Complex Analysis Preliminary Exam
January 13, 2024

Justification is required for all statements.

1. Find all Laurent series expansions of $\dfrac{2z+3}{z+1}$ centered at $z=1$, and determine their open annuli of convergence.

2. Find all possible holomorphic functions ϕ on $\mathbb{C}$ such that $\left(\phi\left(e^{i/n}\right)\right)^2 = 5\phi\left(e^{i/n}\right)$ for all positive integers n.

3. Find a conformal map of the set $S = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ to the set $T = \{z \in \mathbb{C} : |z| < 1$ and $\operatorname{Re} z > 0\}$. (Supply only brief justification that your map is conformal.)

4. Calculate
$$\int_0^\infty \frac{dx}{(a+bx^2)^2}, \quad a, b > 0.$$

5. An entire function $f = u + iv$ with u, v real-valued satisfies $u_x + v_y = 0$ for all values of $z = x + iy$ in the complex plane. Prove that f has the form $f(z) = az + b$, where a and b are constants and $\operatorname{Re}(a) = 0$.

6. Compute $\int_0^{2\pi} \exp\left(\exp\left(i\theta\right)\right)\ d\theta$.

7. Let f, g be entire functions such that $|f| \le |g|$. Prove that f is a constant times g.

8. Find all possible entire functions p such that for some constant $A > 0$, $|p\left(z\right)| \le \left(\sqrt{|z|} + A\right)^2$ for $|z| > 2024$.