Complex Analysis Exam **September 4, 2021**

1. Consider the function $u : \mathbb{C} - \{0\} \to \mathbb{R}$ given by $u(x+iy) = \dfrac{2y}{x^2+y^2}$.
(a) Show by a direct calculation that u is harmonic.
(b) Find explicitly a harmonic function $v : \mathbb{C} - \{0\} \to \mathbb{R}$ for which $f = u + iv$ is holomorphic.

2. Let $p(z) = z^5 + 5z^3 - 100$. Prove that $p(z)$ has exactly five zeros (counting multiplicity) and all lie in the annulus $\{z \in \mathbb{C} : 2 < |z| < 4\}$.

3. Find the radius of convergence of the Taylor series at $z = 0$ for the function $g(z) = \dfrac{1}{\sin(z) - 2}$.

4. (a) Find all possible entire functions f such that $|f(z) - e^z| < 2021$ for all z in $\mathbb{C}$.
(b) Prove that a sequence of entire functions that converges uniformly must converge to an entire function.

5. Find a conformal map of the infinite strip $|\text{Im} z| < 1$ onto the unit disk $|z| < 1$.

6. Use a contour integral to compute $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(\alpha x)dx}{x^4+1}$, where $\alpha > 0$.

7. If f is entire and $\displaystyle\lim_{|z|\to\infty} \frac{|f(z)|}{|z|^3}$ is a strictly positive number, show that $f(z)$ is a polynomial of degree exactly three.

8. For $a \in \mathbb{C}$ with $|a| \neq 1$, consider the map $f(z) = \dfrac{z-a}{1-\bar{a}z}$.
(a) Show that $f(z)$ is an invertible map of the extended complex plane.
(b) Show that $f(z)$ takes the unit circle to itself.
(c) Show that $f(z)$ takes the unit disk to itself if and only if $|a| < 1$.