Algebra Preliminary Exam

January 2023

1. Let $S = \{x \in \mathbb{R} : x \neq 2\}$. For x and y in S, define

$$x * y = xy - 2x - 2y + 6.$$

   Prove that $(S, *)$ is an abelian group.

2. Find a direct sum of cyclic groups isomorphic to the quotient group $(\mathbb{Z}_{20} \oplus \mathbb{Z}_2)/\langle(4,1)\rangle$.

3. Let $F \subset K$ be an extension field of degree n and $f(x) \in F[x]$ a polynomial of degree $m > 1$. If $\gcd(m,n) = 1$, show that f has no roots in K.

4. For an $n \times n$ complex matrix M, let M^* be conjugate transpose of M, that is, $(M^*)_{i,j} = \overline{M_{j,i}}$. Prove that if λ is an eigenvalue of M, then $\overline{\lambda}$ is an eigenvalue of M^*.

5. (a) Let A be a nonsingular $n \times n$ complex matrix, and suppose that A^2 and A^3 are diagonalizable. Prove that A is diagonalizable.

   (b) Find an $n \times n$ complex matrix B such that B^2 and B^3 are diagonalizable, but B is not.

6. (a) Prove that $p(x) = 2x^3 + 3x + 4$ is irreducible over $\mathbb{Z}_5$.

   (b) Let α be a zero of $p(x)$ in an extension field of $\mathbb{Z}_5$. Express $(3\alpha+4)^{-1}$ in the form $a\alpha^2 + b\alpha + c$ for a, b, and c in $\mathbb{Z}_5$.

7. A proper ideal P of a commutative ring R is *prime* if $ab \in P$ implies that either $a \in P$ or $b \in P$. Prove that every nonzero prime ideal in a principal ideal domain is maximal.

8. Let a and b be integers, and suppose $p(x) = x^4 + ax^2 + b^2$ is irreducible over $\mathbb{Q}$.

   (a) If α is a root of $p(x)$ in an extension field, show that $K = \mathbb{Q}(\alpha)$ is the splitting field for $p(x)$.

   (b) Compute the Galois group of K over $\mathbb{Q}$.