Algebra Preliminary Exam
January 27, 2024

Justification is required for all statements.

(1) Let G be a group, and let $h \in G$ be fixed. Define $F_h : G \to G$ by $F_h(g) = h^{-1}gh$.

  (a) Prove that F_h is an isomorphism.

  (b) Let $H = \{h \in G : F_h \text{ is the identity}\}$. Prove that H is the center of G.

(2) Show that if K is finite dimensional field extension of F, then K is algebraic over F.

(3) Show that every subgroup of a cyclic group is cyclic.

(4) Let F be a field, and let $p(x) \in F[x]$. Prove that an element $a \in F$ satisfies $p(a) = 0$ if and only if $x - a$ is a factor of $p(x)$.

(5) Let $F \supset K$ be an extension field, and let u be an element of F. Show that if $[K(u) : K]$ is an odd number, then $K(u^2) = K(u)$.

(6) Let $T : V \to V$ be a linear transformation on a real vector space of dimension n such that, for all $v \in V$, there exists a positive integer k such that $(T - 2I)^k v = 0$, where I is the identity transformation.

  (a) Prove that $(T - 2I)^n v = 0$ for all $v \in V$.

  (b) Prove that if W is a subspace of V such that $T(W) \subseteq W$, then $T(W) = W$.

(7) Let $q(x) = x^5 - 6x + 3$.

  (a) Show $q(x)$ is irreducible over $\mathbb{Q}$.

  (b) Prove that $q(x)$ has exactly 3 real roots.

  (c) Show that the Galois group of $q(x)$ over $\mathbb{Q}$ contains elements of order 2 and 5.

(8) Prove that $\mathbb{Z}[\sqrt{-3}]$ is an integral domain but not a unique factorization domain.