REAL ANALYSIS PRELIMINARY EXAMINATION
AUGUST 13, 2013

(1) Let C be the curve parametrized by

$$\vec{r}(t) = e^{\sqrt{t}}\,\vec{i} + \arctan(t^3)\,\vec{j}, \quad 0 \leq t \leq 1.$$

Evaluate

$$\int_C (6xy+2)\,dx + (3x^2+8y)\,dy.$$

(2) Determine the maximum and minimum values of the quantity $xy + 4z$ on the half ellipsoid $x^2 + 4y^2 + 2z^2 = 64, \quad z \geq 0.$

(3) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by the formula

$$f(x) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{xy^2}{x^2+y^2} & (x,y) \neq (0,0). \end{cases}$$

(a) Prove that f is continuous at $(0,0)$.

(b) Prove that if $\vec{u} = a\,\vec{i} + b\,\vec{j}$ is a unit vector, then the directional derivative of f at $(0,0)$ in the direction of $\vec{u}$ exists, and compute its value.

(c) Is f is differentiable at $(0,0)$?

(4) Let $\{f_n\}$ be a sequence of continuous functions on $[0,1]$.

(a) Suppose $\{f_n\}$ converges uniformly to a function f. Prove that f is continuous.

(b) Give an example where $\{f_n\}$ converges pointwise to a function f that is not continuous.

(5) Suppose that $\{a_n\}$ be a decreasing sequence of positive real numbers. Prove that the infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if the infinite series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

(6) Suppose that $h : [0,1] \longrightarrow \mathbb{R}$ is bounded and has the property that h is Riemann integrable on $[\epsilon, 1]$ for every $0 < \epsilon < 1$. Using the definition of the Riemann integral, prove that h is Riemann integrable on the interval $[0,1]$.

(7) Let h be a real-valued function and differentiable function on $[0, \infty)$ such that $h(0) = 1$ and $3 \leq h'(x) \leq 4$ for all $x \geq 0$. Prove that there exists a constant c such that

$$1 \leq \frac{h(x)}{\sqrt{9x^2+1}} \leq c$$

for all $x \geq 0$.

(8) Let f be a continuous function of period 2π, and suppose that

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of f.

(a) Prove that the sum $\displaystyle\sum_{n=1}^{\infty} |a_n|^2$ converges.

(b) Suppose also that $\displaystyle\sum_{n=1}^{\infty} n \max\{|a_n|, |b_n|\}$ converges. Prove that f is differentiable and that the integral $\displaystyle\int_{-\pi}^{\pi} (f'(x))^2 \, dx$ is finite.