Find the point on the surface

$$xy + 3x + z^2 = 9$$

closest to the origin.
Find the point on the surface

\[ xy + 3x + z^2 = 9 \]

closest to the origin.

Minimise distance.
Distance formula:

\[ d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}. \]

Optimisation tip: if everything is under a root, just get rid of it ⋆.

Minimise \( D = x^2 + y^2 + z^2. \)
Distance formula:

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Minimise

\[ D = x^2 + y^2 + z^2. \]
The problem

Formulae

Domain

Calculus

Boundary

Three variables are not independent:

\[ xy + 3x + z^2 = 9 \iff z^2 = 9 - xy - 3x, \]

so

\[ D = x^2 + y^2 + (9 - xy - 3x). \]
Three variables are not independent:

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\[ D = x^2 + y^2 + (9 - xy - 3x). \]
The problem

<table>
<thead>
<tr>
<th>Formulae</th>
<th>Domain</th>
<th>Calculus</th>
<th>Boundary</th>
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xy + 3x + z^2 = 9 \implies xy + 3x \leq 9.
The problem

**Formulae**

\[ \text{Boundary} \]

\[ xy + 3x + z^2 = 9 \implies xy + 3x \leq 9. \]

*Boundary* is where one or more inequalities become equalities:

\[ xy + 3x = 9. \]
\[ D = x^2 + y^2 + (9 - xy - 3x). \]
The problem

**Formulae**

\[ D = x^2 + y^2 + (9 - xy - 3x). \]

\[ \nabla D = \langle 2x - y - 3, 2y - x \rangle = \langle 0, 0 \rangle. \]
The problem

Formulae

Domain

Calculus

Boundary

2y − x = 0 \Rightarrow x = 2y.
The problem

Formulae

Domain

Calculus

Boundary

$$2y - x = 0 \Rightarrow x = 2y.$$  

*Simultaneous* solution:

$$2x - y - 3 = 0$$

$$2(2y) - y - 3 = 0$$

$$3y - 3 = 0$$

$$y = 1.$$
The problem

Formulæ

Domain

Calculus

Boundary

2y - x = 0 \Rightarrow x = 2y.

**Simultaneous solution:**

\[ 2x - y - 3 = 0 \]

\[ 2(2y) - y - 3 = 0 \]

\[ 3y - 3 = 0 \]

\[ y = 1. \]

Always look for \((x, y)\) pairs: \((x, y) = (2, 1)\).
\[ 2y - x = 0 \Rightarrow x = 2y. \]

*Simultaneous* solution:

\[ 2x - y - 3 = 0 \]

\[ 2(2y) - y - 3 = 0 \]

\[ 3y - 3 = 0 \]

\[ y = 1. \]

Always look for \((x, y)\) pairs: \((x, y) = (2, 1)\).

Must deal with the *boundary.*
When $xy + 3x = 9$
When \( xy + 3x = 9 \), then \( x = \frac{9}{y + 3} \):

\[
D = x^2 + y^2 + (9 - xy - 3x)
= \left( \frac{9}{y + 3} \right)^2 + y^2.
\]
When \( xy + 3x = 9 \), then \( x = \frac{9}{y + 3} \):

\[
D = x^2 + y^2 + (9 - xy - 3x) = \left( \frac{9}{y + 3} \right)^2 + y^2.
\]

Find critical points for this one-variable problem:

\[
\frac{dD}{dy} = -\frac{2 \cdot 9^2}{(y + 3)^3} + 2y = 0.
\]
\[- \frac{2 \cdot 9^2}{(y + 3)^3} + 2y = 0\]

\[2y = \frac{2 \cdot 9^2}{(y + 3)^3}\]

\[2y(y + 3)^3 = 2 \cdot 9^2\]

\[2y^4 + 18y^3 + 54y^2 + 54 - 162 = 0.\]
\[-\frac{2 \cdot 9^2}{(y + 3)^3} + 2y = 0\]

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Ugh!
\[ 2y^4 + 18y^3 + 54y^2 + 54 - 162 = 0. \]

By CAS: \( y \approx -5.458, 1.141. \)
\[2y^4 + 18y^3 + 54y^2 + 54 - 162 = 0.\]

By CAS: \(y \approx -5.458, 1.141.\)

Always looking for \((x, y)\) pairs: \((x, y) \approx (-3.662, -5.458), (x, y) = (1.141, 2.173).\)
Already had critical point \((x, y) = (2, 1)\)
Already had critical point \((x, y) = (2, 1)\); now have \((x, y) \approx (1.141, 2.173)\).

Now, take *all* critical points and plug in to

\[
D = x^2 + y^2 + (9 - xy - 3x)
\]

to find minimum distance ⋆⋆:

<p>| | | |</p>
<table>
<thead>
<tr>
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Minimum $\star\star$ when $(x, y) = (2, 1)$. 
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Minimum ★★ when \((x, y) = (2, 1)\).

Looking for a point \((x, y, z)\). Since \(xy + 3x + z^2 = 9\), get

\[
z = \pm \sqrt{9 - xy - 3x} = \pm \sqrt{9 - (2)(1) - 3(2)} = \pm 1.
\]
The problem

Formulae

Domain

Calculus

Boundary

\[
\begin{array}{ccc}
\hline
x & y & D \\
\hline
2 & 1 & 6 \\
1.141 & 2.173 & 6.026 \\
\hline
\end{array}
\]

Minimum \(\star\star\) when \((x, y) = (2, 1)\).

Looking for a point \((x, y, z)\). Since \(xy + 3x + z^2 = 9\), get

\[
z = \pm \sqrt{9 - xy - 3x} = \pm \sqrt{9 - (2)(1) - 3(2)} = \pm 1.
\]

Closest points are \((2, 1, \pm 1)\), and minimum distance is \(\sqrt{D} = \sqrt{6}\).