Algebra Preliminary Exam
March 1, 2013

Work all eight problems, justifying your work. Each is worth 10 points. You have four hours.

(1) Let $A = \begin{pmatrix} -1 & 4 & 5 \\ -4 & 10 & 8 \\ 5 & -12 & -9 \end{pmatrix}$.

  (a) Determine a matrix P and a diagonal matrix D such that $D = P^{-1}AP$.
  (b) Compute A^{2013}.

(2) Let B be a real matrix that is diagonalizable over the reals. Prove that B is symmetric if and only if eigenvectors corresponding to distinct eigenvalues are orthogonal.

(3) Let G be a group and let Z be its center. Prove that if G/Z is cyclic, then G is abelian.

(4) Suppose F is a field, K is an algebraic extension of F, and L is an algebraic extension of K. Prove L is an algebraic extension of F.

(5) (a) Prove that the only two-sided ideals in the ring of real $n \times n$ matrices are the zero ideal and the ring itself.
  (b) For each $n \geq 2$, exhibit a nontrivial proper one-sided ideal in the ring of real $n \times n$ matrices.

(6) Let G be a group of order 56. Prove that G has a nontrivial proper normal subgroup.

(7) For all pairs of positive integers m and n, find the number of
  (a) group homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$;
  (b) ring homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

(8) Let K be the splitting field of $x^4 - 2$ over $\mathbb{Q}$.
  (a) Determine the Galois group $G(K/\mathbb{Q})$.
  (b) Determine all intermediate fields E such that $\mathbb{Q} \subset E \subset K$, writing each explicitly in the form $\mathbb{Q}(a, b, \ldots)$.