Algebra Preliminary Exam
August 19, 2015
4 hours

(1) Let H and K be subgroups of a group G. For $x, y \in G$, prove that the double cosets HxK and HyK are either equal or disjoint.

(2) Suppose the matrix A has rank r. Prove that there exist r rows and r columns such that the $r \times r$ submatrix consisting of the entries from the chosen rows and columns is invertible.

(3) (a) Prove that the eigenvalues in $\mathbb{C}$ of an orthogonal matrix have absolute value 1.
(b) Prove that orthogonal matrices are diagonalizable over $\mathbb{C}$.

(4) Prove that the fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{6})$ are not isomorphic.

(5) Prove that there are at least four nonisomorphic groups of order 20.

(6) Let α and β be complex numbers such that $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are Galois extensions of $\mathbb{Q}$. (This implies $\mathbb{Q}(\alpha, \beta)$ is also a Galois extension of $\mathbb{Q}$.)
(a) Prove that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = [\mathbb{Q}(\beta) : \mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta)]$.
(b) Using part (a) or otherwise, prove $\mathrm{Gal}(\mathbb{Q}(\alpha, \beta)/\mathbb{Q}(\alpha)) \cong \mathrm{Gal}(\mathbb{Q}(\beta)/\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta))$.

(7) Let R_1 and R_2 be rings.
(a) If R_1 and R_2 have multiplicative identities, prove that every (two-sided) ideal in $R_1 \times R_2$ has the form $I_1 \times I_2$, where I_1 and I_2 are ideals in R_1 and R_2, respectively.
(b) Construct an example to show the conclusion in (a) may be false if R_1 and R_2 do not have multiplicative identities.

(8) Let $\mathbb{Z}_n$ denote the integers mod n. For what $n > 1$ is it true that the number of distinct roots in $\mathbb{Z}_n$ of every monic polynomial over $\mathbb{Z}_n$ is at most the degree of the polynomial?