COMPLEX ANALYSIS PRELIM EXAM
AUGUST, 2015

(1) Find all values of $(\sqrt{3}+i)^{2-i}$.

(2) Compute $\int_C \frac{z^3-6}{2z-i}\,dz$

(a) around the circle $|z-2|=1$, oriented counterclockwise

(b) around the circle $|z-1|=2$, oriented clockwise.

(3) Expand the function

$$\frac{1}{1-z^2}+\frac{1}{3-z}$$

as a Laurent series centered at 0 that converges when $z=2i$.

(4) (a) Exhibit a conformal map from $\{z : \mathrm{Re}z>0, \mathrm{Im}z>0\}$ onto $U=\{z : \mathrm{Re}(z)>1\}$.

(b) Could any such map in (a) be a linear fractional transformation?

(c) Is there a non-constant holomorphic function $f : \mathbb{C} \to U$?

(5) How many solutions does the equation $z^5+15z=1$ have inside the annulus $1.5<|z|<2$?

(6) Let D be a bounded connected open set in $\mathbb{C}$, and consider points $P_1,\ldots,P_n$ in $\mathbb{C}$. Show that the function $f(z)=|z-P_1|\cdots|z-P_n|$ defined on $\overline{D}$ attains its maximum at a point of ∂D.

(7) Let $n\geq 2$ be an integer. Show that

$$\int_0^\infty \frac{dx}{1+x^n}=\frac{\pi/n}{\sin \pi/n}$$

by utilizing the contour formed by the segment $[0,R]$, the arc represented by Re^{it}, $0\leq t\leq 2\pi/n$, and the segment $re^{\frac{2\pi i}{n}}$, $0\leq r\leq R$.

(8) Let f,g be holomorphic functions on the open unit disk D. Assume that neither f nor g has a zero in D. If

$$\frac{f'\left(\frac{1}{n}\right)}{f\left(\frac{1}{n}\right)}=\frac{g'\left(\frac{1}{n}\right)}{g\left(\frac{1}{n}\right)}$$

for $n=1,2,3,\ldots$, find another simple relation between f and g.