Algebra Preliminary Exam
January 16, 2016
4 hours

(1) Prove Lagrange's Theorem: If H is a subgroup of a finite group G, then the order of H divides the order of G.

(2) Let $T : V \to W$ be a linear transformation of finite-dimensional vector spaces.
   (a) If T is injective, prove there exists a linear transformation $T_1 : W \to V$ such that $T_1 \circ T : V \to V$ is the identity map.
   (b) If T is surjective, prove there exists a linear transformation $T_2 : W \to V$ such that $T \circ T_2 : W \to W$ is the identity map.

(3) Let R be an integral domain. Show that there cannot be principal prime ideals $\langle x \rangle$ and $\langle y \rangle$ such that
$$\{0\} \subsetneqq \langle y \rangle \subsetneqq \langle x \rangle \subsetneqq R.$$

(4) Find the minimal polynomial of $\alpha = \sqrt{5} - \sqrt{2}$ over $\mathbb{Q}$.

(5) Let S be a nonempty subset of a group G such that $xy^{-1}z \in S$ for all $x, y, z \in S$. Prove that there are a subgroup H and an element g in G such that $S = Hg$.

(6) (a) Prove that the subset H of S_4 consisting of the identity and all products of pairs of disjoint transpositions is a normal subgroup of S_4.
   (b) Prove that $S_4/H \cong S_3$.

(7) Let $F(\alpha)$ and $F(\beta)$ be cubic extensions of a field F such that $F(\alpha, \beta)$ is a degree 6 extension of F. Prove that $F(\alpha, \beta)$ is a Galois extension of F with Galois group isomorphic to S_3.

(8) Let A and B be real, commuting matrices. Suppose further that A is diagonalizable over the reals and that B has minimal polynomial $x^4 - 1$. Prove that at least one eigenspace for A has dimension greater than 1.