

# PRELIMINARY COMPLEX ANALYSIS EXAM

AUGUST, 2010

Work all problems, justifying everything, clearly identifying any major theorems used. In the unlikely event that you find an error in one of the problems, clearly state what the error is. If the statement of a problem is false, provide a counterexample. If you believe there is a typographical error, notify the proctor.

- (1) Find the Taylor expansion of  $\frac{1}{z^2 - 5z}$  at  $z = 2$ , and find the largest disk on which it converges to the function.
- (2) Suppose that  $g(x, y) = e^{2y} \cos(cx) + ih(x, y)$  for  $x, y \in \mathbb{R}$ . Find all possible constants  $c \in \mathbb{R}$  and real-valued functions  $h(x, y)$  so that  $g(x, y)$  is an entire function of  $z = x + iy$ .
- (3) Express all values of  $(1 - i\sqrt{3})^{3-4i}$  in the form  $a + bi$ , with  $a, b \in \mathbb{R}$ .

- (4) Give an example of a meromorphic function  $\psi$  that has a pole of order five at  $0 \in \mathbb{C}$  and such that

$$\int_C \psi(z) dz = 4,$$

where  $C$  is the unit circle, oriented counterclockwise.

- (5) Find all analytic functions  $h$  defined on the half-plane  $\{z = x + iy : x > 0, y \in \mathbb{R}\}$  satisfying the formula

$$h(e^{i/n}) = \frac{i}{3n} + e^{2i/n}, n \geq 1.$$

- (6) Find all possible  $b > 0$  such that

$$i \int_{|z|=b} \frac{\alpha'(z)}{\alpha(z)} dz > 0,$$

where

$$\alpha(z) = \frac{z^2 e^{3z} (z+1)^2 (z-3)^2}{(z-2i)^5 (z+4i)^3},$$

and where the circles  $|z| = b$  are oriented counterclockwise.

- (7) Use contour integrals to evaluate

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 1} dx.$$

- (8) Find a conformal map from the quadrant  $\{z = x + yi : x > 0, y > 0\}$  to the half-circle  $\{z = x + yi : |z| < 1, y < 0\}$ .
- (9) Find all entire functions for which  $|f(z)| \leq |z|^2$  for  $|z| \leq 1$  and  $|f(z)| \leq |z|^3$  for  $|z| \geq 1$ .
- (10) Find the number of zeros of  $f(z) = z^5 - 20z^4 + 5z^3 - z^2 + 50z - 17$  inside the annulus  $1 \leq |z| \leq 5$ .