Algebra Preliminary Exam
January 31, 2017
4 hours

(1) Suppose that B is a real $n \times n$ matrix such that $B^3 - B$ is invertible. Prove that 1 is not an eigenvalue of B.

(2) Let G be a group and let Aut G denote its group of automorphisms. Define an operation on the set $H = G \times \text{Aut } G$ by $(x, \phi) \cdot (y, \psi) = (x\phi(y), \phi \circ \psi)$ (where $(\phi \circ \psi)(x) = \phi(\psi(x))$).
  (a) Prove that $(H, \cdot)$ is a group under this operation.
  (b) Prove or disprove that if G is abelian then $(H, \cdot)$ is an abelian group under this operation.

(3) Let $GL_n(\mathbb{R})$ denote the multiplicative group of real $n \times n$ matrices. Let $SL_n(\mathbb{R})$ be the subgroup whose determinant is 1. Show that $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ is isomorphic to the multiplicative group $\mathbb{R}^\times$.

(4) Let G be a finite abelian group of order n. Prove that G is not cyclic if and only if there exists a positive integer $k < n$ such that $g^k = e$ for every $g \in G$.

(5) Let A be an $n \times n$ matrix over a field F. Prove that there exist invertible matrices P and Q with entries in F such that $Q^{-1}AP$ is a diagonal matrix.

(6) Suppose that $p(x)$ and $q(x)$ are two polynomials in $\mathbb{Q}[x]$. Let $I = \langle p(x), q(x) \rangle$ be the ideal generated by $p(x)$ and $q(x)$.
  (a) Provide necessary and sufficient conditions on the polynomials $p(x)$ and $q(x)$ so that I is a prime ideal.
  (b) Provide necessary and sufficient conditions on the polynomials $p(x)$ and $q(x)$ so that $\mathbb{Q}[x]/I$ is a field.

(7) Let F be a field and let $F(\alpha)$ be an extension of F of degree 7. Prove that $F(\alpha^3) = F(\alpha)$.

(8) (a) Prove that $\mathbb{Q}(\sqrt{-3})$ is the splitting field of $x^3 - 1$ over $\mathbb{Q}$.
  (b) Find the minimal polynomial $\alpha = \sqrt[3]{2} + \sqrt{-3}$ over $\mathbb{Q}(\sqrt{-3})$ and over $\mathbb{Q}$.
  (c) Find the Galois group of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.