Complex Analysis Preliminary Exam

August 2016

INSTRUCTIONS: Work all eight problems. Justify your work. Each problem is worth 10 points. You have four hours to complete the exam.

1. Solve the equation $z^2 - (1+3i)z - 2 = 0$, expressing your solutions in the form $a + bi$, where a and b are explicit real numbers.

2. If $\Sigma a_n z^n$ has radius of convergence R, what are the radii of convergence of the series $\Sigma a_n z^{2n}$ and $\Sigma a_n^2 z^n$?

3. Let a and b be positive real numbers. Use the method of contour integrals to evaluate $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+b^2}\,dx$.

4. Let Ω be an open, connected set in the complex plane. Let $u : \Omega \to \mathbb{R}$ be a non-constant harmonic function. Prove that the set of critical points of u, $\mathcal{C}_u = \{z \in \Omega : \nabla u(z) = 0\}$, has no limit point in Ω.

5. If f is holomorphic in an open set containing the closed unit disk centered at 0 and $|f(z)| < 1$ for $|z| = 1$, prove that, for every integer $n \geq 1$, the equation $f(z) = z^n$ has exactly n solutions, counting multiplicities, in the open unit disk centered at 0.

6. (a) Construct a bijective, holomorphic function f from the open first quadrant

$$\{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$$

   onto the open unit disk.

   (b) Assume that such a function f (not necessarily your example) can be analytically continued to a holomorphic function g on the union of the open first quadrant and a neighborhood of 0. Prove that $g'(0) = 0$.

7. Let n be a positive integer. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying $|f(z)| \leq |z|^n + 1$. Show that f is a polynomial of degree at most n.

8. Suppose that g is a holomorphic function defined on $\{z \in \mathbb{C} : z \neq 0\}$, and suppose that

$$|g'(z)| \leq \frac{1}{|z|^{3/2}}$$

   for $0 < |z| \leq 1$. Prove that $z = 0$ is a removable singularity for g.