j) converges uniformly to the zero function.

(7.9) Fill in the following statement with the most appropriate adverb, and then prove it: *If (f_n) is a sequence of real-valued, continuous functions on R, and if $\sum f_n$ converges ___________ to f, then f is continuous.*

8. Taylor Series

(8.1) State and prove Taylor's Theorem with the remainder in a (standard) form of your choosing.

(8.2) Estimate the integral

$$I = \int_0^{1/2} \frac{\sin x}{x}\, dx$$

to an accuracy of two decimal places.

(8.3) Let f have a continuous second derivative. Show that for any x and any $h > 0$, there exists ξ between x and $x + h$ such that

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(\xi).$$

(8.4) For which positive real values of x is $\sin x \geq x - x^3/6$?

(8.5) Prove that $x \leq e^{x/2} - e^{-x/2}$ for $x \geq 0$.

(8.6) Find an integer k and a nonzero real number β such that

$$\lim_{\Delta\to 0} \Delta^k \int_0^{\Delta} \int_0^{\Delta} (e^{xy} - 1)\, dy\, dx = \beta.$$

(8.7) Show that

$$\int_0^1 \frac{1 + x^{30}}{1 + x^{60}} dx = 1 + \frac{C}{31}$$

for some C such that $0 < C < 1$.

(8.8) Show that given any positive real number x, there exists a positive integer n such that

$$\sqrt[n]{n!} > x.$$

9. Topology of $\mathbb{R}^n$

(9.1) Let $S \subset \mathbb{R}^n$ be a subset which is uncountable. Prove that there is a sequence of distinct points in S converging to a point of S.

(9.2) Let the collection $\mathcal{U}$ of open subsets of $\mathbb{R}$ cover the interval $[0, 1]$. Prove that there is a positive number δ such that any two points x and y of $[0, 1]$ satisfying $|x - y| < \delta$ belong together to some member of the cover $\mathcal{U}$.

(9.3) Fill in and prove: The only subsets of $\mathbb{R}$ that are both open and closed are ______________________.

(9.4) Prove that the set $\left\{\frac{a}{b} : a \in \mathbb{Z} \setminus 2\mathbb{Z},\ b \in \mathbb{Z}_{>0}\right\}$ is dense in $\mathbb{R}$.

(9.5) Let $h : \mathbb{R} \to \mathbb{R}$ be continuous. Let $S = \{h(c) : 0 < c < 1\}$. True or False:
- (a) S is connected.
- (b) S is open.
- (c) S is bounded.

(9.6) Let a real-valued function $f(x, y)$ of two variables satisfy the *intermediate value property* if the following holds. For any two points $(a, b), (c, d)$ in a connected component of the domain of f and a real number β that is between $f(a, b)$ and $f(c, d)$, for every continuous curve $\alpha : [0, 1] \to$domain(f) such that $\alpha(0) = (a, b)$, $\alpha(1) = (c, d)$, then there exists $t \in [0, 1]$ such that $f(\alpha(t)) = \beta$.
- (a) Prove that a continuous function on $\mathbb{R}^2$ satisfies the intermediate value property.
- (b) Give an example of a function on $\mathbb{R}^2$ that satisfies the intermediate value property but is not continuous on $\mathbb{R}^2$.

10. Multivariable Differentiation

(10.1) Let $S \subset \mathbb{R}^3$ denote the ellipsoidal surface defined by

$$2x^2 + (y - 1)^2 + (z - 10)^2 = 1.$$

Let $T \subset \mathbb{R}^3$ be the surface defined by

$$z = \frac{1}{x^2 + y^2 + 1}.$$

Prove that there exist points $p \in S$, $q \in T$, such that the line $\overline{pq}$ is perpendicular to S at p and to T at q.

(10.2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$f(x, y) = \begin{cases} y/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,. \end{cases}$$

Determine all points at which f is differentiable.

(10.3) Let f be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume that there is a differentiable function g from $\mathbb{R}^n$ to $\mathbb{R}$ having no critical points such that $g \circ f$ vanishes identically. Prove that the Jacobian determinant of f vanishes identically.

(10.4) Let

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Is f continuous?
(b) Do f_x and f_y exist?
(c) Is f differentiable?

(10.5) Classify the critical points of the function $f(x,y) = xy - x^3 - y^3$.

(10.6) Let $g(x,y)$ be a function on $\mathbb{R}^2$ whose second partial derivatives are continuous at each point. Prove that

$$\frac{\partial^2 g}{\partial x \partial y}(0,0) = \frac{\partial^2 g}{\partial y \partial x}(0,0).$$

(10.7) Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with continuous first partial derivatives, then f is differentiable at each $(x_0, y_0) \in \mathbb{R}^2$.

11. Inverse and Implicit Function Theorems

(11.1) Discuss the number of solutions in (x,y) to

$$u = x + y^2$$
$$v = y + xy$$

for (u,v) sufficiently close to $(0,0)$.

(11.2) State the implicit and inverse function theorems. Assuming knowledge of the implicit function theorem, prove the inverse function theorem.

(11.3) State the implicit and inverse function theorems. Assuming knowledge of the inverse function theorem, prove the implicit function theorem.

(11.4) Let $A \subseteq \mathbb{R}^3$ be the set defined by $x^3y + y^3z^2 - 2xz^4 = 2$. Prove or disprove that there exists $\delta > 0$ and a curve $\alpha : (-1-\delta, -1+\delta) \to A$ such that $\alpha(t) = (t, g(t), h(t))$ with g and h differentiable.

(11.5) Let $B = \{(x,y,z) \in \mathbb{R}^3 : z^5 = (x-1)^4 + y^4\}$.

(a) Prove or disprove that there exists a disk D of some radius $r > 0$ centered at $(s,t) = (-1,2)$ and a differentiable function $f : D \to \mathbb{R}$ such that the function $G(s,t) = (f(s,t), s, t)$ maps D to B.
(b) Prove or disprove that there exists a disk D' of some radius $r' > 0$ centered at $(s,t) = (1,0)$ and a differentiable function $p : D' \to \mathbb{R}$ such that the function $H(s,t) = (s,t,p(s,t))$ maps D' to B.
(c) Prove or disprove that there exists a disk D'' of some radius $r'' > 0$ centered at $(s,t) = (1,0)$ and a differentiable function $u : D'' \to \mathbb{R}$ such that the function $H(s,t) = (s, u(s,t), t)$ maps D'' to B.

12. Theorems and Integration of Vector Calculus

(12.1) Use Green's Theorem to evaluate $\int\int_E (x^2+y^2)\ dx\ dy$, where E is the solid ellipse $x^2/a^2 + y^2/b^2 \leq 1$.

(12.2) Find the volume of the intersection of the cylinders $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1\}$ and $\{(x,y,z) \in \mathbb{R}^3 : x^2+z^2 \leq 1\}$.

(12.3) Compute

$$\int_{\mathbb{R}^n} e^{-|x|^n} dx.$$

(12.4) For $k \geq 0$, let B_k be the volume of the unit ball of dimension k.
  (a) Derive a formula for B_k for $k \geq 0$.
  (b) Prove that

$$\sum_{n=0}^{\infty} B_{2n} = e^{\pi}.$$

(12.5) Find the line integral

$$\oint \left(-2y\ dx + x^2\ dy\right)$$

around the curve $x^2 + y^2 = 9$, oriented clockwise.

(12.6) Find the average value of x_1^2 on the unit sphere S^5 centered at the origin in $\mathbb{R}^6$.

(12.7) Find the average length of line segments selected at random with endpoints on the unit 3-dimensional sphere in $\mathbb{R}^4$.

(12.8) Let B^4 be the unit ball centered at the origin in $\mathbb{R}^4$. If A is an 4×4 real symmetric matrix, prove that

$$\int_{B^4} \|Ax\|^2 dx_1 dx_2 dx_3 dx_4 = \frac{V}{24} \mathrm{Tr}(A^2),$$

where V is the (3-dimensional) volume of the unit 3-sphere.

13. Fourier Series

(13.1) Prove that if $\int_0^{2\pi} (f(x))^2\ dx$ converges and f is Riemann integrable, then

$$\int_0^{2\pi} f(x)\ dx \leq \sqrt{2\pi} \sqrt{\int_0^{2\pi} (f(x))^2\ dx}.$$

(13.2) Let the function g be periodic with period 6, and let $g(x) = |x - 3|$ on the interval $[0, 6]$.
  (a) Find the corresponding Fourier series for g.
  (b) Does the series converge uniformly to g?
  (c) Use the series to find a formula for

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

  (d) Use the series to find a formula for

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}.$$

(13.3) The function h is periodic of period 4 and satisfies

$$h(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 4 \end{cases}$$

  (a) Find the corresponding Fourier series for h.
  (b) Prove or disprove that the Fourier series converges uniformly to h on $[1, 2]$.

(c) Prove or disprove that the Fourier series converges uniformly to h on $[2, 3]$.

(13.4) Let f be a C^∞ function on $\mathbb{R}$ that is periodic of period 2π. Let $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ be its Fourier series.

(a) Prove that the coefficients a_n are uniquely determined by the equation.

(b) Prove that for every positive integer k, there exists a positive real number C such that for all $n \in \mathbb{Z}$, $|a_n| < C|n|^k + C$.

(13.5) Prove that if g is a real-valued continuous function on $\mathbb{R}$ that is periodic of period 2, and if h is the sum of the first 12 terms of the Fourier series of g, then

$$\int_{-1}^{1} h(t)g(t)\,dt = \int_{-1}^{1} (h(t))^2\,dt.$$

14. Inequalities and Estimates

(14.1) Prove the triangle inequality:

(a) For real numbers.

(b) For vectors in $\mathbb{R}^n$.

(c) For the space $B(\mathbb{R})$ of bounded continuous functions on the real line, where we define $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$ if $f \in B(\mathbb{R})$.

(d) For the space $L^2(\mathbb{R})$.

(14.2) Prove that there exists a real number $K > 0$ such that for any $\varepsilon > 0$, if $a, b \in \mathbb{R}$, then

$$K|ab| \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2.$$

Furthermore, find all possible constants K with this property.

(14.3) Prove that there exists a constant $C > 0$ such that $(x^6 + 2x^2) \leq Cx^7$ for all $x > 1$, and find the smallest such constant.

(14.4) Prove or disprove that there exists constants $p > 0$ and $\varepsilon > 0$ such that

$$p - p\cos(x) + p^2x^2 > x$$

for all x such that $0 < x < \varepsilon$.

(14.5) Prove that if $a > 1$ and $b > 1$, there exists a real number M such that $a^x > x^b$ for all $x > M$. Does there always exist a positive real number m such that $a^m \leq m^b$?

15. Suggested Practice Exams

(A) Do the following problems:
(1.7),(2.2),(3.6),(4.1),(6.6),(7.7),(10.7),(5.3)

(B) Do the following problems:
(3.3),(5.4),(7.4),(8.7),(10.5),(11.4),(13.3),(14.5)

(C) Do the following problems:
(2.3),(7.3),(8.5),(9.6),(11.3),(12.7),(13.2),(14.3)