Algebra Preliminary Exam

August, 2023

Justification is required for all statements.

1. Let H and K be subgroups are of a group G such that K is a normal subgroup. Prove that $HK = \{hk : h \in H, k \in K\}$ is a subgroup of G.

2. Let $q(x) = (x - 3)^2$ be the minimum polynomial of the $n \times n$ matrix S, and suppose that the $n \times n$ matrix T is diagonalizable. Also, assume that $ST = TS$.

   (a) Prove that if v is a (nontrivial) eigenvector of T corresponding to eigenvalue λ, then Sv is also a (nontrivial) eigenvector of T.

   (b) Prove that at least one eigenspace of T has dimension at least two.

3. Let A and B be $n \times n$ matrices over $\mathbb{C}$. Suppose the null space of A is contained in the image [or column space] of B. Prove that $\operatorname{rank}(AB) = \operatorname{rank}(A) + \operatorname{rank}(B) - n$.

4. Prove that an ideal I of a commutative ring R is prime if and only if R/I is an integral domain.

5. Prove that for every element g of the permutation group S_N is conjugate to g^{-1}.

6. Let K be a field extension of F of degree n and let $f(x) \in F[x]$ be an irreducible polynomial of degree $m > 1$. Show that if m is relatively prime to n, then f has no root in K.

7. Let $\alpha = \sqrt{2} - \sqrt{5}$, and let $K = \mathbb{Q}(\alpha)$.

   (a) Find the minimal polynomial $p(x)$ of α over $\mathbb{Q}$.

   (b) Find the Galois group of K over $\mathbb{Q}$

   (c) Find all subfields of K, expressing them explicitly in the form $\mathbb{Q}(\alpha)$. Prove that you have listed them all.

8. Prove that, up to isomorphism, there are four groups of order 70.