Decomposing nuclear maps

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Abstract. We show that the strengthened version of the completely positive approximation property of Brown, Carrión, and White—where the downward maps are asymptotically order zero and the upward maps are convex combinations of order zero maps—is enjoyed by every nuclear order zero map.

Choi and Effros [9] and Kirchberg [16] proved that nuclearity for C^*-algebras is equivalent to the completely positive approximation property: a C^*-algebra A is nuclear if and only if there exist nets of finite-dimensional C^*-algebras F_i and of contractive completely positive (c.c.p.) maps

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A,$$

the composition of which tends to the identity map on A in the point-norm topology, in the sense that

$$(1) \qquad \|(\phi_i \psi_i)(a) - a\| \to 0 \quad \text{for all } a \in A.$$

See [6] for the general theory of nuclearity for C^*-algebras.

This approximation property has been strengthened by several authors in various cases [2, 24, 15, 5]. Notably, in [5], building on techniques from [15], it was shown that all nuclear C^*-algebras enjoy completely positive approximations as in (1), where, in addition, all of the maps ϕ_i are convex combinations of order zero maps and the maps ψ_i are asymptotically order zero, in the sense that $\|\psi_i(a)\psi_i(b)\| \to 0$ whenever a and b are positive elements of A with $ab = 0$.

In this short note, we show that this improved approximation property holds for any c.c.p. nuclear order zero map between two C^*-algebras. Recall that a c.c.p. map $\theta \colon A \to B$ has *order zero* if $\theta(a)\theta(b) = 0$ whenever a and b are positive elements in A with $ab = 0$; see [23].

Theorem 1. *Let A and B be C^*-algebras. If $\theta \colon A \to B$ is a c.c.p. nuclear order zero map, then there are nets of wfinite-dimensional C^*-algebras (F_i) and of c.c.p. maps $A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} B$ satisfying*

   (i) *$\|(\phi_i \psi_i)(a) - \theta(a)\| \to 0$ for all $a \in A$,*
   (ii) *$\|\psi_i(a)\psi_i(b)\| \to 0$ for all $a, b \in A$ such that $ab = 0$, and*
   (iii) *every ϕ_i is a convex combination of c.c.p. order zero maps.*

The refined versions of the completely positive approximation property originate in a search for a theory of non-commutative covering dimension of C^*-algebras in the work of Kirchberg and Winter on decomposition rank [18] and of Winter and Zacharias on nuclear dimension [23]. In both [18] and [23] one finds conditions asking for an approximate factorization of the identity map similar to that in Theorem 1, but with more control on the upward maps ϕ_i.

The refined approximation property given in [5]—which is Theorem 1 in the special case when θ is the identity map on a nuclear C^*-algebra—is a crucial part of [8], where it is shown that unital, separable, simple, nuclear, $\mathcal{Z}$-stable C^*-algebras have nuclear dimension at most 1. This latter result plays an important role in the classification theory for nuclear C^*-algebras through the results of [22]. The improved approximation properties mentioned above have also seen application to near-inclusions of C^*-algebras [15]. Given the growing interest in the structure and classification of *-homomorphisms (as opposed to C^*-algebras) in papers such as [10, 13, 20, 8, 7, 3], we hope that the results presented here will prove just as useful as their counterparts for C^*-algebras have.

The outline of the proof of Theorem 1 is not altogether new: the same broad steps used in [15] and [5] apply. We point out, however, that the proofs here, even once restricted to the case of the identity map on a nuclear C^*-algebra, are new. For instance, we avoid the use of amenable and quasidiagonal traces, which are critical in [5].

In order to prove Theorem 1, we will prove the following von Neumann algebraic version first. Theorem 1 will then follow from a Hahn–Banach argument similar to that used in [5] and [15].

Theorem 2. *Let A be a C^*-algebra and let N be a von Neumann algebra. If $\theta\colon A \to N$ is a weakly nuclear *-homomorphism, then there are nets of finite-dimensional C^*-algebras (F_i) and of c.c.p. maps $A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} N$ satisfying*

- (i) *$(\phi_i\psi_i)(a) \to \theta(a)$ in the σ-strong* topology (and therefore also in the σ-weak topology) for all $a \in A$,*
- (ii) *$\|\psi_i(a)\psi_i(b)\| \to 0$ for all $a, b \in A$ such that $ab = 0$, and*
- (iii) *every ϕ_i is a *-homomorphism.*

In fact, the approximate factorizations satisfying conditions (i) and (ii) of Theorem 2 can be arranged for all c.c.p. weakly nuclear maps.

Proposition 3. *Let A be a C^*-algebra and let N be a von Neumann algebra. If $\theta\colon A \to N$ is a c.c.p. weakly nuclear map, then there are nets of finite-dimensional C^*-algebras (F_i) and of c.c.p. maps $A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} N$ satisfying*

- (i) *$(\phi_i\psi_i)(a) \to \theta(a)$ in the σ-strong* topology for all $a \in A$, and*
- (ii) *$\|\psi_i(a)\psi_i(b)\| \to 0$ for all $a, b \in A$ such that $ab = 0$.*

If A is quasidiagonal, we may further arrange for

- (ii′) *$\|\psi_i(a)\psi_i(b) - \psi_i(ab)\| \to 0$ for all $a, b \in A$.*

Remark 4. If N is a von Neumann algebra and ρ is a normal state on N, then the semi-norm $\|\cdot\|_{2,\rho}^{\sharp}$ on N is defined by

$$\|x\|_{2,\rho}^{\sharp} = \rho\Big(\frac{xx^* + x^*x}{2}\Big)^{1/2}, \quad x \in N.$$

If $\{\rho_i\}$ is a separating family of normal states on N, then the topology generated by $\{\|\cdot\|_{2,\rho_i}^{\sharp}\}$ agrees with the σ-strong* topology on any bounded subset of N; see [1, III.2.2.19].

Proof of Proposition 3. First we assume that A is quasidiagonal and prove the stronger approximation condition. We may assume A is unital. Let $\mathcal{F} \subseteq A\backslash\{0\}$ and $\mathcal{S} \subseteq S(N) \cap N_*$ be finite sets, and let $\varepsilon > 0$. Because θ is weakly nuclear, there exist an integer $\ell \geq 1$ and u.c.p. maps $A \xrightarrow{\psi'} M_\ell(\mathbb{C}) \xrightarrow{\phi'} N$ satisfying

$$(2) \qquad \|\phi'(\psi'(x)) - \theta(x)\|_{2,\rho}^{\sharp} < \|x\|\varepsilon$$

for all $x \in \mathcal{F}$ and $\rho \in \mathcal{S}$. We will prove the existence of an integer k and of u.c.p. maps $A \xrightarrow{\psi} M_k(\mathbb{C}) \xrightarrow{\phi} N$ satisfying

    (a) $\|\phi(\psi(x)) - \theta(x)\|_{2,\rho}^{\sharp} < 5\|x\|\varepsilon$ for all $x \in \mathcal{F}$ and $\rho \in \mathcal{S}$, and

    (b) $\|\psi(x)\psi(y) - \psi(xy)\| < \|x\|\|y\|\varepsilon$ for all $x, y \in \mathcal{F}$.

Replacing A with $C^*(\mathcal{F} \cup \{1_A\}) \subseteq A$, we may assume A is separable. Fix a separable Hilbert space H and a faithful representation $A \subseteq \mathcal{B}(H)$ with $A \cap \mathcal{K}(H) = 0$. After identifying $M_\ell(\mathbb{C})$ with $\mathcal{B}(\mathbb{C}^\ell)$, Voiculescu's theorem[1] provides an isometry $v \colon \mathbb{C}^\ell \to H$ such that $\|v^*xv - \psi'(x)\| < \|x\|\varepsilon$ for all $x \in \mathcal{F}$.

Since A is quasidiagonal, there is a finite rank projection $p \in \mathcal{B}(H)$ such that $\|px - xp\| < \varepsilon\|x\|$ for all $x \in \mathcal{F}$ and $\|pv - v\| < \varepsilon$. Identify $\mathcal{B}(pH)$ with $M_k(\mathbb{C})$ for some integer $k \geq 1$ and define $\psi \colon A \to M_k(\mathbb{C})$ by $\psi(a) = pap$. Then

$$(3) \qquad \|v^*\psi(x)v - \psi'(x)\| < 3\|x\|\varepsilon$$

and

$$\|\psi(x)\psi(y) - \psi(xy)\| < \|x\|\|y\|\varepsilon$$

for all $x, y \in \mathcal{F}$. Note that (2) and (3) imply

$$(4) \qquad \|\phi'(v^*\psi(x)v) - \theta(x)\|_{2,\rho}^{\sharp} < 5\|x\|\varepsilon$$

for all $x \in \mathcal{F}$ and $\rho \in \mathcal{S}$. To complete the proof (in the quasidiagonal case), define $\phi = \phi'(v^*(\,\cdot\,)v)$.

Now we handle the case of a general C^*-algebra A. Let $\pi \colon C_0(0,1] \otimes A \to A$ be the *-homomorphism given by $\pi(f \otimes a) = f(1)a$ for all $f \in C_0(0,1]$, $a \in A$. By the homotopy invariance of quasidiagonality [21], $C_0(0,1] \otimes A$ is quasidiagonal. Applying the first part of the proof to the c.c.p. weakly nuclear map $\theta\pi$, there is a net $C_0(0,1] \otimes A \xrightarrow{\psi_i'} F_i \xrightarrow{\phi_i} N$ satisfying (i) and (ii'). The result follows by defining $\psi_i \colon A \to F_i$ by $\psi_i(a) = \psi_i'(\mathrm{id}_{(0,1]} \otimes a)$. $\qquad\square$

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[1]See [11, Theorem II.5.3] for the version we are using here.

Remark 5. If A and B are C^*-algebras and $\theta \colon A \to B$ is a c.c.p. nuclear map, then a statement analog to Proposition 3 holds with the approximations in (i) taken in the operator norm. The proof is the same except with operator norm estimates in (2), (4), and (a).

Proof of Theorem 2. Through a standard argument, we may assume A is separable and N has separable predual. Note that N decomposes as a direct sum $N = N_1 \oplus N_\infty$ with N_1 finite and N_∞ properly infinite. Then the map θ decomposes accordingly as $\theta = \theta_1 \oplus \theta_\infty$ for weakly nuclear *-homomorphisms $\theta_i \colon A \to N_i$, $i \in \{1, \infty\}$. By handling each summand separately, we may assume that either N is finite or N is properly infinite.

Properly Infinite Case: We may assume A is unital. Fix a faithful normal state ρ on N, a finite set $\mathcal{F} \subseteq A$ of unitaries, and a finite set $\mathcal{G} \subseteq A \times A$ such that $xy = 0$ for all $(x, y) \in \mathcal{G}$, and let $\varepsilon > 0$. By Proposition 3, there are a finite-dimensional C^*-algebra F and c.c.p. maps $A \xrightarrow{\psi} F \xrightarrow{\phi'} N$ satisfying

(a) $\|\phi'(\psi(x)) - \theta(x)\|_{2,\rho}^\sharp < \varepsilon$ for all $x \in \mathcal{F}$, and
(b) $\|\psi(x)\psi(y)\| < \varepsilon$ for all $(x, y) \in \mathcal{G}$.

At this point we can follow the proof of [5, Lemma 2.4], which borrows heavily from [14, Theorem 2.2]. Indeed the last two paragraphs of the proof of [5, Lemma 2.4], with θ and ϕ' in place of π_∞ and θ, produce a *-homomorphism $\phi \colon F \to M$ such that

$$\|(\phi\psi)(x) - \theta(x)\|_{2,\rho}^\sharp < 2\varepsilon^{1/2} \quad \text{for all } x \in \mathcal{F}.$$

Finite Case: Let τ_N be a faithful normal trace on N so that on bounded subsets of N, the σ-strong* topology is induced by the norm

$$\|x\|_{2,\tau_N} = \tau_N(x^*x)^{1/2}, \quad x \in N.$$

Let $\tau_A = \tau_N\theta$, and note that θ induces a faithful normal *-homomorphism $\bar{\theta} \colon \pi_{\tau_A}(A)'' \to N$. As N is finite, there is normal expectation $N \to \bar{\theta}(\pi_{\tau_A}(A)'')$ (see [6, Lemma 1.5.11]). Therefore, the corestriction of π_{τ_A} to $\pi_{\tau_A}(A)''$ is weakly nuclear. It follows that $\pi_{\tau_A}(A)''$ is hyperfinite by the equivalence of (5) and (6) in [4, Theorem 3.2.2]. After replacing N with $\pi_{\tau_A}(A)''$ and θ with π_{τ_A}, we may assume that N is hyperfinite, which we do for the rest of the proof.

Let F_i be an increasing sequence of finite-dimensional C^*-subalgebras of N with σ-strong* dense union and let $\phi_i \colon F_i \to N$ denote the inclusion maps. Let $\tau_i = \tau_N|_{F_i}$ be the induced trace on F_i. Fix a trace-preserving expectation $E_i \colon N \to F_i$ and set $\psi_i' = E_i\theta$. The sequence (ψ_i') induces a *-homomorphism to the tracial von Neumann algebra ultraproduct

$$\psi^\omega \colon A \to F^\omega = \prod_{i}^{\omega}(F_i, \tau_i)$$

and a c.c.p. map to the C^*-algebra ultraproduct

$$\psi_\omega \colon A \to F_\omega = \prod_{\omega} F_i.$$

Let $q \colon F_\omega \to F^\omega$ be the quotient map and note that $q\psi_\omega = \psi^\omega$. Let $J = \ker q$.

Proposition 4.6 of [17] proves that the uniform 2-norm null sequences in a norm ultrapower form a σ-ideal [17, Definition 4.4], by a standard application of Kirchberg's ϵ-test. A very slight modification of the proof shows that J is a σ-ideal of F_ω. Then there is a positive contraction $e \in J \cap \psi_\omega(A)'$ such that $ex = x$ for all $x \in J \cap C^*(\psi_\omega(A))$.

Consider the c.c.p. map $\Psi \colon A \to F_\omega$ defined by $\Psi(x) = (1-e)\psi_\omega(x)$, $x \in A$. We claim Ψ is order zero. Indeed, if $x, y \in A_+$ satisfy $xy = 0$, then $q(\psi_\omega(x)\psi_\omega(y)) = \psi^\omega(xy) = 0$, so $\psi_\omega(x)\psi_\omega(y) \in J$. Then $\Psi(x)\Psi(y) = (1-e)^2\psi_\omega(x)\psi_\omega(y) = (1-e)(\psi_\omega(x)\psi_\omega(y) - \psi_\omega(x)\psi_\omega(y)) = 0$, as required.

Represent e by a sequence of positive contractions (e_i) with $e_i \in F_i$, and define
$$\psi_i = (1-e_i)^{1/2}\psi_i'(\,\cdot\,)(1-e_i)^{1/2} \colon A \to F_i.$$
Then the maps $A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} N$ have the desired properties. $\qquad\square$

Finally, we prove Theorem 1.

Proof of Theorem 1. The structure theorem for order zero maps (see [23, Corollary 4.1]) implies there is a *-homomorphism $\acute{\theta} \colon C_0(0,1] \otimes A \to B$ such that $\theta(a) = \acute{\theta}(\mathrm{id}_{(0,1]} \otimes a)$ for $a \in A$. By [12, Theorem 2.10], $\acute{\theta}$ is nuclear (since θ is nuclear). Hence, after replacing A with $C_0(0,1] \otimes A$ and θ with $\acute{\theta}$, we may assume θ is a *-homomorphism.

Let $\mathcal{F}$ be a finite subset of A and let $\varepsilon > 0$. Let K be the subset of $\mathcal{B}(A,B)$ consisting of all c.c.p. maps $\eta \colon A \to B$ for which there exist a wfinite-dimensional C^*-algebra F and c.c.p. maps $A \xrightarrow{\psi} F \xrightarrow{\phi} B$ satisfying:

- $\eta = \phi\psi$,
- $\|\psi(a)\psi(b)\| < \varepsilon$ for all $a, b \in \mathcal{F}$ with $ab = 0$, and
- ϕ is a convex combination of c.c.p. order zero maps.

To prove the theorem, it suffices to show that θ is in the point-norm closure of K.

By Theorem 2, there are nets of wfinite-dimensional C^*-algebras (F_i) and c.c.p. maps $A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} B^{**}$ satisfying conditions (i)–(iii) of Theorem 2. Lemma 1.1 of [15] implies that for each i, there is a net $(\phi_i^\lambda \colon F_i \to B)_{\lambda \in \Lambda}$ of c.c.p. order zero maps such that $\phi_i^\lambda(x) \to \phi_i(x)$ in the σ-strong* topology (and therefore in the σ-weak topology) for all $x \in F_i$. Therefore, we may assume that the image of ϕ_i is contained in B for all i. Then, for all sufficiently large i, we have that $\phi_i \circ \psi_i \in K$. Thus, θ is in the point-weak closure of K. As K is a convex set in $\mathcal{B}(A,B)$, its point-norm and point-weak closures coincide, by the Hahn–Banach Theorem.[2] Next, recall that the weak topology on B is the same as the restriction to B of the weak-* topology on B^{**} (with B regarded as a subspace of B^{**} via the standard map). But the weak*-topology on B^{**} is precisely the σ-weak topology (see, e.g., [1, I.8.6]), so the theorem follows. $\quad\square$

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[2]In fact, this is consequence of the Hahn–Banach Separation Theorem. See [6, Lemma 2.3.4] for a proof.

For a nuclear *-homomorphism $\theta\colon A \to B$ as in Theorem 1, it is natural to consider a stronger approximation property where the maps ψ_i are required to be approximately multiplicative in the sense that

(ii$'$) $\|\psi_i(a)\psi_i(b) - \psi_i(ab)\| \to 0$ for all $a, b \in A$.

Question 6. For which nuclear *-homomorphisms $\theta\colon A \to B$ between C^*-algebras A and B are there approximate factorizations $A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} B$ of θ as in Theorem 1 that also satisfy (ii$'$)?

A characterization of such θ is likely related to some appropriate form of quasidiagonality. In particular, note that if θ is faithful and (ϕ_i), (ψ_i), and (F_i) are as in Question 6, then for all $a \in A$, we have

$$\|\psi_i(a)\| \geq \|\phi_i(\psi_i(a))\| \to \|\theta(a)\| = \|a\|.$$

Now, the net (ψ_i) is an approximate embedding of A into finite-dimensional C^*-algebras, and this shows that A is quasidiagonal. We end with some further comments and partial results related to Question 6.

First, [5, Theorem 2.2] states that, for a separable nuclear C^*-algebra A, id_A enjoys approximate factorizations as in Question 6 if and only if A is quasidiagonal and all traces on A are quasidiagonal. It is not hard to show that the separability assumption is superfluous.

Second, if A is quasidiagonal and B has no traces, then every nuclear *-homomorphism $\theta\colon A \to B$ enjoys approximate factorizations as in Question 6. This follows as in the proof of Theorem 1 with only slight modifications. The key observations are that

(a) because A is quasidiagonal, the approximate factorizations of the composition $A \xrightarrow{\theta} B \hookrightarrow B^{**}$ given in Proposition 3 can be taken to satisfy (ii$'$), and

(b) because B has no traces, B^{**} is properly infinite, and so the finite case in the proof of Theorem 2 is not needed.

Third, one might also compare Question 6 with the recent results of [3], which address the nuclear dimension and decomposition rank of $\mathcal{O}_\infty$-stable and $\mathcal{O}_2$-stable *-homomorphisms. In particular, [3, Theorem D] states that a full $\mathcal{O}_2$-stable *-homomorphism with a separable and exact domain has nuclear dimension zero if and only if it is nuclear and its domain is quasidiagonal. Noting that decomposition rank zero is equivalent to nuclear dimension zero, [3, Proposition 1.7] implies such a map enjoys approximate factorizations as in Question 6. Again, via standard reductions, the separability of A is not needed in this result.

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