COMPLEX ANALYSIS PRELIMINARY EXAMINATION

AUGUST 2022

In the questions below, for $a \in \mathbb{C}$ and $R > 0$, let $D(a, R)$ denote the open disk of radius R centered at a.

(1) Find the Laurent expansion of $h(z) = \dfrac{1}{z^2 - z - 2}$ centered at $z = 2$ that converges at $z = 5i$. For which $z \in \mathbb{C}$ does this expansion converge to $h(z)$?

(2) Compute

$$\int_\gamma \frac{z+\pi}{(z-\pi)(e^z+1)} dz,$$

where γ is the boundary of the rectangle $\{x + iy : x, y \in \mathbb{R}, |x| < 1, -1 < y < 10\}$, oriented counterclockwise.

(3) Find all possible holomorphic functions g defined on the unit disk $D(0, 1)$ such that $g(z) = \overline{g(z)}$ whenever $|z| < \frac{1}{4}$.

(4) Prove that if $z \in \mathbb{C}$ and $\mathrm{Re}(z^n) \geq 0$ for all $n \in \mathbb{N}$, then $z \in [0, \infty)$.

(5) Let f be an entire holomorphic function that maps $D(0, 5)$ into $D(4i, 2)$. Prove that the equation $f(z) + 9z^3 - z = 0$ has exact 3 solutions in the unit disk $D(0, 1)$.

(6) Find all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$|f(z)| \leq \sqrt{|z|^2 + |z|}$$

for all $z \in \mathbb{C}$.

(7) Use the method of residues to compute

$$\int_0^\infty \frac{dx}{4 + x^4}.$$

(8) (a) Find an orientation-preserving conformal map $\phi : S \to H$ from the set

$$S = \{x + iy : x, y \in \mathbb{R}, x^2 + y^2 < 4, y > 0\}$$

onto

$$H = \{x + iy : x, y \in \mathbb{R}, x > 0\}$$

such that $\phi(i) = 2 + i$.

(b) Prove or disprove that the map is unique.

(c) Does there exist an orientation-preserving, surjective conformal map $G : \mathbb{C} \to H$?