Quasi-representations of surface groups

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Abstract

By a quasi-representation of a group \( G \), we mean an approximately multiplicative map of \( G \) to the unitary group of a unital \( C^* \)-algebra. A quasi-representation induces a partially defined map at the level \( K \)-theory.

In the early 1990s, Exel and Loring associated two invariants with almost-commuting pairs of unitary matrices \( u \) and \( v \): one is a \( K \)-theoretic invariant, which may be regarded as the image of the Bott element in \( K_0(C(T^2)) \) under a map induced by quasi-representation of \( \mathbb{Z}^2 \) in \( U(n) \); the other is the winding number in \( \mathbb{C} \setminus \{0\} \) of the closed path \( t \mapsto \det(te^{i\theta} + (1-t)uv) \). The so-called Exel–Loring formula states that these two invariants coincide if \( \|uv - vu\| \) is sufficiently small.

A generalization of the Exel–Loring formula for quasi-representations of a surface group taking values in \( U(n) \) was given by the second-named author. Here, we further extend this formula for quasi-representations of a surface group taking values in the unitary group of a tracial unital \( C^* \)-algebra.

1. Introduction

Let \( G \) be a discrete countable group. In [3, 4], the second-named author studied the question of how deformations of the group \( G \) (or of the group \( C^* \)-algebra \( C^*(G) \)) into the unitary group of a (unital) \( C^* \)-algebra \( A \) act on the \( K \)-theory of the algebras \( \ell^1(G) \) and \( C^*(G) \). By a deformation, we mean an almost-multiplicative map, a quasi-representation, which we will define precisely in a moment. Often, matrix-valued multiplicative maps are inadequate for detecting the \( K \)-theory of the aforementioned group algebras. In fact, if a countable, discrete, torsion-free group \( G \) satisfies the Baum–Connes conjecture, then a unital finite-dimensional representation \( \pi : C^*(G) \to M_r(\mathbb{C}) \) induces the map \( r \cdot \iota_* \) on \( K_0(C^*(G)) \), where \( \iota \) is the trivial representation of \( G \) (see [4, Proposition 3.2]). It turns out that almost-multiplicative maps detect \( K \)-theory quite well for large classes of groups: one can implement any group homomorphism of \( K_0(C^*(G)) \) to \( \mathbb{Z} \) on large swaths of \( K_0(C^*(G)) \) using quasi-representations (see [4, Theorem 3.3]).

Knowing that quasi-representations may be used to detect \( K \)-theory, we turn to how it is that they act. An index theorem of Connes, Gromov and Moscovici [2] is very relevant to this topic, in the following context. Let \( M \) be a closed Riemannian manifold with fundamental group \( G \) and let \( D \) be an elliptic pseudo-differential operator on \( M \). The equivariant index of \( D \) is an element of \( K_0(\ell^1(G)) \). Connes, Gromov and Moscovici showed that the push-forward of the equivariant index of \( D \) under a quasi-representation of \( G \) coming from parallel transport in an almost-flat bundle \( E \) over \( M \) is equal to the index of \( D \) twisted by \( E \).

At around the same time, Exel and Loring studied two invariants associated to pairs of almost-commuting scalar unitary matrices \( u, v \in U(r) \). One is a \( K \)-theory invariant, which may be regarded as the push-forward of the Bott element \( \beta \) in the \( K_0 \)-group of \( C(T^2) \cong C^*(\mathbb{Z}^2) \) by a quasi-representation of \( \mathbb{Z}^2 \) into the unitary group \( U(r) \). The Exel–Loring formula...
proved in [6] states that this invariant equals the winding number in $\mathbb{C} \setminus \{0\}$ of the path $t \mapsto \det((1-t)uv + tvu)$. An extension of this formula for almost-commuting unitaries in a $C^*$-algebra of tracial rank 1 is due to Lin and plays an important role in the classification theory of amenable $C^*$-algebras. In a different direction, the Exel–Loring formula was generalized in [3] to finite-dimensional quasi-representations of a surface group using a variant of the index theorem of [2]. Remark 2.4 discusses the Exel–Loring formula in more detail.

In [3], the second-named author used the Mishchenko–Fomenko index theorem to give a new proof and a generalization of the index theorem of Connes, Gromov and Moscovici that allows $C^*$-algebra coefficients. In this paper, we use this generalization to address the question of how a quasi-representation $\pi$ of a surface group in the unitary group of a tracial $C^*$-algebra acts at the level of $K$-theory. We extend the Exel–Loring formula to a surface group $\Gamma_g$ (with canonical generators $\alpha_i, \beta_i$) and coefficients in a unital $C^*$-algebra $A$ with a trace $\tau$. Briefly, writing $K_0(\ell^1(\Gamma_g)) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[\mu]\Sigma_g$ we have

$$
\tau(\pi(\mu|\Sigma_g))) = \frac{1}{2\pi i} \tau \left( \log \left( \prod_{i=1}^{g} [\pi(\alpha_i), \pi(\beta_i)] \right) \right),
$$

where $[1]$ is the $K_0$ class of the unit 1 $\in \ell^1(\Gamma_g)$, $[\Sigma_g]$ is the fundamental class in $K$-homology of the genus $g$ surface $\Sigma_g$ and $\mu: K_0(\Sigma_g) \to K_0(\ell^1(G))$ is the $\ell^1$-version of the assembly map of Lafforgue. For a complete statement, see Theorem 2.3. In the proof, we make use of Chern–Weil theory for connections on Hilbert $A$-module bundles as developed by Schick [15] and the de la Harpe–Skandalis determinant [7] to calculate the first Chern class of an almost-flat Hilbert module $C^*$-bundle associated to a quasi-representation (Theorem 5.2).

The paper is organized as follows. In Section 2, we define quasi-representations and the invariants we are interested in, and state our main result, Theorem 2.3. The invariants make use of the Mishchenko line bundle, which we discuss in Section 3. The push-forward of this bundle by a quasi-representation is considered in Section 4. Section 5 contains our main technical result, Theorem 5.2, which computes one of our invariants in terms of the de la Harpe–Skandalis determinant [7]. To obtain the formula given in Theorem 2.3, we must work with concrete triangulations of oriented surfaces; this is contained in Section 6. Assembling these results in Section 7 yields a proof of Theorem 2.3.

2. The main result

In this section, we state our main result. It depends on a result in [3] that we revisit. Let us provide some notation and definitions first.

Let $G$ be a discrete countable group and $A$ be a unital $C^*$-algebra.

**Definition 2.1.** Let $\epsilon > 0$ and let $F$ be a finite subset of $G$. An $(F, \epsilon)$-representation of $G$ in $U(A)$ is a function $\pi: G \to U(A)$ such that for all $s, t \in F$ we have

$$
\begin{align*}
\pi(1) &= 1, \\
\|\pi(s^{-1}) - \pi(s)^*\| &< \epsilon \text{ and} \\
\|\pi(st) - \pi(s)\pi(t)\| &< \epsilon.
\end{align*}
$$

We refer to the third condition by saying that $\pi$ is $(F, \epsilon)$-multiplicative. Let us note that the second condition follows from the other two if we assume that $F$ is symmetric, that is, $F = F^{-1}$. A quasi-representation is an $(F, \epsilon)$-representation, where $F$ and $\epsilon$ are not necessarily specified.
A quasi-representation \( \pi: G \to U(A) \) induces a map (also denoted by \( \pi \)) of the Banach algebra \( \ell^1(G) \) to \( A \) by \( \sum \lambda_s s \mapsto \sum \lambda_s \pi(s) \). This map is a unital linear contraction. We also write \( \pi \) for the extension of \( \pi \) to matrix algebras over \( \ell^1(G) \).

### 2.1. Pushing-forward via quasi-representations

A group homomorphism \( \pi: G \to U(A) \) induces a map \( \pi_*: K_0(\ell^1(G)) \to K_0(A) \) (via its Banach algebra extension). We think of a quasi-representation \( \pi \) as inducing a partially defined map \( \pi_* \) at the level of \( K \)-theory, in the following sense. If \( e \) is an idempotent in some matrix algebra over \( \ell^1(G) \) such that \( \| \pi(e) - \pi(e)^2 \| < \frac{1}{4} \), then the spectrum of \( \pi(e) \) is disjoint from the line \( \{ \text{Re} z = \frac{1}{2} \} \). Writing \( \chi \) for the characteristic function of \( \{ \text{Re} z > \frac{1}{2} \} \), it follows that \( \chi(\pi(e)) \) is an idempotent and we set

\[
\pi_*(e) = [\chi(\pi(e))] \in K_0(A).
\]

For an element \( x \) in \( K_0(\ell^1(G)) \), we make a choice of idempotents \( e_0 \) and \( e_1 \) in some matrix algebra over \( \ell^1(G) \) such that \( x = [e_0] - [e_1] \). If \( \| \pi(e_i) - \pi(e_i)^2 \| < \frac{1}{4} \) for \( i \in \{0, 1\} \), then write \( \pi_*(x) = \pi_*(e_0) - \pi_*(e_1) \). The choice of idempotents is largely inconsequential: given two choices of representatives one finds that if \( \pi \) is multiplicative enough, then both choices yield the same element of \( K_0(A) \).

Of course, the more multiplicative \( \pi \) is, the more elements of \( K_0(\ell^1(G)) \) we can push-forward into \( K_0(A) \).

### 2.2. An index theorem

Fix a closed oriented Riemannian surface \( M \) and let \( G \) be its fundamental group. Fix also a unital \( C^* \)-algebra \( A \) with a tracial state \( \tau \). Write \( K_0(M) \) for \( KK(C(M), \mathbb{C}) \). Because the assembly map \( \mu: K_0(M) \to K_0(\ell^1(G)) \) is known to be an isomorphism in this case (see [10]), we have

\[
K_0(\ell^1(G)) \cong \mathbb{Z}[1] \oplus \mathbb{Z}\mu[M],
\]

where \([M]\) is the fundamental class of \( M \) in \( K_0(M) \) (see [1, Lemma 7.9]) and \([1]\) is the class of the unit of \( \ell^1(G) \). Since we are interested in how a quasi-representation of \( G \) acts on \( K_0(\ell^1(G)) \), we would like to study the push-forward of the generator \( \mu[M] \) by a quasi-representation.

2.2.1. Consider the universal cover \( \tilde{M} \to M \) and the diagonal action of \( G \) on \( \tilde{M} \times \ell^1(G) \) giving rise to the so-called Mishchenko line bundle \( \ell, \tilde{M} \times_G \ell^1(G) \to M \). We will discuss it in more detail in Section 3, where we will give a description of it as the class of a specific idempotent \( e \) in some matrix algebra over \( C(M) \otimes \ell^1(G) \).

If \( \pi \) is a quasi-representation of \( G \) in \( U(A) \), then \( \text{id}_{C(M)} \otimes \pi \) is an almost-multiplicative unital linear contraction on \( C(M) \otimes \ell^1(G) \) with values in \( C(M) \otimes A \). Assuming that \( \pi \) is sufficiently multiplicative, we may define the push-forward of the idempotent \( e \) by \( \text{id}_{C(M)} \otimes \pi \), just as in Subsection 2.1. We set

\[
\ell_\pi := (\text{id}_{C(M)} \otimes \pi)_2(\ell) := (\text{id}_{C(M)} \otimes \pi)_2(e) \in K_0(C(M) \otimes A).
\]

Let \( D \) be an elliptic operator on \( M^n \) and let \( \mu[D] \in K_0(\ell^1(G)) \) be its image under the assembly map. Let \( q_0 \) and \( q_1 \) be idempotents in some matrix algebra over \( \ell^1(G) \) such that \( \mu[D] = [q_0] - [q_1] \) and write \( \pi_*(\mu[D]) := \pi_*(q_0) - \pi_*(q_1) \). By [3, Corollary 3.8], if \( \pi: G \to A \) is sufficiently multiplicative, then

\[
\tau(\pi_*(\mu[D])) = (-1)^{n(n+1)/2} \langle p_\tau \text{ch}(\sigma(D)) \cup \text{Td}(TM \otimes \mathbb{C}) \cup \text{ch}_\tau(\ell_\pi), [M] \rangle,
\]

(2.1)
It follows that $\nabla M$ implies $p$ where $504$

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the following.

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On the other hand, it follows from the Atiyah–Singer index theorem that the Chern character in homology $\text{ch}$:

This gives an obvious choice of idempotents $g$ genus $G$ that

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Theorem 2.2 (cf. [3, Corollary 3.8]). Let $M$ be a closed oriented Riemannian surface of genus $g$ with fundamental group $G$. Let $q_0$ and $q_1$ be idempotents in some matrix algebra over $\ell^1(G)$ such that $\mu[M] = [q_0] - [q_1]$. Then there exist a finite subset $G$ of $G$ and $\omega > 0$ satisfying the following.

Let $A$ be a unital $C^*$-algebra with a tracial state $\tau$ and let $\pi: G \to U(A)$ be a $(G, \omega)$-representation. Write $\pi_\tau(\mu[M]) := \pi_\tau(q_0) - \pi_\tau(q_1)$. Then

In the case of surfaces, this formula specializes to the following statement.

Here $\text{ch}_\tau: K_0(C(M) \otimes A) \to H^2(M, \mathbb{R})$ is a Chern character associated to $\tau$ (see Section 5), and $[M] \in H_2(M, \mathbb{R})$ is the fundamental class of $M$.

Proof. Given another pair of idempotents $q_0', q_1'$ in some matrix algebra over $\ell^1(G)$ such that $\mu[M] = [q_0'] - [q_1']$, there is an $\omega_0 > 0$ such that if $0 < \omega < \omega_0$, then for any $(G, \omega)$-representation $\pi$ we have $\pi_\tau(q_0) - \pi_\tau(q_1) = \pi_\tau(q_0') - \pi_\tau(q_1')$. We are therefore free to prove the theorem for a convenient choice of idempotents.

It is known that the fundamental class of $M$ in $K_0(M)$ coincides with $[\bar{\partial}_g] + (g - 1)[i]$, where $\bar{\partial}_g$ is the Dolbeault operator on $M$ and $\iota: C(M) \to \mathbb{C}$ is a character (see [1, Lemma 7.9]). Let $e_0, e_1, f_0, f_1$ be idempotents in some matrix algebra over $\ell^1(G)$ such that

This gives an obvious choice of idempotents $q_0$ and $q_1'$ in some matrix algebra over $\ell^1(G)$ so that $\mu[M] = [q_0'] - [q_1']$. We want to prove

for $z = [M] \in K_0(M)$. Because of the additivity of this last equation, the fact that $[M] = [\bar{\partial}_g] + (g - 1)[i]$, and equation (2.2), it is enough to prove

By [3, Corollary 3.5],

We can represent $\epsilon_\pi$ by a projection $f$ in matrices over $C(M, A)$. The definition of the Kasparov product implies

$$
\langle \langle \epsilon_\pi, [i] \otimes 1_A \rangle \rangle = \iota_\epsilon[f] = [f(x_0)] \in K_0(A).
$$
On the other hand, the definition of $\text{ch}_r$ (see [15, Definition 4.1]) implies $\text{ch}_r(f) = \tau(f(x_0)) + \text{a term in } H^2(M, \mathbb{R})$. Since $\text{ch}[i] = 1 \in H_0(M, \mathbb{R})$, we get

$$\langle \text{ch}_r(f), \text{ch}[i] \rangle = \tau(f(x_0)).$$  \hspace{1cm} (2.4)

\hfill \Box

2.3. Statement of the main result

We will often write $\Sigma_g$ for the closed oriented surface of genus $g$ and $\Gamma_g$ for its fundamental group. It is well known that $\Gamma_g$ has a standard presentation

$$\Gamma_g = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] \rangle,$$

where we write $[\alpha, \beta]$ for the multiplicative commutator $\alpha \beta \alpha^{-1} \beta^{-1}$.

Our main result is the following.

**Theorem 2.3.** Let $g \geq 1$ be an integer and let $q_0$ and $q_1$ be idempotents in some matrix algebra over $\ell^1(\Gamma_g)$ such that $\mu[S_g] = [q_0] - [q_1] \in K_0(\ell^1(\Gamma_g))$. There exists $\epsilon_0 > 0$ and a finite subset $F_0$ of $\Gamma_g$ such that for every $0 < \epsilon < \epsilon_0$ and every finite subset $F \supseteq F_0$ of $\Gamma_g$ the following holds.

If $A$ is a unital C$^*$-algebra with a trace $\tau$ and $\pi: \Gamma_g \to U(A)$ is an $(\mathcal{F}, \epsilon)$-representation, then

$$\tau(\pi_* \mu[S_g]) = \frac{1}{2\pi i} \tau \left( \log \left( \prod_{i=1}^{g} [\pi(\alpha_i), \pi(\beta_i)] \right) \right),$$  \hspace{1cm} (2.5)

where $\pi_* \mu[S_g] := \pi_\sharp(q_0) - \pi_\sharp(q_1)$.

The rest of the paper is devoted to the proof.

**Remark 2.4.** The case when $g = 1$ and $A = M_n(\mathbb{C})$ recovers the Exel–Loring formula of [6]. As mentioned in Section 1, this formula states that two integer-valued invariants $\kappa(u, v)$ and $\omega(u, v)$ associated to a pair of unitary matrices $u, v \in U(n)$ coincide as long as $\|\{u, v\} - 1\| < c$, where $c$ is a small constant that is independent of $n$. Exel observed (in [5, Lemma 3.1]) that the ‘winding-number’ invariant $\omega(u, v)$ equals

$$\frac{1}{2\pi i} \text{tr}((u, v))).$$

This corresponds to the right-hand side of (2.5).

The ‘$K$-theory invariant’ $\kappa(u, v)$ is an element in the $K_0$-group of $M_n(\mathbb{C})$ and is regarded as an integer after identifying this group with $\mathbb{Z}$ (the isomorphism given by the usual trace on $M_n(\mathbb{C})$). This invariant was first introduced by Loring [12]. We briefly recall its definition.

Recall that $K_0(C(T^2)) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[\beta]$, where $[1]$ is the class of the unit of $C(T^2)$ and $\beta$ is the Bott element. Given unitaries $U$ and $V$ in a unital C$^*$-algebra $B$, one defines a self-adjoint matrix

$$e(U, V) = \begin{pmatrix} f(V) & g(V) + h(V)U^* \\ g(V) + Uh(V) & 1 - f(V) \end{pmatrix},$$

where $f$, $g$ and $h$ are certain continuous functions on the circle. These are chosen in such a way that when $U = e^{2\pi i x}$, $V = e^{2\pi i y} \in C(T^2)$ we have that $e(U, V)$ is idempotent and has $K_0$-class $[1] + \beta$ (cf. [12, 14]).

We assume that $\|\{u, v\} - 1\|$ is small enough so that the corresponding matrix $e(u, v)$ is nearly idempotent; in particular, its spectrum does not contain $\frac{1}{2}$. Writing $\chi$ for the
characteristic function of \( \{ \text{Re} \, z > \frac{1}{2} \} \), we have that \( \chi(e(u,v)) \) is a projection in \( M_n(C) \). Define
\[
\kappa(u,v) = \text{tr}(\chi(e(u,v))) - n.
\]
Subtracting \( n \) means cancelling out the \( K \)-theoretic contribution of \([1]\), leaving only the contribution of the push-forward of \( \beta \) under a quasi-representation determined by \( u \) and \( v \). (Proposition 2.5 makes this statement precise.) This corresponds to the left-hand side of (2.5).

Formula (2.5) also recovers its extension by Lin \([11]\) for \( C^* \)-algebras of tracial rank 1. Lin’s strategy was a reduction to the finite-dimensional case of \([6]\) using approximation techniques.

The following proposition says that we may associate quasi-representations with unitaries that nearly satisfy the group relation in the standard presentation of \( \Gamma_g \) mentioned above. The proof is in Section 7.

**Proposition 2.5.** For every \( \epsilon > 0 \) and every finite subset \( F \) of \( \Gamma_g \), there is a \( \delta > 0 \) such that if \( A \) is a unital \( C^* \)-algebra with a trace \( \tau \) and \( u_1, v_1, \ldots, u_g, v_g \) are unitaries in \( A \) satisfying
\[
\left\| \prod_{i=1}^g [u_i, v_i] - 1 \right\| < \delta,
\]
then there exists an \( (F, \epsilon) \)-representation \( \pi: \Gamma_g \to U(A) \) with \( \pi(\alpha_i) = u_i \) and \( \pi(\beta_i) = v_i \), for all \( i \in \{1, \ldots, g\} \).

**Example 2.6.** To revisit a classic example, consider the noncommutative 2-torus \( A_\theta \), regarded as the universal \( C^* \)-algebra generated by unitaries \( u \) and \( v \) with \([u,v] = e^{2\pi i \theta} \cdot 1\). This is a tracial unital \( C^* \)-algebra. If \( \theta \) is small enough, then we may apply Proposition 2.5 and Theorem 2.3 to obtain
\[
\tau(\pi_\sharp(\beta)) = \frac{1}{2\pi i} \tau(\log e^{-2\pi i \theta}) = -\theta,
\]
where \( \beta \in K_0(C(T^2)) \) is the Bott element, \( \tau \) is a unital trace of \( A_\theta \) and \( \pi: \mathbb{Z}^2 \to U(A_\theta) \) is a quasi-representation obtained from Proposition 2.5.

3. The Mishchenko line bundle

Recall our setup: \( M \) is a closed oriented surface with fundamental group \( G \) and universal cover \( \tilde{M} \to M \). In this section, we give a picture of the Mishchenko line bundle that will enable us to explicitly describe its push-forward by a quasi-representation.

The Mishchenko line bundle is the bundle \( \tilde{M} \times_G \ell^1(G) \to M \), obtained from \( \tilde{M} \times \ell^1(G) \) by passing to the quotient with respect to the diagonal action of \( G \). We write \( \ell \) for its class in \( K_0(C(M) \otimes \ell^1(G)) \).

3.1. Triangulations and the edge-path group

We adapt a construction found in the appendix of \([13]\). It is convenient to work with a triangulation \( \Lambda \) of \( M \). Let \( \Lambda^{(0)} = \{x_0, \ldots, x_{N-1}\} \) be the 0-skeleton of \( \Lambda \) and let \( \Lambda^{(1)} \) be the 1-skeleton. To each edge, we assign an element of \( G \) as follows. Fix a root vertex \( x_0 \) and a maximal (spanning) tree \( T \) in \( \Lambda \). Let \( \gamma_i \) be the unique path along \( T \) from \( x_0 \) to \( x_i \), and for two adjacent vertices \( x_i \) and \( x_j \) let \( x_ix_j \) be the (directed) edge from \( x_i \) to \( x_j \). For two such adjacent vertices, write \( s_{ij} \in G \) for the class of the loop \( \gamma_i \ast x_i x_j \ast \gamma_j^{-1} \).
Let $F$ be the (finite) set $\{s_{ij}\}$. For example, if $M = T^2$ so that $G = \Z^2 = \langle \alpha, \beta : [\alpha, \beta] = 1 \rangle$, then we have $F = \{1, \alpha \pm 1, \beta \pm 1, (\alpha \beta) \pm 1\}$ for the triangulation and tree pictured in Figure 3.

**Definition 3.1.** For a vertex $x_{i_k}$ in a 2-simplex $\sigma = \langle x_{i_0}, x_{i_1}, x_{i_2} \rangle$ of $\Lambda$, define the dual cell block to $x_{i_k}$, $U_{i_k}^\sigma := \left\{ \sum_{l=0}^2 t_l x_{i_l} : t_l \geq 0, \sum_{l=0}^2 t_l = 1 \right\}$.

Define the dual cell to the vertex $x_i \in \Lambda^{(0)}$ by $U_i = \bigcup\{U_{i_k}^\sigma : x_i \in \sigma\}$.

Let $U_{ij}^\sigma = U_{i}^\sigma \cap U_{j}^\sigma$ etc. (see Figure 1).

Since $p: \tilde{M} \to M$ is a covering space of $M$, we may fix an open cover of $M$ such that for every element $V$ of this cover, $p^{-1}(V)$ is a disjoint union of open subsets of $\tilde{M}$, each of which is mapped homeomorphically onto $V$ by $p$. We require that $\Lambda$ be fine enough so that every dual cell $U_i$ is contained in some element of this cover.

**Lemma 3.2.** The Mishchenko line bundle $\tilde{M} \times_G \ell^1(G) \to M$ is isomorphic to the bundle $E$ obtained from the disjoint union $\bigsqcup U_i \times \ell^1(G)$ by identifying $(x, a)$ with $(x, s_{ij} a)$ whenever $x \in U_i \cap U_j$.

**Proof.** Lift $x_0$ to a vertex $\tilde{x}_0$ in $\tilde{M}$. By the unique-path-lifting property, every path $\gamma_i$ lifts (uniquely) to a path $\tilde{\gamma}_i$ from $\tilde{x}_0$ to a lift $\tilde{x}_i$ of $x_i$. In this way, lift $T$ to a tree $\tilde{T}$ in $\tilde{M}$. Each $U_i$ also lifts to a dual cell to $\tilde{x}_i$, denoted by $\tilde{U}_i$, which $p$ maps homeomorphically onto $U_i$.

We first describe the cocycle (transition functions) for the Mishchenko line bundle. Identify the fundamental group $G$ of $M$ with the group of deck transformations of $\tilde{M}$; see, for example, [8, Proposition 1.39]. Use this to write $p^{-1}(U_i)$ as the disjoint union $\bigsqcup \{s\tilde{U}_i : s \in G\}$.
Consider the isomorphism \( \Phi_i: p^{-1}(U_i) \times_G \ell^1(G) \to U_i \times \ell^1(G) \) described by the following diagram:

\[
\begin{array}{ccc}
(s\tilde{x}, a) & \mapsto & (p(\tilde{x}), s^{-1}a) \\
\| & & \downarrow \\
p^{-1}(U_i) \times G \ell^1(G) & \dashrightarrow & U_i \times \ell^1(G)
\end{array}
\]

If \( U_{ij} := U_i \cap U_j \neq \emptyset \), then we obtain the cocycle \( \phi_{ij}: U_{ij} \to \text{Aut}(\ell^1(G)) \):

\[
U_{ij} \times \ell^1(G) \xrightarrow{\Phi_i^{-1}} p^{-1}(U_{ij}) \times_G \ell^1(G) \xrightarrow{\Phi_i} U_i \times \ell^1(G),
\]

\[
(x, a) \mapsto (x, \phi_{ij}(x)a).
\]

Observe that \( \widetilde{M} \times_G \ell^1(G) \) is isomorphic to the bundle obtained from the disjoint union \( \bigsqcup U_i \times \ell^1(G) \) by identifying \( (x, a) \) with \( (x, \phi_{ij}(x)a) \) whenever \( x \in U_{ij} \). We only need to prove that \( \phi_{ij} \) is constantly equal to \( s_{ij} \).

Let \( x \in U_{ij} \) and let \( \tilde{x} \in \hat{U}_j \) be a lift of \( x \). Then \( \Phi_j([\tilde{x}, a]) = (x, a) \). Because \( p(\tilde{x}) \in U_{ij} \) there is a (unique) \( s \in G \) such that \( \tilde{x} \in s\hat{U}_i \cap \hat{U}_j \neq \emptyset \). Thus, \( \Phi_i([\tilde{x}, a]) = (x, s^{-1}a) \). Now, the path \( s_{ij} * s\tilde{x}_i \tilde{x}_j * s_{ij}^{-1} \) starts at \( s\tilde{x}_0 \) and ends at \( \tilde{x}_0 \). Its projection in \( M \) is the loop defining \( s_{ij} \), so \( s^{-1} = s_{ij} \). Thus, \( \phi_{ij}(x) = s_{ij} \).

\[
\square
\]

3.2. The push-forward of the line bundle

We will need an open cover of \( M \), so we dilate the dual cells \( U_i \) to obtain one. Let \( 0 < \delta < \frac{1}{2} \) and define \( V_i^\sigma \) to be the \( \delta \)-neighborhood of \( U_i^\sigma \) intersected with \( \sigma \). As before, set \( V_i = \bigcup_\sigma V_i^\sigma \).

Let \( \{\chi_i\} \) be a partition of unity subordinate to \( \{V_i\} \).

By Lemma 3.2, the class of the Mishchenko line bundle in \( K_0(C(M) \otimes \ell^1(G)) \), denoted earlier by \( \ell \), corresponds to the class of the projection

\[
e := \sum_{i,j} e_{ij} \otimes \chi_i^{1/2} \chi_j^{1/2} \otimes s_{ij} \in M_N(\mathbb{C}) \otimes C(M) \otimes \ell^1(G),
\]

where \( \{e_{ij}\} \) are the canonical matrix units of \( M_N(\mathbb{C}) \) and \( N \) is the number of vertices in \( \Lambda \).

We may fix a pair of idempotents \( q_0 \) and \( q_1 \) in some matrix algebra over \( \ell^1(G) \) satisfying \( [q_0] = [q_1] = [\mu[M]] \in K_0(\ell^1(G)) \). Let \( \omega > 0 \) be given by Theorem 2.2. (We may assume \( \omega < \frac{1}{4} \).)

Fix \( 0 < \epsilon < \omega \) and an \( (\mathcal{F}, \epsilon) \)-representation \( \pi: G \to U(A) \). We recall the following notation from Section 1.

**Notation 3.3.** For an \( (\mathcal{F}, \epsilon) \)-representation \( \pi: G \to U(A) \) as above, let

\[
\ell_\pi := (\text{id}_{C(M)} \otimes \pi)_{\mathcal{F}}(\epsilon).
\]

4. Hilbert-module bundles and quasi-representations

As mentioned in Section 1, in [2] a quasi-representation (with scalar values) of the fundamental group of a manifold is associated to an ‘almost-flat’ bundle over the manifold. In this section, we instead define a canonical bundle \( E_\pi \) over \( M \) associated with quasi-representation \( \pi \). Its class in \( K_0(C(M) \otimes A) \) will be the class \( \ell_\pi \) of the push-forward of the Mishchenko line bundle by \( \pi \). Our construction will be explicit enough so that we can use Chern–Weil theory for such bundles to analyze \( \text{ch}_\tau(\ell_\pi) \); see [15].

Recall that \( A \) is a \( C^* \)-algebra with trace \( \tau \).
**Definition 4.1.** Let $X$ be a locally compact Hausdorff space. A *Hilbert $A$-module bundle* $W$ over $X$ is a topological space $W$ with a projection $W \to X$ such that the fiber over each point has the structure of a Hilbert $A$-module $V$, and with local trivializations $W|_U \sim U \times V$ which are fiberwise Hilbert $A$-module isomorphisms.

We should point out that the $K_0$-group of the $C^*$-algebra $C(M) \otimes A$ is isomorphic to the Grothendieck group of isomorphism classes of finitely generated projective Hilbert $A$-module bundles over $M$. We identify the two groups.

### 4.1. Constructing bundles

We adapt a construction found in [13].

First we define a family of maps \{\(u_{ij}: U_{ij} \to \text{GL}(A)\}\) satisfying

\[
\begin{align*}
u_{ij}(x) &= u_{ij}^{-1}(x), & x & \in U_{ij}, \\
u_{ik}(x) &= u_{ij}(x)u_{jk}(x), & x & \in U_{ij}.
\end{align*}
\]

These maps will be then extended to a cocycle defined on the collection \(\{V_{ij}\}\).

Following [13], we will find it convenient to fix a partial order $\prec$ on the vertices of $\Lambda$ such that the vertices of each simplex form a totally ordered subset. We then call $\Lambda$ a *locally ordered simplicial complex*. One may always assume that such an order exists by passing to the first barycentric subdivision of $\Lambda$: if $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are the barycenters of simplices $\sigma_1$ and $\sigma_2$ of $\Lambda$, then define $\bar{\sigma}_1 < \bar{\sigma}_2$ if $\sigma_1$ is a face of $\sigma_2$.

Consider a simplex $\sigma = (x_{i_0}, x_{i_1}, x_{i_2})$ (with vertices written in increasing $\prec$-order). Observe that in this case $U_{i_0}^\sigma \cap U_{i_2}^\sigma = U_{i_0 i_2}^\sigma$, may be described using a single parameter $t_1$:

\[
U_{i_0 i_2}^\sigma = \left\{ \sum_{t=1}^{2} t_i x_{i_1} : t_0 = t_2 = \frac{1 - t_1}{2}, 0 \leq t_1 \leq \frac{1}{3} \right\}.
\]

Define

\[
u_{i_0 i_1}^\sigma = \text{the constant function on } U_{i_0 i_1}^\sigma \text{ equal to } \pi(s_{i_0 i_1}),
\]

\[
u_{i_1 i_2}^\sigma = \text{the constant function on } U_{i_1 i_2}^\sigma \text{ equal to } \pi(s_{i_1 i_2}),
\]

\[
u_{i_0 i_2}^\sigma(t_1) = (1 - 3t_1)\pi(s_{i_0 i_2}) + 3t_1\pi(s_{i_0 i_1})\pi(s_{i_1 i_2}), & 0 \leq t_1 \leq \frac{1}{3}.
\]

Define $\nu_{i_0 i_2}^\sigma$ etc. to be the pointwise inverse of $\nu_{i_0 i_2}^\sigma$. For fixed $i$ and $j$, the maps $\nu_{ij}^\sigma: U_{ij}^\sigma \to \text{GL}(A)$ define a map $u_{ij}: U_{ij} \to \text{GL}(A)$. Indeed, if $x_i x_j$ is a common edge of two simplices $\sigma$ and $\sigma'$, then $U_{ij}^\sigma \cap U_{ij}^{\sigma'}$ is the barycenter of $(x_i, x_j)$, where by definition both $u_{ij}^\sigma$ and $u_{ij}^{\sigma'}$ take the value $\pi(s_{ij})$. By construction, the family $\{u_{ij}\}$ has the desired properties.

#### 4.1.1. Recall the sets $V_i$ etc. from Subsection 3.2. To define the smooth transition function $\nu_{i_0 i_2}^\sigma: V_{i_0 i_2}^\sigma \to \text{GL}(A)$ that will replace $u_{i_0 i_2}^\sigma$, let us assume for simplicity that the simplex $\sigma$ is the triangle with vertices $v_{i_0} = (-\frac{1}{2}, 0), v_{i_1} = (0, 1)$ and $v_{i_2} = (\frac{1}{2}, 0)$. (It may be helpful to consider Figure 1(a).)

Define $v_{i_0 i_2}^\sigma$ as follows:

\[
v_{i_0 i_2}^\sigma(x, y) = \begin{cases} \pi(s_{i_0 i_1})\pi(s_{i_1 i_2}), & 1/3 - \delta \leq y \leq 1/3 + \delta, \\
1/3 - \delta \leq y \leq 1/3 + \delta, & \frac{y}{1/3 - \delta}\pi(s_{i_0 i_1})\pi(s_{i_1 i_2}), \quad 0 \leq y \leq \frac{1}{3} - \delta.
\end{cases}
\]
(so $v_{i_0 i_2}^\sigma$ is constant along the horizontal segments in $V_{i_0 i_2}$). The remaining two transition functions remain constant:

$$v_{i_0 i_1}^\sigma = \pi(s_{i_0 i_1}),$$
$$v_{i_1 i_2}^\sigma = \pi(s_{i_1 i_2}).$$

Again, for fixed $i$ and $j$ the maps $v_{ij}^\sigma : V_{ij}^\sigma \to \text{GL}(A)$ define a map $v_{ij} : V_{ij} \to \text{GL}(A)$. Since $v_{i_0 i_2}^\sigma$ is constant and equal to $\pi(s_{i_0 i_1})\pi(s_{i_1 i_2})$ in $V_{i_0} \cap V_{i_2} \cap V_{i_2}$, we indeed obtain a family $\{v_{ij}\}$ of transition functions.

**Definition 4.2.** The Hilbert $A$-module bundle $E_\pi$ is constructed from the disjoint union $\bigsqcup V_i \times A$ by identifying $(x, a)$ with $(x, v_{ij}(x) a)$ for $x$ in $V_{ij}$.

**Proposition 4.3.** The class of $E_\pi$ in $K_0(C(M) \otimes A)$ coincides with $\ell_\pi$, the class of the push-forward of $e$ by $\text{id}_{C(M) \otimes \pi}$ (see Subsection 3.2).

**Proof.** The bundle $E_\pi$ is a quotient of $\bigsqcup V_i \times A$ and from its definition it is clear that for each $i$ the quotient map is injective on $V_i \times A$. The restriction of the quotient map to $V_i \times A$ has an inverse, call it $\psi_i$, and $\psi_i$ is a trivialization of $E_\pi|_{V_i}$. Recalling that $N$ is the number of vertices in $\Lambda$ (which is the same as the number of sets $V_i$ in the cover), we define an isometric embedding

$$\theta : E_\pi \to M \times A^N,$$
$$[x, a] \mapsto (\chi_i^{1/2}(x)\psi_i([x, a]))_{i=0}^{N-1}.$$

Let $e_\pi : M \to M_N(A)$ be the function

$$x \mapsto \sum_{i,j} e_{ij} \otimes \chi_i^{1/2}(x)\chi_j^{1/2}(x)v_{ij}(x).$$

Because $\psi_i\psi_j^{-1}(x, a) = (x, v_{ij}(x) a)$ for $x \in V_{ij}$, it is easy to check that $e_\pi(x)$ is the matrix representing the orthogonal projection of $A^N$ onto $\theta(E_\pi|_x)$. In this way, we see that $[E_\pi] = [e_\pi] \in K_0(C(M) \otimes A)$.

Since $\mathcal{F} = \{s_{ij}\}$ and $\pi$ is an $(\mathcal{F}, \epsilon)$-representation, it follows immediately that the transition functions $v_{ij}$ satisfy $\|v_{ij}(x) - \pi(s_{ij})\| < \epsilon$ for all $x \in V_{ij}$. Thus,

$$\|e_\pi - (1 \otimes \pi)(e)\| = \left\|e_\pi - \sum_{i,j} e_{ij} \otimes \chi_i^{1/2}\chi_j^{1/2}\pi(s_{ij})\right\| < \epsilon,$$

as well. Recall that $\ell_\pi$ is obtained by perturbing $(1 \otimes \pi)(e)$ to a projection using functional calculus and then taking its $K_0$-class (see Subsection 2.1). The previous estimate shows that this class must be $[e_\pi]$.

**Remark 4.4.** The previous proposition shows that the class $[E_\pi]$ is independent of the order $\mathbf{o}$ on the vertices of $\Lambda_0$.

**4.2. Connections arising from transition functions**

We now define a canonical connection on $E_\pi$ associated with the family $\{v_{ij}\}$ of transition functions. This connection will be used in the proof of Theorem 5.2.
4.2.1. The smooth sections $\Gamma(E_\pi)$ of $E_\pi$ may be identified with
$$\left\{ (s_i) \in \bigoplus_i \Omega^0(V_i, E_\pi) : s_j = v_{ji}s_i \text{ on } V_{ij} \right\}.$$ Let $\nabla_i : \Omega^0(V_i, A) \to \Omega^1(V_i, A)$ be given by
$$\nabla_i(s) = ds + \omega_is \quad \forall s \in \Omega^0(V_i, A),$$
where
$$\omega_i = \sum_k \chi_kv_{ki}^{-1}dv_{ki}.$$ Note that $v_{ki} \in \Omega^0(V_{ik}, \text{GL}(A))$ and so $\omega_i$ may be regarded as an $A$-valued 1-form on $V_i$, which can be multiplied fiberwise by the values of the section $s$.

We define a connection $\nabla$ on $E_\pi$ by
$$\nabla(s_i) = (\nabla_is_i).$$ That $\nabla$ takes values in $\Omega^1(M, E_\pi)$ follows from a straightforward computation verifying
$$\nabla_js_j = v_{ji}\nabla_is_i.$$ It is just as straightforward to verify that $\nabla$ is $A$-linear and satisfies the Leibniz rule.

4.2.2. Define $\Omega_i = d\omega_i + \omega_i \wedge \omega_i \in \Omega^2(V_i, A)$. One checks that $\Omega_i = v_{ji}^{-1}\Omega_jv_{ji}$ and so $(\Omega_i)$ defines an element $\Omega$ of $\Omega^2(M, \text{End}_A(E_\pi))$. This is nothing but the curvature of $\nabla$ (see [15, Proposition 3.8]).

5. The Chern character

In this section, we prove our main technical result, Theorem 2.3. It computes the trace of the push-forward of $\mu[M]$ in terms of the de la Harpe–Skandalis determinant by using that the cocycle conditions almost hold for the elements $\pi(s_{ij})$.

5.1. The de la Harpe–Skandalis determinant

The de la Harpe–Skandalis determinant [7] appears in our formula below. Let us recall the definition. Write $\text{GL}_\infty(A)$ for the (algebraic) inductive limit of $(\text{GL}_n(A))_{n \geq 1}$ with standard inclusions. For a piecewise smooth path $\xi : [t_1, t_2] \to \text{GL}_\infty(A)$, define
$$\tilde{\Delta}_\tau(\xi(t)) = \frac{1}{2\pi i} \tau \left( \int_{t_1}^{t_2} \xi(t)^{-1} \xi'(t) dt \right) = \frac{1}{2\pi i} \int_{t_1}^{t_2} \tau(\xi(t)^{-1}) dt.$$ We will make use of some of the properties of $\tilde{\Delta}_\tau$ stated below.

**Lemma 5.1** (cf. [7, Lemme 1]).

(i) Let $\xi_1, \xi_2 : [t_1, t_2] \to \text{GL}_\infty^0(A)$ be two paths and $\xi$ be their pointwise product. Then $\tilde{\Delta}_\tau(\xi) = \tilde{\Delta}_\tau(\xi_1) + \tilde{\Delta}_\tau(\xi_2)$.

(ii) Let $\xi : [t_1, t_2] \to \text{GL}_\infty^0(A)$ be a path with $\|\xi(t) - 1\| < 1$ for all $t$. Then
$$2\pi i \cdot \tilde{\Delta}_\tau(\xi) = \tau(\log \xi(t_2)) - \tau(\log \xi(t_1)).$$

(iii) The integral $\int_{t_1}^{t_2} \tilde{\Delta}_\tau(\xi)$ is left invariant under a fixed-endpoint homotopy of $\xi$. 
5.2. The Chern character on $K_0(C(M) \otimes A)$

Assume that $\tau$ is a trace on $A$. Then $\tau$ induces a map on $\Omega^2(V_i, \text{End}_A(E_\tau|_{V_i}))$ and, by the trace property, $\tau(\Omega_j) = \tau(\Omega_j)$ on $V_{ij}$. We obtain in this way a globally defined form $\tau(\Omega) \in \Omega^2(M, \mathbb{C})$.

Since the fibers of our bundle are all equal to $A$, and our manifold is two-dimensional, the definition of the Chern character associated with $\tau$ (from [15, Definition 4.1], but we have included a normalization coefficient) reduces to

$$
\text{ch}_\tau(\ell_\pi) = \tau \left( \exp \left( \frac{i\Omega}{2\pi} \right) \right) = \tau \left( \sum_{k=0}^{\infty} \frac{i\Omega/2\pi \wedge \cdots \wedge i\Omega/2\pi}{k!} \right) = \tau \left( \frac{i\Omega}{2\pi} \right) \in \Omega^2(M, \mathbb{C}).
$$

(5.1)

This is a closed form whose cohomology class does not depend on the choice of the connection $\nabla$ (see [15, Lemma 4.2]).

A few remarks are in order before stating the next result.

Because $\Lambda$ is a locally ordered simplicial complex (recall the partial order $o$ from Subsection 4.1), every 2-simplex $\sigma$ may be written uniquely as $\langle x_i, x_j, x_k \rangle$ with the vertices written in increasing $o$-order. Whenever we write a simplex in this way, it is implicit that the vertices are written in increasing $o$-order. We may write $\sigma$ for $\sigma$ along with this order.

The orientation $[M]$ induces an orientation of the boundary of the dual cell $U_i$ and in particular of the segment $U^\sigma_{ik}$. Let $s(\sigma) = 0$ if the initial endpoint of $U^\sigma_{ik}$ under this orientation is the barycenter of $\sigma$, and let $s(\sigma) = 1$ otherwise.

**Theorem 5.2.** For a simplex $\sigma = \langle x_i, x_j, x_k \rangle$ of $\Lambda$, let $\xi_\sigma$ be the linear path

$$
\xi_\sigma(t) = (1-t)\pi(s_{ik}) + t\pi(s_{ij})\pi(s_{jk}), \quad t \in [0, 1]
$$

in $\text{GL}(A)$. Then

$$
\tau(\pi_2(\mu[M])) = \sum_{\sigma} (-1)^{s(\sigma)} \tilde{\Delta}_r(\xi_\sigma),
$$

where the sum ranges over all 2-simplices $\sigma$ of $\Lambda$.

**Proof.** The path $\xi_\sigma$ lies entirely in $\text{GL}(A)$ because $\|\pi(s_{ik}) - \pi(s_{ij})\pi(s_{jk})\| < \varepsilon$. It follows from Theorem 2.2 and equation (5.1) that

$$
\tau(\pi_2(\mu[M])) = \langle \text{ch}_\tau(\ell_\pi), [M] \rangle = -\frac{1}{2\pi i} \int_M \tau(\Omega).
$$

We compute this integral.

First observe that by the trace property of $\tau$, we have $\tau(\omega_l \wedge \omega_l) = 0$ for every $l$. Thus,

$$
\int_M \tau(\Omega) = \sum_l \int_{U_l} \tau(\Omega_l) = \sum_l \int_{U_l} \tau(d\omega_l + \omega_l \wedge \omega_l)
$$

$$
= \sum_l \int_{U_l} \tau(d\omega_l) = \sum_l \int_{U_l} d\tau(\omega_l) = \sum_l \int_{\partial U_l} \tau(\omega_l),
$$

where we used Green's theorem for the last equality and $\partial U_l$ has the orientation induced from $[M]$. Recall that $U_l$ is the dual cell to $v_l$. Write this as a sum over the 2-simplices of $\Lambda$:

$$
\sum_l \int_{\partial U_l} \tau(\omega_l) = \sum_l \sum_{\sigma} \int_{(\partial U_l) \cap \sigma} \tau(\omega_l) = \sum_{\sigma} \sum_l \int_{(\partial U_l) \cap \sigma} \tau(\omega_l).
$$

Exactly three dual cells meet a 2-simplex $\sigma = \langle x_i, x_j, x_k \rangle$: $U_i$, $U_j$, and $U_k$, so for each simplex there are three integrals we need to account for. Let us treat each of these in turn.
The definition of the connection forms (see Paragraph 4.2.1) implies that $\omega_i$ restricted to $\sigma$ equals

$$\omega_i = \chi k v_{ki}^{-1} dv_{ki} + \chi j v_{ji}^{-1} dv_{ji} = \chi k v_{ki}^{-1} dv_{ki},$$

where the last equality follows from the fact that $v_{ji}$ is constant. Now, $(\partial U_i) \cap \sigma$ is the union of the two segments $U_{ij}^\sigma$ and $U_{ik}^\sigma$. Observe that $v_{ik}$ is constantly equal to $\pi(s_{ij})\pi(s_{jk})$ on $V_i \cap V_j \cap V_k$ (see Paragraph 4.1.1). Since $U_{ij}^\sigma \subseteq V_i \cap V_j \cap V_k$ and $\chi_k$ vanishes outside $V_k$, we get

$$\int_{(\partial U_i) \cap \sigma} \tau(\omega_i) = \int_{U_{ij}^\sigma} \tau(\chi k v_{ki}^{-1} dv_{ki}) + \int_{U_{ik}^\sigma} \tau(\chi k v_{ki}^{-1} dv_{ki}) = \int_{U_{ik}^\sigma} \tau(\chi k v_{ki}^{-1} dv_{ki}).$$

The second integral, $\int_{(\partial U_j) \cap \sigma} \tau(\omega_j)$, vanishes. This is because $v_{ij}$ and $v_{jk}$ are constant and so

$$\omega_j = \chi i v_{ij}^{-1} dv_{ij} + \chi k v_{kj}^{-1} dv_{kj} = 0.$$

The third integral may be calculated just as the first, with the roles of $i$ and $k$ reversed. We obtain

$$\int_{(\partial U_k) \cap \sigma} \tau(\omega_k) = \int_{U_{ik}^\sigma} \tau(\chi i v_{ik}^{-1} dv_{ik}).$$

Combining the three integrals, we get

$$\sum_{\sigma} \sum_I \int_{(\partial U_l) \cap \sigma} \tau(\omega_l) = \sum_{\sigma} \left( \int_{U_{ik}^\sigma} \tau(\chi k v_{ki}^{-1} dv_{ki}) + \int_{U_{ik}^\sigma} \tau(\chi i v_{ik}^{-1} dv_{ik}) \right)$$

$$= \sum_{\sigma} \int_{U_{ik}^\sigma} \tau(\chi k v_{ki}^{-1} dv_{ki} - \chi i v_{ik}^{-1} dv_{ik}),$$

where the last equality is due to the opposite orientations of the segment $U_{ik}^\sigma$ in the preceding two integrals.

It follows from $v_{ik}v_{ki} = 1$ that $dv_{ik}v_{ik}^{-1} + v_{ki}^{-1} dv_{ki} = 0$. Therefore, the last line in the equation above is equal to

$$\sum_{\sigma} \int_{U_{ik}^\sigma} \tau(\chi k v_{ki}^{-1} dv_{ki} + \chi i v_{ik}^{-1} dv_{ki}) = \sum_{\sigma} \int_{U_{ik}^\sigma} \tau(v_{ki}^{-1} dv_{ki}) = - \sum_{\sigma} \int_{U_{ik}^\sigma} \tau(v_{ik}^{-1} dv_{ik}).$$

To arrive at the conclusion of the theorem, consider the restriction of $v_{ik}$ to the segment $U_{ik}^\sigma$. This is the segment between the barycenter of $\sigma$, where $v_{ik}$ takes the value $\pi(s_{ij})\pi(s_{jk})$, and the barycenter of $\langle x_i, x_k \rangle$, where $v_{ik}$ takes the value $\pi(s_{ik})$ (see Paragraph 4.1.1). Then

$$\int_{U_{ik}^\sigma} \tau(v_{ki}^{-1} dv_{ki}) = (-1)^{s(\sigma)} 2\pi i \cdot \Delta_\tau(\xi_\sigma).$$

This concludes the proof. \(\square\)

6. Oriented surfaces

For the proof of Theorem 2.3, we will use a convenient triangulation $\Lambda_g$ of the orientable genus $g$ surface $\Sigma_g$ that we proceed to describe. The covering space of $\Sigma_g$ is the open disk and we may take as a fundamental domain a regular $4g$-gon, call it $\Sigma_g$, drawn in the hyperbolic plane.

Figure 2 depicts a procedure to obtain $\Sigma_g$ by gluing together two copies of $\Sigma_1$. (We will give a more explicit description of $\Sigma_g$ in a moment.) It also illustrates the labeling we use for the (oriented) sides of $\Sigma_1$ and $\Sigma_2$. To obtain $\Sigma_1$, for example, we identify the side $a$ with $*a$ and the side $b$ with $*b$. To obtain the double torus $\Sigma_2$, we identify $a_k$ with $*a_k$ and $b_k$ with $*b_k$ for $k \in \{1, 2\}$.
6.1. Triangulations

Let us first define a triangulation $\tilde{\Lambda}_g$ of the fundamental domain $\tilde{\Sigma}_g$. We do this by gluing $g$ triangulated copies of $\tilde{\Sigma}_1$ together. Figure 4 shows the triangulation for the $k$th copy of $\tilde{\Sigma}_1$ (with a hole); call it $\tilde{\Lambda}_k^1$. Ignore the labels on the edges and the highlighted edges for now. The vertex labeling also indicates how to glue $\tilde{\Lambda}_k^1$ to $\tilde{\Lambda}_{k-1}^1$ and $\tilde{\Lambda}_{k+1}^1$, with addition modulo $g$. Figure 4(b) illustrates the result of this gluing, the end result being $\tilde{\Lambda}_g$ by definition.

The underlying space of $\tilde{\Lambda}_g$ is $\tilde{\Sigma}_g$. Identifying all the vertices $v^k_i$, as well as identifying $a^k_i$ with $\ast a^k_i$ and $b^k_i$ with $\ast b^k_i$, for each $i \in \{1, 2\}$ and $k \in \{1, \ldots, n\}$, yields a triangulation $\Lambda_g$ of $\Sigma_g$.

6.2. Surface groups

We identify the fundamental group $\Gamma_g$ of $\Sigma_g$ with the group of deck transformations of the universal covering space of $\Sigma_g$. We give a more concrete description of this group now.

The fundamental domain $\tilde{\Sigma}_g$ is a regular $4g$-gon. We write $a_k, b_k, \ast a_k$ and $\ast b_k, k \in \{1, \ldots, n\}$ for its (oriented) sides. The triangulation $\tilde{\Lambda}_g$ gives a subdivision of the side $a_k$ into the three edges in the path $(v^k_0, a^k_1, a^k_2, v^k_1)$ (with orientation given by the directed edge $(a^k_1, a^k_2)$). The subdivision of the sides $b_k, \ast a_k$ and $\ast b_k$ is similar; see Figure 4(a).

The group of deck transformations $\Gamma_g$ is generated by the hyperbolic isometries $\alpha_k$ and $\beta_k, k \in \{1, \ldots, g\}$ defined as follows: $\alpha_k$ maps $\ast a_k$ to $a_k$ in such a way that, locally, the half-plane bounded by $\ast a_k$ containing $\tilde{\Sigma}_g$ is mapped to the half-plane bounded by $a_k$ but opposite $\tilde{\Sigma}_g$. The transformation $\beta_k$ is defined analogously, mapping $\ast b_k$ to $b_k$. We refer the reader to [9, Chapter VII] for more details. When $g = 1$, for example, the transformations $\alpha_1$ and $\beta_1$ are just translations. See Figure 3, where we have omitted the sub- and superscripts corresponding to $k = 1$, since $g = 1$.

For $k \in \{1, \ldots, g\}$, let

$$\kappa_k = \prod_{j=1}^{k} [\alpha_j, \beta_j],$$

and let $\kappa_0 = 1$. We have that $\kappa_g = 1$.

6.3. Local orders and trees

We need $\Lambda_g$ to be locally ordered, so we proceed to fix a partial order on the vertices of $\Lambda_g$ such that the vertices of every simplex form a totally ordered set. Let us define an order on the vertices of $\tilde{\Lambda}_g$ that drops down to the order we need. On the $k$th copy $\tilde{\Lambda}_k^1$, the corresponding
order is indicated in Figure 4(a) by arrows on the edges, always pointing from a smaller vertex to a larger one. It is defined as follows.

(1) For the ‘inner’ vertices, we go ‘counterclockwise’: for fixed \( k \in \{1, \ldots, g\} \), \( w_i^k < w_j^k \) if \( i < j \), except when \( k = g \) and \( j = 4 \) (in which case \( w_4^g = w_1^1 \) and we already have \( w_0^1 < w_1^1 \)).

(2) The ‘inner’ vertices are larger than the ‘outer’ ones: \( w_i^k > v_j^l \), \( a_j^l \), \( b_j^l \), \( a_j^l \), \( b_j^l \) for all \( i, j, k \) and \( l \).

(3) For the ‘outer’ vertices, \( v_i^k < a_j^l \), \( b_j^l \), \( a_j^l \), \( b_j^l \) for all \( i, j, k \) and \( l \); for every \( k \), \( a_1^k < a_2^k \), \( *a_1^k < *a_2^k \), and similarly for the \( b_j^k \).

Finally, we will need a spanning tree \( T_g \) of \( \Lambda_g \), and a lift \( \tilde{T}_g \) to the triangulation \( \tilde{\Lambda}_g \) of the fundamental domain \( \tilde{\Sigma}_g \). Again, we define \( \tilde{T}_g \) first. It is obtained as the union of the edge between \( w_0^1 \) and \( v_0^1 \) (including those two vertices) and trees in each copy \( \Sigma_1^k \). The tree in \( \Sigma_1^k \) is depicted in Figure 4(a) by highlighted (heavier) edges. This drops to a spanning tree \( T_g \) of \( \Sigma_g \). We regard \( T_g \) as ‘rooted’ at the vertex \( v_0^1 \). (In the notation of Subsection 3.1, where the vertices were labeled consecutively as \( x_0, \ldots, x_N \), we have that \( v_0^1 = x_0 \).)

7. Proof of the main result

This section contains the proof of Theorem 2.3. The proof is split into a number of lemmas.

To apply Theorem 5.2, we will first compute the group element \( s_{ij} \) corresponding to each edge \( x_i x_j \) of \( \Lambda_g \), in the sense discussed in Subsection 3.1. Equivalently, we compute group
Figure 4. (a) The triangulation we use for $\tilde{\Sigma}_g^k$, the $k$th copy of $\tilde{\Sigma}_1$ (with a hole). Every edge is labeled with the element of $\Gamma_g$ corresponding to the loop it induces. (b) How the simplicial complex $\tilde{\Sigma}_g$ is defined. The $k$th ‘wedge’ is pictured in (a).

Elements corresponding to edges in the cover $\tilde{\Lambda}_g$, keeping in mind that the lifts of any edge of $\Lambda_g$ will all correspond to the same group element.

A concise way of stating the result of these computations is to label each edge in Figure 4(a) with the corresponding group element.

**Lemma 7.1.** The labels in Figure 4(a) are correct.

**Proof.** We carry out the computations in three separate claims.

Claim 1. An edge of the form $a_i^k w_j^k$ corresponds to $\alpha_{k}^{-1} \in \Gamma_g$. Similarly, an edge of the form $b_i^k w_j^k$ corresponds to $\beta_{k}^{-1} \in \Gamma_g$. 
Consider \( a^k_i w^k_j \) first. When we add this edge to the forest that is the union of all the lifts of \( T_g \) (that is, translates of \( \tilde{T}_g \)), we obtain a unique path \( P \) between \( v^1_0 \), our root vertex and some translate \( sv^1_0 \), where \( s \in \Gamma_g \). We regard \( P \) as directed in the direction of the edge \( a^k_i w^k_j \) that we started with, so it is a path from \( sv^1_0 \) to \( v^1_0 \). It therefore drops down to a loop in \( \Sigma_g \) whose class is \( s^{-1} \), the group element we want to compute (see [8, Proposition 1.39], for example). Now note that because \( *a^k_i \) belongs to \( \tilde{T}_g \), its translation \( \alpha_k(*a^k_i) = a^k_i \) belongs to the translate \( \alpha_k \tilde{T}_g \) of \( \tilde{T}_g \). Thus, \( P \) is a path between \( v^1_0 \) and \( \alpha_k v^1_0 \). The corresponding group element is therefore \( \alpha_k^{-1} \). An entirely similar argument applies to the edge \( b^k_i w^k_j \).

**Claim 2.** Any edge between inner vertices (vertices of the form \( w^k_i \)) corresponds to \( 1 \in \Gamma_g \). The edges \( a^k_i a^k_2, b^k_1 b^k_2, *a^k_2 *a^k_3 \) and \( *b^k_1 *b^k_2 \) all correspond to \( 1 \in \Gamma_g \).

We proceed as in the previous claim. Any edge between inner vertices is either in \( \tilde{T}_g \) or between two vertices that are in \( \tilde{T}_g \). The associated path we get is therefore from \( v^k_i \) to itself. The same is true of the edges \( b^k_1 b^k_2 \) and \( *a^k_2 *a^k_3 \). It follows that the corresponding group element is \( 1 \). Since \( a^k_i a^k_2 \) and \( *a^k_2 *a^k_3 \) are both lifts of the same edge, they correspond to the same element. Similarly, \( *b^k_1 *b^k_2 \) corresponds to \( 1 \).

**Claim 3.** An edge that is incident to \( v^k_i \) and to a vertex \( z \) in the tree \( \tilde{T}_g \) corresponds to the element \( s \in \Gamma_g \) such that \( v^1_0 = sv^k_i \). (The edge is given the orientation induced by the order on the vertices, as usual.) For \( k \in \{1, \ldots, g\} \),

\[
\begin{align*}
v^1_0 &= \kappa_{k-1} \cdot v^k_0, \\
v^1_1 &= \kappa_{k-1} \beta_k \alpha_k^{-1} \cdot v^k_1, \\
v^1_0 &= \kappa_{k-1} \beta_k \cdot v^k_2, \\
v^1_1 &= \kappa_{k-1} \alpha_k \cdot v^k_3.
\end{align*}
\]

(Recall that \( \kappa_k \) is the product of commutators \([\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_k, \beta_k] \) for \( k \in \{1, \ldots, g\} \), and that \( \kappa_0 = 1 \).)

Observe that, because of how the order was defined, \( v^k_i < z \) always holds. When we add the edge \( v^k_i z \) to the tree \( \tilde{T}_g \), we obtain a path from \( v^k_i \) to \( v^1_0 \). (See Figure 4, but keep in mind that in the case \( k = 1 \) the edge \( v^1_0 w^1_i \) belongs to the tree.) It follows that the corresponding element is the \( s \in \Gamma_g \) such that \( v^1_0 = sv^k_i \).

To compute these elements \( s \), we argue by induction on \( k \). Assume \( k = 1 \). We observe that

\[
\begin{align*}
v^1_4 \xrightarrow{\beta_1^{-1}} v^1_1 \xrightarrow{\alpha_1^{-1}} v^1_2 \xrightarrow{\beta_1} v^1_3 \xrightarrow{\alpha_1} v^1_0.
\end{align*}
\]

Indeed, from the definition (see Subsection 6.2) we see that the transformation \( \alpha_1 \) takes \( v^1_1 \) to \( v^1_0 \); think of the side \( *a_1 = (v^1_1, *a^1_1, *a^1_2, v^1_2) \) being mapped to the side \( a_1 = (v^1_3, a^1_1, a^1_2, v^1_1) \); the vertex \( *a^1_1 \) is mapped to \( a^1_1 \) and so \( v^1_1 \) is mapped to \( v^1_0 \). We also see from Subsection 6.2 and Figure 4(a) that \( \beta_1 \) maps \( v^1_2 \) to \( v^1_3 \), and so \( v^1_0 = \alpha_1 \beta_1 \cdot v^1_2 \). A similar argument shows \( v^1_0 = \alpha_1 \beta_1 \cdot v^1_1 \) and that

\[
v^1_1 = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdot v^1_1 = \kappa_1 \cdot v^1_4.
\]

Assuming that the computations hold for \( k - 1 \), we prove them for \( k \). In fact, most of the work is already done. The same argument we used for the case \( k = 1 \) shows

\[
\begin{align*}
v^k_4 \xrightarrow{\beta_k^{-1}} v^k_1 \xrightarrow{\alpha_k^{-1}} v^k_2 \xrightarrow{\beta_k} v^k_3 \xrightarrow{\alpha_k} v^k_0.
\end{align*}
\]
The inductive hypothesis implies
\[ \kappa_{k-1}v_0^k = \kappa_{k-1}v_1^{k-1} = v_0^1. \]
This ends the proof of the claim.

These three claims prove that the labels in Figure 4(a) are correct.
(The labels in Figure 3 also follow from these calculations, but may be obtained by more straightforward arguments because the generators of \( \Gamma_1 \cong \mathbb{Z}^2 \) may be regarded as shifts in the plane.)

**Notation 7.2.** For \( k \in \{1, \ldots, g\} \), let
\[ F_k = \{ \alpha_k^{-1}, \beta_k^{-1}, \kappa_{k-1}, \kappa_{k-1} \alpha_k \beta_k, \kappa_{k-1} \alpha_k \beta_k \alpha_k^{-1} \}. \]
Note that the set \( F = \{ s_{ij} \} \) considered in Subsection 3.1 is equal to the union \( F_1 \cup F_1^{-1} \cup \cdots \cup F_g \cup F_g^{-1} \) by Lemma 7.1.

7.1. **Choosing quasi-representations**

We want to apply Theorem 5.2 using the labels obtained in Lemma 7.1 and some convenient choice of a quasi-representation of \( \Gamma_g \) in \( U(A) \). We begin by proving a slightly stronger version of Proposition 2.5, which guarantees the existence of quasi-representations (under certain conditions). Let us set up some notation first.

For certain unitaries \( u_1, v_1, \ldots, u_g, v_g \) in \( A \), we will need to produce a quasi-representation \( \pi \) satisfying
\[ \pi(\alpha_k) = u_k \quad \text{and} \quad \pi(\beta_k) = v_k \quad \forall k \in \{1, \ldots, g\}. \] (7.1)
Write \( F_{2g} = \langle \hat{\alpha}_1, \hat{\beta}_1, \ldots, \hat{\alpha}_g, \hat{\beta}_g \rangle \) for the free group on \( 2g \) generators. Let \( q: F_{2g} \to \Gamma_g \) and \( \hat{\pi}: F_{2g} \to U(A) \) be the homomorphisms given by
\[ q(\hat{\alpha}_k) = \alpha_k, \quad q(\hat{\beta}_k) = \beta_k \]
and
\[ \hat{\pi}(\hat{\alpha}_k) = u_k, \quad \hat{\pi}(\hat{\beta}_k) = v_k, \]
for all \( k \in \{1, \ldots, g\} \). Note that the kernel of \( q \) is the normal subgroup generated by
\[ \hat{\kappa}_g := \prod_{k=1}^{g} [\hat{\alpha}_k, \hat{\beta}_k], \]
and therefore consists of products of elements of the form \( \hat{\gamma} \hat{\kappa}_g^{\pm 1} \hat{\gamma}^{-1} \), where \( \hat{\gamma} \in F_{2g} \).

Choose a set-theoretic section \( s: \Gamma_g \to F_{2g} \) of \( q \) such that \( s(1) = 1 \),
\[ s(\alpha_k) = \hat{\alpha}_k \quad \text{and} \quad s(\beta_k) = \hat{\beta}_k \quad \forall k \in \{1, \ldots, g\}. \]

**Lemma 7.3.** For all \( \epsilon > 0 \), there exists \( \delta(\epsilon) > 0 \) such that if \( A \) is a unital \( C^* \)-algebra and \( u_1, v_1, \ldots, u_g, v_g \in U(A) \) satisfy
\[ \left\| \prod_{i=1}^{g} [u_i, v_i] - 1 \right\| < \delta(\epsilon), \] (7.2)
then \( \pi = \hat{\pi} \circ s \) (with \( s \) as constructed above) is an \((F, \epsilon)\)-representation satisfying equation (7.1).

This lemma obviously implies Proposition 2.5.
Proof. We only need to check that $\pi$ is $(\mathcal{F}, \epsilon)$-multiplicative. Assume that equation (7.2) holds for some $\delta$ in place of $\delta(\epsilon)$.

Because $\hat{\pi}$ is a homomorphism, for all $\gamma, \gamma' \in \Gamma_g$ we have

$$\|\pi(\gamma)\pi(\gamma') - \pi(\gamma\gamma')\| = \|\pi(\gamma)\pi(\gamma')\pi(\gamma\gamma')^* - 1\| = \|\hat{\pi}(s(\gamma)s(\gamma')s(\gamma\gamma')^{-1}) - 1\|.$$ 

Now, $s(\gamma)s(\gamma')s(\gamma\gamma')^{-1}$ is in the kernel of $q$ and is therefore a product of the form

$$\prod_{i=1}^{m} \hat{\gamma}_i \hat{k}_i^{-1},$$

where $m$ depends on $\gamma$ and $\gamma'$ and $\epsilon_i \in \{1, -1\}$. Thus,

$$\|\pi(\gamma)\pi(\gamma') - \pi(\gamma\gamma')\| = \left\| \hat{\pi} \left( \prod_{i=1}^{m} \hat{\gamma}_i \hat{k}_i^{-1} \right) - 1 \right\| \leq \sum_{i=1}^{m} \|\hat{\pi}(\hat{\gamma}_i)\hat{\pi}(\hat{k}_i)^{\epsilon_i}\hat{\pi}(\hat{\gamma}_i)^* - 1\| \leq m \left\| \prod_{i=1}^{g} [u_i, v_i] - 1 \right\| < m\delta.$$

Since $\mathcal{F}$ is a finite set, there is a positive integer $M$ such that if $\gamma, \gamma' \in \mathcal{F}$, then $s(\gamma)s(\gamma')s(\gamma\gamma')^{-1}$ is a product of at most $M$ elements of the form $\hat{\gamma}_i \hat{k}_i^{-1}$ as above. It follows that $\pi$ is an $(\mathcal{F}, M\delta)$-representation. Choose $\delta(\epsilon) = \epsilon/M$. \hfill \qed

Notation 7.4. Recall the set $\mathcal{F}_k$ defined in Notation 7.2. Let $s_0 : \Gamma_g \rightarrow \mathbb{F}_{2g}$ be a set-theoretic section of $q$ such that

$$s_0(\alpha_k^{\pm 1}) = \hat{\alpha}_k^{\pm 1}, \quad s_0(\beta_k^{\pm 1}) = \hat{\beta}_k^{\pm 1}, \quad s_0(\kappa_k^{-1}) = \hat{k}_k^{-1},$$

for all $k \in \{1, \ldots, g\}$, and

$$s_0(\kappa_k^{-1} \alpha_k \beta_k \alpha_k^{-1}) = \hat{k}_k^{-1} \hat{\alpha}_k \hat{\beta}_k \hat{\alpha}_k^{-1}$$

for all $k \in \{1, \ldots, g - 1\}$. That such a section exists follows from the fact that all the words in the list $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_g \cup \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ are distinct, with two exceptions: $\alpha_1 = \kappa_0 \alpha_1 \in \mathcal{F}_1$ appears twice, as does $\beta_2 = \kappa_2^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \in \mathcal{F}_g$.

Define $\pi_0 = \hat{\pi} \circ s_0 : \Gamma_g \rightarrow U(A)$.

Lemma 7.5. If $\langle x_i, x_j, x_k \rangle$ is any 2-simplex in $\Lambda_g$ different from $\langle v_1^g, a_2^g, w_1^g \rangle$, then $\pi_0(s_{ik}) = \pi_0(s_{ij})\pi_0(s_{jk})$.

If $\langle x_i, x_j, x_k \rangle = \langle v_1^g, a_2^g, w_1^g \rangle$, then $\pi_0(s_{ik}) = v_g$ and

$$\pi_0(s_{ij})\pi_0(s_{jk}) = \left( \prod_{i=1}^{g} [u_i, v_i] \right) v_g.$$ 

Proof. The definition of $s_0$ implies that the image under $s_0$ of any ‘word’ in the list $\mathcal{F}_k$ is the word obtained by replacing $\alpha_k^{\pm 1}$ by $\hat{\alpha}_k^{\pm 1}$ and $\beta_k^{\pm 1}$ by $\hat{\beta}_k^{\pm 1}$, with one exception: the image of $\kappa_{g-1} \alpha_g \beta_g \alpha_g^{-1} = \beta_g$ under $s_0$ is $\hat{\beta}_g$. 

This observation along with inspection of Figure 4(a) shows $s_0(s_{ik}) = s_0(s_{ij})s_0(s_{jk})$ for every 2-simplex in $\Lambda_g$ different from $\langle v_1^g, a_2^g, w_1^g \rangle$. For instance, let $l \in \{1, \ldots, g\}$ and consider the simplex

$$\langle v_0^l, a_1^l, w_0^l \rangle = \langle x_i, x_j, x_k \rangle.$$ 

The corresponding group elements are

$$s_{ij} = \kappa_{l-1} \alpha_l,$$

$$s_{jk} = \alpha_l^{-1}$$

and

$$s_{ik} = \kappa_{l-1}.$$ 

Then

$$s_0(s_{ik}) = \hat{\kappa}_{l-1} = \kappa_{l-1} \hat{\alpha}_l \cdot \hat{\alpha}_l^{-1} = s_0(\kappa_{l-1} \alpha_l) \cdot s_0(\alpha_l^{-1}) = s_0(s_{ij}) \cdot s_0(s_{jk}).$$

The computations in all other 2-simplices apart from $\langle v_1^g, a_2^g, w_1^g \rangle$ are very similar. For this exceptional simplex, we get

$$s_0(s_{ik}) = s_0(\kappa_{g-1} \alpha_g \beta_g \alpha_g^{-1}) = s_0(\beta_g) = \hat{\beta}_g,$$

but

$$s_0(s_{ij})s_0(s_{jk}) = s_0(\kappa_{g-1} \alpha_g \beta_g) s_0(\alpha_g^{-1}) = \hat{\kappa}_{g-1} \hat{\alpha}_g \hat{\beta}_g \hat{\alpha}_g^{-1} = \hat{\kappa}_g \beta_g.$$ 

Since $\pi_0 = \hat{\pi} \circ s_0$ and $\hat{\pi}$ is a homomorphism, the lemma follows. \hfill $\square$

Recall that we used Theorem 2.2 to define $\omega > 0$ in Subsection 3.2.

**Lemma 7.6.** If $0 < \epsilon < \omega$ and equation (7.2) holds (so that $\pi_0$ is an $\langle \mathcal{F}, \epsilon \rangle$-representation), then

$$\tau(\pi_0\mu(\Sigma_g)) = \frac{1}{2\pi i} \tau \left( \log \left( \prod_{i=1}^{g} [u_i, v_i] \right) \right).$$

**Proof.** We apply Theorem 5.2. For each simplex $\langle x_i, x_j, x_k \rangle$, we compute $\hat{\Delta}_\tau(\xi)$ where $\xi_\sigma$ is the path

$$\xi_\sigma(t) = (1 - t)\pi(s_{ik}) + t\pi(s_{ij})\pi(s_{jk}), \quad t \in [0, 1].$$

Observe that the value of $\hat{\Delta}_\tau$ on a constant path is 0. Lemma 7.5 implies that there is only one 2-simplex $\sigma$ such that $\xi_\sigma$ is not constant: $\sigma_0 = \langle v_1^g, a_2^g, w_1^g \rangle$. By Lemma 7.5, it yields the linear path $\xi_{\sigma_0}$ from $v_g$ to

$$\left( \prod_{i=1}^{g} [u_i, v_i] \right) v_g.$$ 

Using Lemma 5.1, we obtain

$$\hat{\Delta}_\tau(\xi_{\sigma_0}) = \frac{1}{2\pi i} \tau \left( \log \left( \prod_{i=1}^{g} [u_i, v_i] \right) \right).$$

Finally, Theorem 5.2 implies

$$\tau(\pi_0\mu(\Sigma_g)) = (-1)^{s(\sigma_0)} \frac{1}{2\pi i} \tau \left( \log \left( \prod_{i=1}^{g} [u_i, v_i] \right) \right),$$

where the sign $(-1)^{s(\sigma_0)}$ depends on the orientation $[\Sigma_g]$. The standard orientation on $\Sigma_g$ gives $s(\sigma_0) = 1$. \hfill $\square$
By putting these lemmas together, we can prove Theorem 2.3.

Proof of Theorem 2.3. Recall that the statement of the theorem fixes a positive integer $g$ and idempotents $q_0$ and $q_1$ in some matrix algebra over $\ell^1(\Gamma_g)$ such that $\mu_{\Sigma_g} = [q_0] - [q_1] \in K_0(\ell^1(\Gamma_g))$.

Let $\mathcal{F}_0$ be the finite set $\{s_{ij}\}$ defined in Subsection 3.1 and described explicitly in Notation 7.2. Theorem 2.2 provides an $\omega > 0$ so small that if $\pi: \Gamma_g \to U(A)$ is an $(\mathcal{F}_0, \omega)$-representation, then $\pi_\sharp(\mu_{\Sigma_g}) := \pi_\sharp(q_0) - \pi_\sharp(q_1)$ is defined and

$$\tau(\pi_\sharp(\mu_{\Sigma_g})) = \langle \text{ch}_r(\ell_\pi), [\Sigma_g] \rangle.$$

Lemma 7.3 shows that by making this quantity smaller we can make $\pi_0$ more multiplicative on $\mathcal{F}_0$. Therefore, because $\pi$ and $\pi_0$ agree on the generators of $\Gamma_g$, there exists an $0 < \epsilon_0 < \omega$ so small that if $\pi$ is an $(\mathcal{F}_0, \epsilon_0)$-representation, then $\pi_\sharp$ and $\pi_0\sharp$ agree on $\{q_0, q_1\} \subset K_0(\ell^1(\Gamma_g))$.

Finally,

$$\tau(\pi_\sharp(\mu_{\Sigma_g})) = \tau(\pi_0\sharp(\mu_{\Sigma_g})) = \frac{1}{2\pi i} \tau \left( \log \left( \prod_{i=1}^g [u_i, v_i] \right) \right),$$

by Lemma 7.6.

\[\square\]

References
