Algebra Preliminary Exam
August 22, 2014

Work all eight problems, justifying your work. Each is worth 10 points. You have four hours.

(1) Let G be a group with subgroup H and normal subgroup N.
  (a) Prove that HN is a subgroup of G.
  (b) Prove the Second Isomorphism Theorem: the groups $(HN)/N$ and $H/(H \cap N)$ are isomorphic.

(2) Let R be a ring with identity, and let a be an element of R with a right inverse. Prove that the following are equivalent:
  (i) a has more than one right inverse,
  (ii) a is a zero divisor,
  (iii) a is not a unit.

(3) (a) Find the centralizer of $(1, 2, 3, 4, 5)$ in S_6 .
  (b) Find the normalizer of $\langle(1, 2, 3, 4, 5)\rangle$ in S_6.

(4) Let F be the splitting field of $(x^2 - 3)(x^3 - 5)$ over $\mathbb{Q}$. Find the degree of F over $\mathbb{Q}$.

(5) For an $n \times n$ matrix A, prove the following are equivalent.
  (i) $A^T A = I$.
  (ii) For every $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x}$ and $\mathbf{x}$ have the same length.
  (iii) For every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the dot product of $A\mathbf{x}$ and $A\mathbf{y}$ equals the dot product of $\mathbf{x}$ and $\mathbf{y}$.

(6) Let F, K, L, M be fields such that $[K : F] = 3$, $[L : F] = 4$, $[M : F] = 12$, $K \subset M$, $L \subset M$, and L and M are Galois over F. Prove that there exists a field N with $F \subset N \subset M$ and $[N : F] = 2$.

(7) Let A be a matrix over an algebraically closed field whose characteristic polynomial has distinct roots. Let B be a matrix that commutes with A.
  (a) Prove that there exists a matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal.
  (b) Prove that B is a polynomial in A.

(8) Prove that, up to isomorphism, there are exactly five groups of order 8.