Algebra Preliminary Exam

August 2016

INSTRUCTIONS: Work all eight problems. Justify your work. Each problem is worth 10 points. You have four hours to complete the exam.

1. Let H and K be normal subgroups of a group G and suppose that $H \cap K = \{e\}$. Prove that $hk = kh$ for every $h \in H$ and $k \in K$.

2. Find a product of cyclic groups that is isomorphic to the factor group
$$(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1, 2)\rangle.$$

3. Let V be a finite-dimensional real vector space.
   (a) Suppose S is a finite set that spans V. Prove that some subset of S is a basis for V.
   (b) Suppose that L is a linearly independent subset of V. Prove that exists a basis B for V such that L is a subset of B.

4. Let I be an ideal in a commutative ring R. Define the *radical* of I to be
$$\operatorname{Rad} I = \{a \in R : a^n \in I \text{ for some positive integer } n\}.$$
Prove that $\operatorname{Rad} I$ is an ideal in R.

5. Prove that $d(a + bi) = a^2 + b^2$ makes the ring $\mathbb{Z}[i]$ into a Euclidean domain.

6. Find the Galois group of $x^4 + 4x^2 + 2$ over $\mathbb{Q}$, and give the Galois correspondence between the subgroups and the subfields of the splitting field.

7. (a) List four nonisomorphic groups of order 2^3. Be sure to state how you know that each one of the four that you list is not isomorphic to the other three.
   (b) For each prime p, construct a nonabelian group of order p^3.

8. Let A be an $n \times n$ matrix with entries in $\mathbb{C}$, and suppose that $A^3 = A$.
   (a) Prove that A is diagonalizable.
   (b) Is A necessarily diagonalizable if $\mathbb{C}$ is replaced by an arbitrary field F?