TECHNOLOGY Some graphing utilities, such as Derive, Maple, Mathcad, Mathematica, and the TI-89, perform symbolic differentiation. Others perform numerical differentiation by finding values of derivatives using the formula

\[ f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} \]

where \( \Delta x \) is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of \( f(x) = |x| \) when \( x = 0 \)?

If \( f \) is differentiable at \( x = c \), then \( f \) is continuous at \( x = c \).

Proof You can prove that \( f \) is continuous at \( x = c \) by showing that \( f(x) \) approaches \( f(c) \) as \( x \to c \). To do this, use the differentiability of \( f \) at \( x = c \) and consider the following limit.

\[
\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left[ (x - c) \frac{f(x) - f(c)}{x - c} \right] \\
= \left[ \lim_{x \to c} (x - c) \right] \left[ \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right] \\
= (0)[f'(c)] \\
= 0
\]

Because the difference \( f(x) - f(c) \) approaches zero as \( x \to c \), you can conclude that \( \lim_{x \to c} f(x) = f(c) \). So, \( f \) is continuous at \( x = c \).

The following statements summarize the relationship between continuity and differentiability.

1. If a function is differentiable at \( x = c \), then it is continuous at \( x = c \). So, differentiability implies continuity.
2. It is possible for a function to be continuous at \( x = c \) and not be differentiable at \( x = c \). So, continuity does not imply differentiability.

Exercises for Section 2.1

Exercises 1 and 2, estimate the slope of the graph at the points \((x_1, y_1)\) and \((x_2, y_2)\).

(a)  
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(b)  
\[ 
\begin{array}{c}
\text{(b)} \\
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\]

In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

3. Identify or sketch each of the quantities on the figure:
   (a) \( f(1) \) and \( f(4) \)
   (b) \( f(4) - f(1) \)
   (c) \( y = \frac{f(4) - f(1)}{4 - 1} + f(1) \)

4. Insert the proper inequality symbol (< or >) between the given quantities:
   (a) \( \frac{f(4) - f(1)}{4 - 1} \text{ and } \frac{f(4) - f(3)}{4 - 3} \)
   (b) \( \frac{f(4) - f(1)}{4 - 1} \text{ and } f'(1) \)
In Exercises 5–10, find the slope of the tangent line to the graph of the function at the given point.

5. \( f(x) = 3 - 2x \), \((-1, 5)\)
6. \( g(x) = \frac{3}{2}x + 1 \), \((-2, -2)\)
7. \( g(x) = x^2 - 4 \), \((1, -3)\)
8. \( g(x) = 5 - x^2 \), \((2, 1)\)
9. \( f(x) = 3x - 2 \), \((0, 0)\)
10. \( h(x) = x^2 + 3 \), \((-2, 7)\)

In Exercises 11–24, find the derivative by the limit process.

11. \( f(x) = 3 \)
12. \( g(x) = -5 \)
13. \( f(x) = -5x \)
14. \( f(x) = 3x + 2 \)
15. \( h(x) = 3 + \frac{3}{x} \)
16. \( f(x) = 9 - \frac{1}{x} \)
17. \( f(x) = 2x^2 + x - 1 \)
18. \( f(x) = 1 - x^2 \)
19. \( f(x) = x^3 - 12x \)
20. \( f(x) = x^3 + x^2 \)
21. \( f(x) = \frac{1}{x - 1} \)
22. \( f(x) = \frac{1}{x^2} \)
23. \( f(x) = \sqrt{x + 1} \)
24. \( f(x) = \frac{4}{\sqrt{x}} \)

In Exercises 25–32, (a) find an equation of the tangent line to the graph of \( f \) at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the derivative feature of a graphing utility to confirm your results.

25. \( f(x) = x^2 + 1 \), \((2, 5)\)
26. \( f(x) = x^2 + 2x + 1 \), \((-3, 4)\)
27. \( f(x) = x^3 \), \((2, 8)\)
28. \( f(x) = x^3 + 1 \), \((1, 2)\)
29. \( f(x) = \sqrt{x} \), \((1, 1)\)
30. \( f(x) = \sqrt{x - 1} \), \((5, 2)\)
31. \( f(x) = x + \frac{4}{x^2} \), \((4, 5)\)
32. \( f(x) = \frac{1}{x + 1} \), \((0, 1)\)

In Exercises 33–36, find an equation of the line that is tangent to the graph of \( f \) and parallel to the given line.

<table>
<thead>
<tr>
<th>Function</th>
<th>Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = x^3 )</td>
<td>( 3x - y + 1 = 0 )</td>
</tr>
<tr>
<td>( f(x) = x^3 + 2 )</td>
<td>( 3x - y - 4 = 0 )</td>
</tr>
<tr>
<td>( f(x) = \frac{1}{\sqrt{x}} )</td>
<td>( x + 2y - 6 = 0 )</td>
</tr>
<tr>
<td>( f(x) = \frac{1}{\sqrt{x - 1}} )</td>
<td>( x + 2y + 7 = 0 )</td>
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In Exercises 37–40, the graph of \( f \) is given. Select the graph of \( f' \).

41. The tangent line to the graph of \( y = g(x) \) at the point \( (1, 1) \) passes through the point \((9, 0)\). Find \( g(5) \) and \( g'(5) \).
42. The tangent line to the graph of \( y = h(x) \) at the point \((-1, 1) \) passes through the point \((3, 6)\). Find \( h(-1) \) and \( h'(-1) \).

Writing About Concepts

In Exercises 43–46, sketch the graph of \( f' \). Explain how you found your answer.

43.  

44.  

45.  

46.  

47. Sketch a graph of a function whose derivative is always negative.
In Exercises 71–80, use the alternative form of the derivative to find the derivative at \( x = c \) (if it exists).

71. \( f(x) = x^2 - 1, \quad c = 2 \)  
72. \( g(x) = x(x - 1), \quad c = 1 \)  
73. \( f(x) = x^3 + 2x^2 + 1, \quad c = -2 \)  
74. \( f(x) = x^3 + 2x, \quad c = 1 \)  
75. \( g(x) = \sqrt{|x|}, \quad c = 0 \)  
76. \( f(x) = 1/x, \quad c = 3 \)  
77. \( f(x) = (x - 6)^{2/3}, \quad c = 6 \)  
78. \( g(x) = (x + 3)^{1/3}, \quad c = -3 \)  
79. \( h(x) = |x + 5|, \quad c = -5 \)  
80. \( f(x) = |x - 4|, \quad c = 4 \)

In Exercises 81–86, describe the \( x \)-values at which \( f \) is differentiable.

81. \( f(x) = \frac{1}{x + 1} \)  
82. \( f(x) = |x^2 - 9| \)

83. \( f(x) = (x - 3)^{2/3} \)  
84. \( f(x) = \frac{x^2}{x^2 - 4} \)

85. \( f(x) = \sqrt{x - 1} \)  
86. \( f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 4 - x^2, & x > 0 \end{cases} \)

In Exercises 91–94, find the derivatives from the left and the right at \( x = 1 \) (if they exist). Is the function differentiable at \( x = 1 \)?

91. \( f(x) = |x - 1| \)  
92. \( f(x) = \sqrt{1 - x^2} \)  
93. \( f(x) = \begin{cases} (x - 1)^2, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases} \)  
94. \( f(x) = \begin{cases} x^2, & x \leq 1 \\ x, & x > 1 \end{cases} \)

In Exercises 95 and 96, determine whether the function is differentiable at \( x = 2 \).

95. \( f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases} \)  
96. \( f(x) = \begin{cases} x^2 + 1, & x \leq 4 \\ \sqrt{2x}, & x > 4 \end{cases} \)

97. **Graphical Reasoning** A line with slope \( m \) passes through the point \((0, 4)\) and has the equation \( y = mx + 4 \).

(a) Write the distance \( d \) between the line and the point as a function of \( m \).

(b) Use a graphing utility to graph the function \( d \) in part (a) for \( 0 < m < 4 \) and \( m > 4 \). Based on the graph, is the function differentiable at value of \( m \)? If not, where is it not differentiable?

98. **Conjecture** Consider the functions \( f(x) = x^2 \) and \( g(x) = x^3 \).

(a) Graph \( f \) and \( f' \) on the same set of axes.

(b) Graph \( g \) and \( g' \) on the same set of axes.

(c) Identify a pattern between \( f \) and \( g \) and their respective derivatives. Use the pattern to make a conjecture about \( h'(x) \) if \( h(x) = x^n \), where \( n \) is an integer and \( n \geq 2 \).

(d) Find \( f'(x) \) if \( f(x) = x^4 \). Compare the result with your conjecture in part (c). Is this a proof of your conjecture? Explain.

**True or False?** In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

99. The slope of the tangent line to the differentiable function at the point \((2, f(2))\) is \( \frac{f(2 + \Delta x) - f(2)}{\Delta x} \).

100. If a function is continuous at a point, then it is differentiable at that point.

101. If a function has derivatives at both the right and the left at the point, then it is differentiable at that point.

102. If a function is differentiable at a point, then it is continuous at that point.

103. Let \( f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \) and \( g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \).

Show that \( f \) is continuous, but not differentiable, at \( x = 0 \).

Show that \( g \) is differentiable at \( 0 \), and find \( g'(0) \).

**104. Writing** Use a graphing utility to graph the two functions \( f(x) = x^2 + 1 \) and \( g(x) = |x| + 1 \) in the same viewing window. Use the trace feature to analyze the graph near the point \((0, 1)\). What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.