Algebra Preliminary Exam
August 19, 2011

Work all eight problems, justifying your work. Each is worth 10 points. You have four hours.

(1) Let K and M be subgroups of a group G. Define a relation $\sim$ on G by $x \sim y$ if and only if there exist $k \in K$, $m \in M$ such that $x = kym$.
  (a) Prove that $\sim$ is an equivalence relation on G.
  (b) For K and M finite, prove that the cardinality of the equivalence class of x is $\dfrac{|K| \cdot |M|}{|x^{-1}Kx \cap M|}$.

(2) A ring with multiplicative identity is called a *local ring* if it has exactly one maximal ideal. Show that a commutative ring with multiplicative identity is a local ring if and only if its set of non-units is an ideal.

(3) Let L be an algebraic extension of the field F. Show that any ring homomorphism $g : L \to L$ fixing F is an automorphism.

(4) Let A be a real 6×6 matrix of rank 4. What are the possible ranks of A^2?

(5) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Determine (concise) necessary and sufficient conditions on a field F so that A is diagonalizable over F.

(6) Show that the group $(\mathbb{R}, +)$ has no proper subgroup of finite index.

(7) Find all common zeros for the polynomials $p(x) = 2x^5 + 7x^4 + 15x^3 - 6x^2 - 21x - 45$ and $q(x) = x^5 + 3x^4 + 7x^3 - 3x^2 - 9x - 21$ over
  (a) $\mathbb{R}$;
  (b) $\mathbb{F}_{25}$, the field of 25 elements.

(8) Let K be the splitting field of $(x^3 - 2)(x^3 - 3)$ over $\mathbb{Q}$.
  (a) Find $[K : \mathbb{Q}]$.
  (b) Find the Galois group of K over $\mathbb{Q}(e^{2\pi i/3})$.
  (c) Find the Galois group of K over $\mathbb{Q}$.