ALGEBRA PRELIMINARY EXAM

AUGUST, 2010

Work all problems, justifying everything, clearly identifying any major theorems used. In the unlikely event that you find an error in one of the problems, clearly state what the error is. If the statement of a problem is false, provide a counterexample. If you believe there is a typographical error, notify the proctor.

(1) Let G be a group, and let N be a normal subgroup of G of finite index. Suppose that H is a finite subgroup of G and that the order of H is relatively prime to the index of N in G. Prove that H is contained in N.

(2) (a) Define the **rank** of a matrix over a field.

(b) Let A and B be matrices such that AB exists. Prove that the rank of AB is at most the rank of A and at most the rank of B.

(3) Suppose that $AB + BA$ is a matrix with all entries zero. Prove that:

(a) A and B are square matrices of the same size.

(b) If v is an eigenvector of A corresponding to eigenvalue α, then Bv is an eigenvector of A or the zero vector.

(4) Let S be a **subset** of a group G. Suppose that for all $g_1, g_2 \in G$ either $Sg_1 = Sg_2$ or $Sg_1 \cap Sg_2 = \emptyset$. Prove that $S = Hg$ for some subgroup H of G and some element g of G.

(5) We say that a polynomial $p(x) \in \mathbb{Z}[x]$ is **primitive** if the greatest common divisor of the set of its coefficients is 1. Prove that the product of two primitive polynomials is primitive.

(6) Suppose α and β are algebraic over a field $\mathbb{F}$. Prove $\alpha\beta$ is algebraic over $\mathbb{F}$.

(7) Find all groups of order 4 up to isomorphism, and prove that your list is complete.

(8) Which groups of order 4 can be the Galois group of an extension of $\mathbb{Q}$? Of $\mathbb{C}$? Of $\mathbb{F}_p$, the finite field of order p, where p is prime? If there is such an extension, give an example; if not, explain why there is none.

(9) Suppose that $\det(A + xB) = x^5 + 10x + 5$ for 5×5 matrices A and B with complex entries and all $x \in \mathbb{C}$. Prove that B is an invertible matrix.

(10) Let I be an ideal in a ring R and f a surjective homomorphism from I onto a ring S with multiplicative identity. Show that there exists a unique homomorphism $g : R \to S$ such that $g(x) = f(x)$ for all $x \in I$.