# SAMPLE QUESTIONS FOR PRELIMINARY ALGEBRA EXAM

Contents



## 1. Linear Algebra

(1.1) State and prove the Spectral Theorem on the diagonalization of real symmetric matrices.

(1.2) Prove the following are equivalent for an $n \times n$ real matrix $A$.
- (a) The columns of $A$ form an orthonormal basis for $\mathbb{R}^n$.
- (b) The rows of $A$ form an orthonormal basis for $\mathbb{R}^n$.
- (c) For every $x \in \mathbb{R}^n$, $||Ax|| = ||x||$.
- (d) For every $x, y \in \mathbb{R}^n$, $Ax \cdot Ay = x \cdot y$.

(1.3) Let $A$ and $B$ be diagonalizable matrices such that $AB = BA$. Prove there is a matrix $P$ such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal matrices.

(1.4) Let $\mathbb{F}$ be a field. For $m$ and $n$ positive integers, let $M_{m\times n}$ be the vector space of $m \times n$ matrices over $\mathbb{F}$. Fix $m$ and $n$, and fix matrices $A$ and $B$ in $M_{m\times n}$. Define the linear transformation $T$ from $M_{n\times m}$ to $M_{m\times n}$ by

$$T(X) = AXB.$$

Prove that if $m \neq n$, then $T$ is not invertible.

(1.5) Prove or disprove each of the following for square matrices $A$ and $B$ over $\mathbb{C}$:

(a) $A + B$ is nonsingular if $A$ and $B$ are nonsingular.

(b) $A + B$ is nonsingular if $A$ and $B$ are real symmetric matrices and all of their eigenvalues are strictly positive.

(c) $A + B$ is nonsingular if all of the eigenvalues of $A + A^*$ and $B + B^*$ are strictly positive. ($A^*$ denotes the conjugate transpose of $A$.)

(1.6) Suppose that the $r \times r$ upper left minor $M$ of $A$ is nonsingular but that every other minor containing $M$ is singular. Prove that $\operatorname{rank}(A) = r$.

(1.7) Let $V$ and $W$ be finite dimensional vector spaces, let $X$ be a subspace of $W$, and let $T : V \to W$ be a linear map. Prove that the dimension of $T^{-1}(X)$ is at least $\dim V - \dim W + \dim X$.

(1.8) Prove that there exists a real symmetric matrix $A$ such that

$$A^2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

(1.9) Construct a $3 \times 3$ matrix with eigenvalue 1 and mapping $[0,1,2]^T$ and $[1,1,1]^T$ to $[1,2,3]^T$ and $[4,5,6]^T$, respectively.

(1.10) Prove or give a counterexample to each of the following. Unless otherwise noted, $A$ is an $n \times n$ complex matrix.

(a) If $\lambda$ is an eigenvalue of $A$ and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.

(b) If $A$ is real and $\lambda$ is an eigenvalue of $A$, then so is $-\lambda$.

(c) If $A$ is real and $\lambda$ is an eigenvalue of $A$, then so is $\overline{\lambda}$.

(d) If $\lambda$ is an eigenvalue of $A$ and $A$ is nonsingular, then $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

(e) If all the eigenvalues of $A$ are zero, then $A = 0$.

(f) If $A$ is diagonalizable and all its eigenvalues are equal, then $A$ is diagonal.

(1.11) Suppose the matrix $A$ satisfies $A^2 = A$. Prove $A$ is diagonalizable.

(1.12) Prove that if $A$ is a real symmetric $n \times n$ matrix, then

$$\inf_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$$

is necessarily an eigenvalue of $A$. Is the statement true, if $A$ is not required to be symmetric?

(1.13) Find necessary and sufficient conditions that $\{(a,b),(c,d)\}$ is a basis of $\mathbb{R}^2$.

(1.14) For which $m$ is the vector $(1,2,m,5)$ a linear combination of $(0,1,1,1)$, $(0,0,0,1)$, and $(1,1,2,0)$?

(1.15) Find the set of all $6 \times 6$ matrices $M$ such that there exists another matrix $N$ such that $N^{-1}MN$ is a multiple of the identity matrix.

(1.16) Suppose that a $7 \times 7$ complex matrix $M$ satisfies $(M - I)(M^* - 2I) = 0$. Prove that $M^* = M$, and find all possible sets of eigenvalues of $M$.

(1.17) Suppose that $\|\bullet\|_1$ and $\|\bullet\|_2$ are norms on a vector space $V$. Prove or disprove that

(a) $\|v\|_3 = 5\|v\|_1 \|v\|_2$ for $v \in V$ defines a norm on $V$.

(b) $\|v\|_4 = 5\|v\|_1 + \|v\|_2$ for $v \in V$ defines a norm on $V$.

(1.18) Let $P_2 = P_2(\mathbb{R})$ denote the real vector space of real polynomials of degree at most 2.

(a) Prove that $\langle p, q \rangle = \displaystyle\int_{-1}^{1} p(x)\, q(x)\, dx$ is an inner product on $P_2$.

(b) Show that there is a unique $q \in P_2$ such that

$$\int_{-1}^{1} p(x)\ \cos(\pi x)\, dx = \langle p, q \rangle$$

for all $p$ and find it.

(1.19) Let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors corresponding to different eigenvalues of the real symmetric matrix $A$. Prove that $\mathbf{u}$ and$\mathbf{v}$ are orthogonal.

(1.20) Define an inner product on the space of real polynomials by

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx.$$

(a) Find an orthonormal basis for the subspace spanned by 1, $x$, and $x^2$.
(b) Calculate the projection of $x^3$ onto this subspace.

## 2. Groups and Subgroups, Permutations

(2.1) Let $H$ be a nonempty subset of a multiplicative group $G$ such that $H$ is closed under the group operation. Suppose in addition that if $x, y \in H$, then $xy^{-1} \in H$. Prove or disprove that $H$ is necessarily a subgroup of $G$.
(2.2) Prove that any group of order 6 is isomorphic to either $\mathbb{Z}_6$ or $S_3$ (the group of permutations of three objects).
(2.3) Suppose that the group $G$ is generated by elements $x$ and $y$ that satisfy $x^5y^3 = x^8y^5 = 1$. Is $G$ the trivial group?
(2.4) Prove that every finite group is isomorphic to
1. A group of permutations;
2. A group of even permutations.

(2.5) Let $\mathbb{F}_2 = \{0, 1\}$ be the field with two elements. Let $G$ be the group of invertible $2 \times 2$ matrices with entries in $\mathbb{F}_2$. Show that $G$ is isomorphic to $S_3$, the group of permutations of three objects.

## 3. Homomorphisms, Factor Groups

(3.1) Let $p$ be the smallest prime factor of a nontrivial finite group. Prove that any subgroup of index $p$ is normal.
(3.2) Show that there are at least two nonisomorphic nonabelian groups of order 24.
(3.3) Let $G$ be a group and $N$ be a proper normal subgroup of $G$. Suppose that there does not exist a subgroup $H$ of $G$ satisfying $N \subsetneqq H \subsetneqq G$. Prove that the index of $N$ in $G$ is finite and equal to a prime number.

## 4. Classification of Abelian Groups

(4.1) Let $G$ be an abelian group. Suppose $G$ has subgroups of orders $m$ and $n$. Show $G$ has a subgroup of order the least common multiple of $m$ and $n$.
(4.2) Let $A$, $B$, and $C$ be finite abelian groups. Prove that if $A \times B$ is isomorphic to $A \times C$, then $B$ is isomorphic to $C$.

## 5. Advanced group theory, Sylow theorems

(5.1) Prove that every finite group of prime power order has a nontrivial center.
(5.2) Let $G$ be a group of permutations of $\{1, 2, \ldots, n\}$. We say $G$ is *transitive* if for every $i, j$, there exists $\sigma \in G$ such that $\sigma(i) = j$. Prove the order of a transitive group $G$ is divisible by $n$.
(5.3) Let $G$ and $H$ be finite groups of relatively prime order. Show that $\mathrm{Aut}(G \times H)$, the group of automorphisms of $G \times H$, is isomorphic to the direct product of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$.
(5.4) Suppose that $G$ is a group such that every subgroup $H$ of $G$ is normal. Does this imply that $G$ is abelian?

(5.5) Prove that every group of order 99 contains a nontrivial, proper, normal subgroup.

## 6. Rings and Ideals

(6.1) Let $R$ be a ring with 1, and let $I$ be the left ideal of $R$ generated by $\{ab-ba \mid a, b \in R\}$. Prove that $I$ is a two-sided ideal.

(6.2) Let $R$ be a ring with identity, and let $u$ be an element of $R$ with a right inverse. Prove that the following conditions on $u$ are equivalent:
1. $u$ has more than one right inverse;
2. $u$ is a left zero divisor;
3. $u$ is not a unit.

(6.3) Suppose that $R$ is a subring of a commutative ring $S$ and that $R$ is of finite index $n$ in $S$. Let $m$ be an integer that is relatively prime to $n$. Prove that the natural map $R/mR \to S/mS$ is a ring isomorphism.

(6.4) Let $R$ be a ring with 1. Suppose that $A_1, A_2, \ldots, A_n$ are left ideals in $R$ such that $R = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ (as additive groups). Prove that there are elements $u_i \in A_i$ such that for any elements $a_i \in A_i$, $a_i u_i = a_i$ and $a_i u_j = 0$ if $j \neq i$.

(6.5) Let $R = \mathbb{Q}$ be the ring of rational numbers.
- (a) Give an example of a maximal ideal of $R$ that is proper (i.e. $\neq \{0\}$, $\neq R$), or prove that no such ideal exists.
- (b) Give an example of a principal ideal of $R$ that is proper (i.e. $\neq \{0\}$, $\neq R$), or prove that no such ideal exists.

(6.6) Let $R = \mathbb{Q}[x, y]$ be the ring of polynomials in two variables with rational coefficients.
- (a) Give an example of a maximal ideal of $R$ that is proper (i.e. $\neq \{0\}$, $\neq R$), or prove that no such ideal exists.
- (b) Give an example of a principal ideal of $R$ that is proper (i.e. $\neq \{0\}$, $\neq R$), or prove that no such ideal exists.

## 7. Fields and Galois Theory

(7.1) (a) Prove that the polynomial $x^4 + x + 1$ is irreducible over $\mathbb{Q}$.
(b) What about over $\mathbb{F}_2$ and $\mathbb{F}_3$?

(7.2) Let $A$ be an invertible matrix over a finite field. Prove there exists an integer $k$ such that $A^k = I$.

(7.3) Show that if a finite subset of a field forms a group, then it must be cyclic.

(7.4) Show that a field extension of degree 2 is normal.

(7.5) Prove that if $K$ is an algebraic extension of $k$, $G$ a finite group of automorphisms of $K$ with fixed field $F$, then $K$ is a Galois extension of $F$.

(7.6) Let $R$ be an integral domain with field of fractions $K$. We say $R$ is *integrally closed* if no element of $K - R$ is a root of a monic polynomial in $R$.
(a) Prove that a unique factorization domain is integrally closed.
(b) For $k$ a field and $t$ an indeterminant, show that $k[t^2, t^3]$ is not integrally closed.

(7.7) Let $K$ be a field containing the field $F$. Let $a$ be an element of $K$ that is algebraic over $F$. Show that the field $F(a)$ equals $F[a]$, the polynomials in $a$ over $F$.

(7.8) Let $K$ be an extension field of $F$. If $a$ and $b$ are elements of $K$ that are algebraic over $F$, prove $a + b$ is algebraic over $F$. (You may not use any theorems about algebraic extensions of algebraic extensions.)

(7.9) Let $p$ be an odd prime and $\mathbf{F}_p$ the field of $p$ elements. How many elements of $\mathbf{F}_p$ have square roots in $\mathbf{F}_p$? How many have cube roots in $\mathbf{F}_p$?
(7.10) Prove or disprove that it is possible for a fourth degree polynomial to a have a Galois group of $\mathbb{Z}_6$.
(7.11) Prove or disprove that it is possible for a fifth degree polynomial to a have a Galois group of $\mathbb{Z}_6$.
(7.12) Let $\mathbb{F}$ be a finite extension field of $\mathbb{Q}$, and let $a, \xi \in \mathbb{F}$ such that $\xi^n = a$. Prove that the Galois group of $x^n - a$ over $\mathbb{F}$ is abelian.
(7.13) Prove or disprove that a $1°$ angle can be constructed using a straightedge and compass alone.
(7.14) Show that $x^3 + 3x + 1$ is irreducible over $\mathbb{Q}$. Let $\alpha$ denote a root. Express $\alpha^{-1}$ and $(1+\alpha)^{-1}$ as a $\mathbb{Q}$-linear combination of $1, \alpha, \alpha^2$.
(7.15) Show that $x^6 + 3$ is irreducible over $\mathbb{Q}$, but is not irreducible over $\mathbb{Q}(\omega)$, where $\omega$ is a primitive sixth root of unity.
(7.16) Prove that every irreducible factor of the $n$th cyclotomic polynomial $\Phi_n$ over a field $K$ has the same degree.
(7.17) Let $L$ be an extension of the field $K$ and $M$ be an extension of $L$. Prove $[M : K] = [M : L] \cdot [L : K]$.
(7.18) Suppose $L_1$ and $L_2$ are two extension fields of $K$. Prove $[K(L_1, L_2) : K] \leq [L_1 : K][L_2 : K]$.
(7.19) Suppose $[K(\alpha) : K]$ and $[L : K]$ are relatively prime. Show that the minimal polynomial of $\alpha$ over $L$ has coefficients in $K$.
(7.20) Suppose $\alpha \notin K$ is algebraic over $K$ and $\beta$ is transcendental over $K$. Prove that $K(\alpha, \beta)$ is not a simple extension of $K$.
(7.21) Suppose $\beta$ is transcendental over $K$. Prove that $K(\beta)$ is not algebraically closed.
(7.22) Prove that a finite extension is simple if and only if it contains finitely many intermediate subfields.
(7.23) Let subfields $M_1$ and $M_2$ of $L$ be normal extensions of $K$. Prove that $K(M_1, M_2)$ and $M_1 \cap M_2$ are also normal extensions of $K$.
(7.24) Give an example, with proof, where $L$ is a normal extension of $K$, $M$ is a normal extension of $L$, and yet $M$ is not a normal extension of $K$.
(7.25) Let $L$ be an algebraic extension of $K$. Show that there is a greatest intermediate field for which $M$ is a normal extension of $K$.
(7.26) Prove that $L$ is a separable extension of $K$ and $M$ is a separable extension of $L$ if and only if $M$ is a separable extension of $K$.
(7.27) Let Char $K = p > 0$ with $f$ irreducible in $K[x]$. Show that $f$ can be written in the form $f(x) = g(x^{p^n})$, where $n$ is a nonnegative integer and $g$ is irreducible and separable.
(7.28) Suppose $L$ and $M$ are finite, separable extensions of $K$. Prove that the following are equivalent:
(i) For every pair of monomorphisms $\sigma : L \to \overline{K}$ and $\tau : M \to \overline{K}$ fixing $K$, there exists a monomorphism $\phi : K(L, M) \to \overline{K}$ fixing $K$ such that $\phi|_L = \sigma$ and $\phi|_M = \tau$.
(ii) For every pair of monomorphisms $\sigma : L \to \overline{K}$ and $\tau : M \to \overline{K}$ fixing $K$, there exists a unique monomorphism $\phi : K(L, M) \to \overline{K}$ fixing $K$ such that $\phi|_L = \sigma$ and $\phi|_M = \tau$.

(iii) $[K(L, M) : K] = [L : K] \cdot [M : K]$.
(iv) $L = K(\alpha)$, where the minimal polynomial of $\alpha$ over $K$ is irreducible in $M[x]$.
(v) $M = K(\beta)$, where the minimal polynomial of $\beta$ over $K$ is irreducible in $L[x]$.

(7.29) (a) Prove that every finite extension of a finite field is Galois.
(b) Prove that the Galois group is cyclic.

(7.30) Find the Galois group of $x^4 - 4x^2 + 2$ over $\mathbb{Q}$.

(7.31) Find the Galois groups of $x^3 - x + 1$ over $\mathbb{F}_2$ and $\mathbb{F}_3$.

(7.32) Let $\mathbb{Q}$ be the base field.
(a) Let $\omega$ be a primitive 7th root of unity. Find the minimal polynomial for $\omega + \omega^2 + \omega^4$.
(b) Show that the intersection of the splitting fields for $\Phi_7$ and $x^4 - 7$ is a quadratic extension of $\mathbb{Q}$.
(c) Find the Galois group for the splitting field of $\Phi_7 \cdot (x^4 - 7)$.

(7.33) Show that the Galois group fo $x^{15} - 2$ over $\mathbb{Q}$ can be generated by elements $\rho$, $\sigma$, and $\tau$ satisfying

$$\rho^{15} = \sigma^4 = \tau^2 = 1,$$
$$\sigma^{-1}\rho\sigma = \rho^7,$$
$$\tau^{-1}\rho\tau = \rho^{14},$$
$$\tau^{-1}\sigma\tau = \sigma.$$

(7.34) Let $f(x) \in K[x]$ be irreducible of degree 6.
(I) Suppose first that $f$ is not separable.
(a) What are the possibilities for the characteristic of $K$?
(b) What is the form of $f$ for these characteristics?
(c) What is the degree of the splitting field of $f$ for these characteristics?
(II) Now suppose instead that $f$ is separable, that $\alpha$ is a root of $f$, and that $f$ has splitting field $L$ of degree 18.
(d) How many automorphisms $K(\alpha) \to K(\alpha)$ are there that fix $K$?
(e) Find the group of automorphisms of $L$ fixing $K$, expressing it as a subgroup of $S_6$.

(7.35) Suppose $\alpha$ is transcendental over $K$. Let $\sigma$ be the automorphism of $K$ which fixes $K$ and takes $\alpha$ to $1/(1-\alpha)$. Verify that $\sigma^3$ is the identity and determine the fixed field of $\sigma$.

(7.36) Suppose $L$ is a Galois extension of $K$. Prove that $L = K(\alpha)$ if and only if the images of $\alpha$ under $G$ are distinct.

(7.37) Given a finite group $G$, show there exists a Galois extension $L : K$ with Galois group isomorphic to $G$.

(7.38) Suppose $L$ and $M$ are distinct cyclic extensions of $K$. Find necessary and sufficient conditions for $K(L, M)$ to be a cyclic extension of $K$.

(7.39) Suppose $L$ is a finite normal extension of $K$ and that $f$ is irreducible in $K[x]$. Suppose $g$ and $h$ are irreducible monic factors of $f$ in $L[x]$. Show that there is an automorphism $\sigma$ of $L$ which fixes $K$ and such that $\sigma(g) = h$.

(7.40) Suppose $K$ has characteristic 0 and the extension $L$ has basis $\{\beta_1, \ldots, \beta_n\}$ over $K$. Let $H$ be a subgroup of the group of automorphisms of $L$ fixing $K$. Let $\gamma_j = \sum_{\sigma \in H} \sigma(\beta_j)$. Prove that $K(\gamma_1, \ldots, \gamma_n)$ is the fixed field of $H$.