Topology Preliminary Examination

January 18, 2013

1. Let A be connected subspace of a topological space X. Prove that the closure of A in X is connected.

2. Let $C([0,\infty))$ denote the space of continuous real–valued functions on the interval $[0,\infty)$. Consider the topology generated by sets of the form
$$U(f,\delta) = \left\{ g \in C([0,\infty)) : \sum_{i=0}^{\infty} |f(i) - g(i)| < \delta \right\}.$$
Prove or disprove that $C([0,\infty))$ is Hausdorff with this topology.

3. Let $\mathbb{R}$ denote the real line **as a set**. Let $\mathcal{B}$ be the collection of subsets of $\mathbb{R}$ of the following two forms
    - sets of the form $(-b,-a) \cup (a,b)$, where $0 < a < b$ and (a,b) denotes the usual interval in $\mathbb{R}$;
    - sets of the form $(-\infty,-d) \cup (-c,c) \cup (d,\infty)$, where $0 < c < d$.

    (a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

    (b) For $\mathbb{R}$ with this topology, decide what are the point(s) of convergence, if any, for the following sequences:

    i. $x_n = 1 - \frac{1}{n}$

    ii. $x_n = n$

4. Let $f : X \to Y$ be a continuous map that is a bijection.

    (a) Show that f is a homeomorphism if X is compact and Y is Hausdorff.

    (b) Given an example of a continuous bijection that is not a homeomorphism.

5. Compute the homology of the Klein bottle.

6. Let M be a surface without boundary, and let $x \in M$. Compute the homology groups $H_*(M, M - x)$.

7. Suppose $A \subset X$ is a (strong) deformation retract, and let $a_0 \in A$. Show that the inclusion $i : (A, a_0) \to (X, a_0)$ induces an isomorphism $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$.

8. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^2$. Compute $\pi_1(X)$, describe the universal cover $\widetilde{X}$, and describe the deck transformations of $\pi_1(X)$ on $\widetilde{X}$.