Algebra Preliminary Exam
January, 2011
Directions: All problems require justification. The time limit is 3.5 hours.

1. Let A be a commutative ring with identity and I a (two-sided) ideal of A that is proper (i.e. not A, not $\{0\}$). Prove that A/I is an integral domain if and only if whenever ab is in I, then either a is in I or b is in I.

2. Let A be an $m \times n$ matrix of rank n. Prove that $P(\mathbf{x}) = A(A^T A)^{-1} A^T \mathbf{x}$ is the orthogonal projection of $\mathbf{x} \in \mathbb{R}^n$ to the column space of A.

3. Let A and N be subgroups of a group G, with N normal.
(a) Prove that AN is a subgroup of G.
(b) Prove that that quotient groups AN/N and $A/A \cap N$ are isomorphic.

4. Consider the polynomial ring $F[x]$ over a field F.
(a) State and prove the division algorithm for $F[x]$.
(b) Prove that every ideal in $F[x]$ is principal.
(c) Let $g(x)$ be a greatest common divisor of $p(x)$ and $q(x)$. Prove that there exist polynomials $a(x)$ and $b(x)$ such that $a(x)p(x) + b(x)q(x) = g(x)$.

5.
(a) Give an example of an 8×8 matrix M such that $(M + 3I)^3 = 0$ and $(M + 3I)^2 \neq 0$. (Here, 0 denotes the zero matrix.)
(b) Must every such M be invertible?

6. Let W_1 and W_2 be finite-dimensional subspaces of a vector space V. Prove that
$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

7. Find the Galois group of $f(x) = (x^2 - 2)(x^3 - 3)$
(a) over $\mathbb{Q}$,
(b) over $\mathbb{F}_7$, the finite field of order 7.

8. Suppose a group G of order 42 has abelian subgroups of orders 6, 14, and 21. Prove that G is cyclic.