COMPLEX ANALYSIS PRELIMINARY EXAMINATION
AUGUST 16, 2013

(1) An entire function $f = u + iv$ with u, v real-valued satisfies $u_x v_y - u_y v_x = 1$ for all values of $z = x + iy$ in the complex plane. Prove that f has the form $f(z) = az + b$, where a and b are constants and $|a| = 1$.

(2) Find $\int_\Delta (2z^4 - z\cos(1/z))\ dz$, where Δ is the unit circle centered at 0, oriented counterclockwise.

(3) Find $\int_0^\infty \frac{3x^2}{(x^2+4)^2}\, dx$.

(4) Find the Laurent expansion of the function $f(z) = \frac{2}{z^2-3z+2}$ valid for $\sqrt{2} < |z+i| < \sqrt{5}$.

(5) Suppose that f and g are entire functions, that $|f(z)| \leq |g(z)|$ for all z, and that $f(1) = g(1) \neq 0$. Prove that $f(z) = g(z)$ for all z.

(6) Suppose that f is a holomorphic function on the open disk of radius 3 centered at 0, and suppose that f maps the closed annulus $\{z : 1 \leq |z| \leq 2\}$ into the open unit disk. Prove that the restriction of f to $D(0,2) = \{z : |z| < 2\}$ has exactly one fixed point.

(7) Use the argument principle to prove Rouche's Theorem.

(8) Find an orientation-preserving conformal map from the open unit disk D to $\{x+iy \in D : y < 0 \text{ and } x < 0\}$.