Algebra Preliminary Exam
Fall, 2013

Work all eight problems, justifying your work. Each is worth 10 points. You have four hours.

(1) Let G be a group, and for each g in G, define a function $\phi_g : G \longrightarrow G$ by the formula $\phi_g(x) = gxg^{-1}$ for every x in G.

(a) Prove that the set $\mathrm{Inn}(G) = \{\phi_g : g \in G\}$ is a group under function composition.

(b) Let $Z(G)$ denote the center of G. Prove that $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$.

(2) Let H be a finite subgroup of G. Prove that the double coset

$$HxH \overset{\text{def}}{=} \{h_1 x h_2 : h_1, h_2 \in H\}$$

has cardinality $|H|$ for all x if and only if H is a normal subgroup of G.

(3) Find a product of cyclic groups that is isomorphic to the factor group

$$(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2, 3)\rangle.$$

(4) For an $n \times n$ matrix A and eigenvalue λ_0, prove the dimension of the eigenspace for λ_0 is at most its multiplicity as a root of the characteristic polynomial.

(5) Let I be a proper nontrivial prime ideal in a principal ideal domain D. Prove that I is a maximal ideal in D.

(6) Let V be an inner product space. Let W be a subspace of V and let $W^\perp$ denote its orthogonal complement.

(a) For V finite-dimensional, prove $\left(W^\perp\right)^\perp = W$.

(b) For $V = \mathbb{R}[x]$, $\langle p, q\rangle = \displaystyle\int_{-1}^{1} p(x)\, q(x)\, dx$, W the subspace of V with basis $\{x^2, x^4, x^6, \ldots\}$, find bases for $W^\perp$ and $\left(W^\perp\right)^\perp$, and show $W \subsetneq \left(W^\perp\right)^\perp$.

(7) Suppose K is a finite extension field of E and that E is a finite extension field of F. Prove that K is a finite extension field of F, and that $[K : F] = [K : E]\,[E : F]$.

(8) Compute the Galois group of the splitting field of $x^4 + x + 1$ over $\mathbb{F}_2$ and over $\mathbb{F}_3$, the finite fields with 2 and 3 elements, respectively.