# Algebra Preliminary Exam
January 15, 2015
2 pm to 6 pm

(1) (a) Prove that every group of order 99 is abelian.

(b) Find all isomorphism classes of groups of order 99.

(2) Let $M$ be a unitary matrix. (A square complex matrix $M$ is unitary if and only if $\overline{M^T}M = I$.) Prove from basic definitions that

(a) All eigenvalues of $M$ have absolute value 1.

(b) Eigenvectors of $M$ corresponding to different eigenvalues are (complex) orthogonal.

(3) Let $R$ be a commutative ring with unity. Suppose the zero divisors and 0 form an ideal $I$. Prove that $R/I$ is an integral domain.

(4) (a) Prove that conjugacy in a group is an equivalence relation.

(b) Prove that a group of prime power order has a non-trivial center.

(5) Let $x_i$, $i = 1, \ldots 4$, be positive real numbers with $\sum x_i < 1$. Prove that the following matrix is invertible.

$$\begin{pmatrix} x_1 - x_1^2 & -x_1x_2 & -x_1x_3 & -x_1x_4 \\ -x_2x_1 & x_2 - x_2^2 & -x_2x_3 & -x_2x_4 \\ -x_3x_1 & -x_3x_2 & x_3 - x_3^2 & -x_3x_4 \\ -x_4x_1 & -x_4x_2 & -x_4x_3 & x_4 - x_4^2 \end{pmatrix}$$

(6) Let $R$ be a ring with unity and let $I$ and $J$ be two-sided ideals of $R$. Define a ring homomorphism $\phi : R \to R/I \times R/J$ by $\phi(r) = (I + r, J + r)$.

(a) Find a relationship between $I$ and $J$ that is equivalent to $\phi$ being injective.

(b) Prove that $\phi$ is surjective if and only if $I + J = R$.

(7) Let $K$ be a degree 4 extension of $\mathbb{Q}$. Prove that $K$ is Galois over $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if there exist integers $a$ and $b$ such that $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$.

(8) Let $p$ be prime.

(a) For how many elements $\alpha \in \mathbb{F}_{p^6}$ is it true that $\mathbb{F}_{p^6} = \mathbb{F}_p(\alpha)$?

(b) How many monic irreducible polynomials of degree 6 are there in $\mathbb{F}_p[x]$?