(1) (**) In class, we considered the numbers

\[ y_1 = \sqrt{2} \approx 2.828, \]
\[ y_2 = 2 \sqrt{2 - \sqrt{2}} \approx 3.061, \]
\[ y_3 = 2^2 \sqrt{2 - \sqrt{2 + \sqrt{2}}} \approx 3.121, \]
\[ y_4 = 2^3 \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 3.136}, \]

and so on. Notice that the first sign is \((-\)), but all others are \((+\)). In the notation of Problem 3 one can think informally of \(y_n\) as being obtained from \(x_n\) by switching the first \((+\)) sign to a \((-\)) sign and multiplying by \(2^n\). Why do the terms of this sequence approach \(\pi\)?

(2) One associates square roots with irrational numbers, but some square roots come out to give nice answers.

(a) Exactly one of \(\sqrt{3} + \sqrt{7} + \sqrt{8 - 2\sqrt{7}}\) and \(\sqrt{3} + \sqrt{7} - \sqrt{8 - 2\sqrt{7}}\) is a rational number. Which one? Explain why that one is, and the other one is not.

(b) Give another example of unexpectedly rational nested square roots. Can you produce many such examples?

(c) (*) If \(C\) and \(D\) are given, then under what circumstances is it possible to find non-0 integers \(m, n, p,\) and \(q\) such that

\[ \sqrt{m + n\sqrt{C} + \sqrt{p + q\sqrt{D}}} \]

is a rational number? (The most interesting part of the answer is to justify your answer for when it is not possible.)

(3) In class, we considered the numbers

\[ x_1 = \sqrt{2}, \]
\[ x_2 = \sqrt{2 + \sqrt{2}}, \]
\[ x_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \]

and so on.

(a) Prove that the sequence \(x_1, x_2, x_3, \ldots\) is increasing, and bounded by 2. (We outlined an inductive proof in class; you are welcome to formalise that proof.)
(b) (⋆) As Alissa pointed out in class, we know by part 3a that the sequence has a limit. What is it? What happens if all the 2’s in the definition of the sequence are replaced by 3’s?

(4) The question came up in class of whether 0.\( \bar{9} \) = 1. The following proofs were suggested to show that equality holds.

<table>
<thead>
<tr>
<th>It is known that 0.( \bar{3} ) = ( \frac{1}{3} ). Multiplying the left-hand side by 3 gives 0.( \bar{9} ), but multiplying the right-hand side by 3 gives 1. Therefore, 0.( \bar{9} ) = 1.</th>
</tr>
</thead>
</table>

| Starting with 0.\( \bar{9} \) = 1, multiply both sides by 10 to get 9.\( \bar{9} \) = 10, then subtract the original equality from both sides to get 9 = 9. Since this is true, the original equality was correct. |

(a) Critique these proofs. Here are some questions to consider:
- Do they convince you?
- Would they convince a high-school student who did not believe in the equality?
- Are they mathematically correct?

(b) Do you believe in this equality? Provide an argument (not necessarily mathematically rigorous) for the point of view opposite to yours, and point out its flaws. (You can consult the literature if you have trouble thinking up a flawed argument on your own.)

(5) How do you (or will you) judge your success as a teacher?