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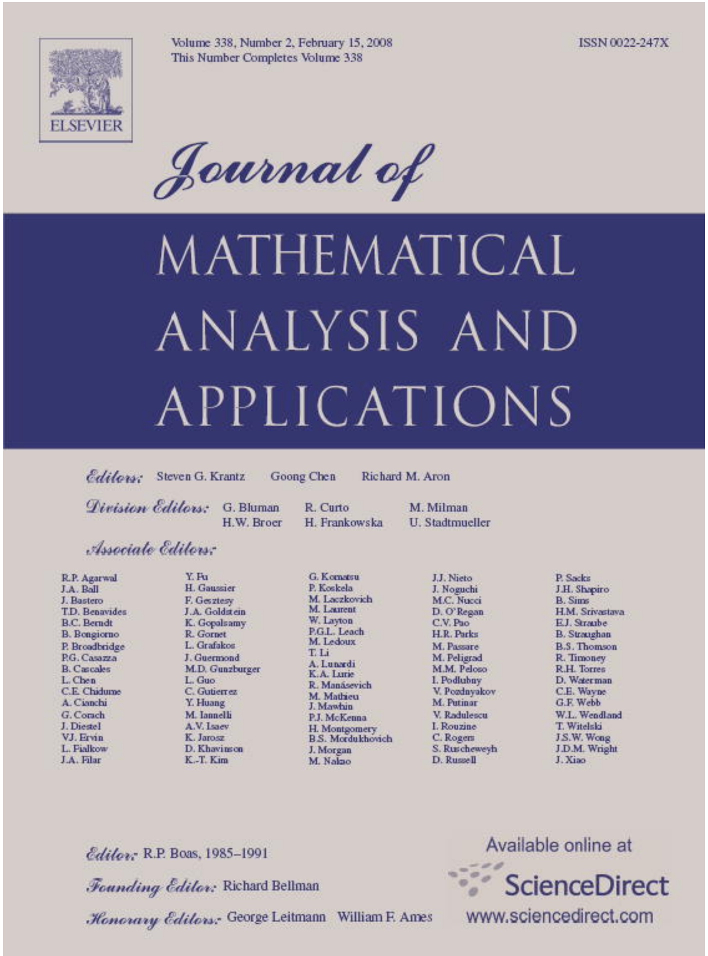

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Approximations of C^*-algebras and the ideal property ☆

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Abstract

We introduce several classes of C^*-algebras (using for this local approximations by "nice" C^*-algebras), that generalize the *AH* algebras, among others. We initiate their study, proving mainly results about the ideal property, but also about the ideals generated by their projections, the real rank zero, the weak projection property, minimal tensor products, extensions, quasidiagonal extensions, ideal structure, the largest ideal with the ideal property and short exact sequences. Some of the previous results of the second named author are generalized.
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Keywords: C^*-algebra; Local approximation; Ideal generated by projections; Ideal property; *LS* algebra; Minimal tensor product; Extension; Quasidiagonal extension; *WLB* algebra; Weak projection property; Real rank zero; Riesz decomposition property; *SLB* algebra; Largest ideal with the ideal property

1. Introduction

A C^*-algebra is said to have the ideal property if each of its ideals is generated (as an ideal) by its projections (in this paper, by an ideal we mean a closed, two-sided ideal). The C^*-algebras with the ideal property can be seen as noncommutative zero-dimensional topological spaces (because a commutative C^*-algebra has the ideal property if and only if its spectrum is zero-dimensional). Every simple, unital C^*-algebra as well as any C^*-algebra of real rank zero [3] has the ideal property. The class of C^*-algebras with the ideal property is important in Elliott's classification program for separable, nuclear C^*-algebras by discrete invariants including K-theory [8].

The *ideal property* first appeared in K. Stevens' Ph.D. thesis, where a certain class of (nonsimple) approximate interval algebras with the ideal property was classified by a K-theoretical invariant; later the second named author has studied this concept extensively (see [14,15,19–32]). In [21], the second named author classified the *AH* algebras with the ideal property and with slow dimension growth up to a shape equivalence by a K-theoretical invariant, and Gong, Jiang, Li and the second named author have proved, in particular, that every *AH* algebra with the ideal property and with the dimensions of the local spectra uniformly bounded can be written as an *AH* algebra over (special) base spaces of dimension at most 3 (see [14,15]). This last result generalizes similar and important reduction theorems of

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Dădărlat [4] and Gong [12] (for real rank zero *AH* algebras) and of Gong [13] (for simple *AH* algebras), which have been essential steps in the classification of the corresponding classes of *AH* algebras (see [5,9,11]). Rørdam and the second named author showed in [31] that the ideal property is not preserved by taking minimal tensor products (even in the separable case) and they proved in [32] (among other things) that a separable, purely infinite C^*-algebra has the ideal property if and only if its primitive spectrum has a basis of compact-open sets, and that if A is a separable C^*-algebra, then $A \otimes \mathcal{O}_2$ has the ideal property if and only if $A \otimes \mathcal{O}_2$ has real rank zero. Also, the second named author has been able to obtain nonstable K-theoretical results for a large class of C^*-algebras with the ideal property. Indeed, if A is an *AH* algebra with the ideal property and with slow dimension growth, we showed that A has stable rank one (which means, when A is unital, that the set of the invertible elements of A is dense in A) [21], that $K_0(A)$ is weakly unperforated in the sense of Elliott and is also a Riesz group [21,22] and that the strict comparability of projections in A is determined by the tracial states of A, if A is unital [21]. The second named author gives also in [21] several characterizations of the ideal property for a given *AH* algebra; two of these characterizations involve the spectra of connecting $*$-homomorphisms. Also, as it was first observed by the second named author in [22], large classes of C^*-algebras with the ideal property have an interesting K-theoretical description of their ideal lattice (see also [24,28,31,34]).

It is important to find large and interesting classes of C^*-algebras for which the ideal property can be characterized (or, more generally, for which one can characterize when a given ideal is generated by its projections), and in general, that behave well with respect to the ideal property and some natural operations. In this paper we define several classes of C^*-algebras, using for this local approximations by particular C^*-algebras that have "good" properties. These new classes of C^*-algebras (the *LS* algebras, the *WLB* algebras and the *SLB* algebras) are nontrivial generalizations of the *AH* algebras, among others. We initiate here their study, proving mainly results about the ideal property, and also about the ideals generated by their projections, the real rank zero, the weak projection property, minimal tensor products, extensions, quasidiagonal extensions, ideal structure, the largest ideal with the ideal property and short exact sequences of C^*-algebras. Many of the results of this paper generalize previous results obtained by the second named author.

In Section 2 we prove some general results for C^*-algebras defined by local approximations involving approximate units of projections and ideals (generated by their projections) (see Definitions 2.1 and 2.2). Then we define, using local approximations by particular "nice" C^*-algebras, the *LS algebras*, which are generalizations of the *AH* algebras. We prove, in particular, that if A and B are nonzero *LS* algebras and at least one of them is exact, then the minimal tensor product $A \otimes B$ has the ideal property if and only if A and B have the ideal property (see Theorem 2.15 for a more general result).

In Section 3 we show some general properties concerning quasidiagonal extensions, including the fact that for a quasidiagonal extension of C^*-algebras $0 \to I \to A \to B \to 0$, if I and B have the ideal property, then so does A (Theorem 3.7).

In Section 4 we define, using again local approximations by "nice" C^*-algebras, the class of *WLB algebras* (see Definition 4.3). Note that each *AH* algebra and, more generally any *LB* algebra [28] is a *WLB* algebra. Let A be a *WLB* algebra and I an ideal of A. We prove that I is generated by its projections if and only if the extension $0 \to I \to A \to A/I \to 0$ is quasidiagonal, if and only if I has an approximate unit of projections, if and only if I is a *WLB* algebra (see Theorem 4.4 for a more general statement). We prove, in particular, that if A is a *WLB* algebra, I an ideal of A and B a sub-C^*-algebra of A that contains I, then B has the ideal property (respectively real rank zero) if and only if I and B/I have the ideal property (respectively real rank zero) (see Theorem 4.10). Note that if $A = B$ is an arbitrary C^*-algebra, these results are not necessarily true (see [3,22]). Also, for a separable *WLB* algebra A such that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property, we identify the ideal lattice of $K_0(A)$ with the lattice $\{I \lhd A\colon I \text{ is stably cofinite and generated by its projections}\}$ (see Theorem 4.13).

In Section 5 we use again local approximations by "good" C^*-algebras to define a new class of C^*-algebras, the *SLB algebras* (see Definition 5.2). Note that any *AH* algebra, or more generally, any special *LB* algebra [28], is an *SLB* algebra. We describe the lattice of the ideals generated their projections of an *SLB* algebra A (see Theorem 5.6) and we show (in particular) that A has the ideal property if and only if $A \otimes \mathcal{K}$ has the ideal property (see Remark 5.5 for a more general result). We also point out that the minimal tensor product of *SLB* algebras behaves well with respect to the ideal property: if A and B are nonzero *SLB* algebras of type I, then $A \otimes B$ has the ideal property if and only if A and B have the ideal property (see Theorem 5.8).

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In Section 6 we show that every C^*-algebra A has the largest ideal with the ideal property, denoted $I_{ip}(A)$ (Proposition 6.2). We prove that if I is an ideal in a *WLB* algebra A, then we have an exact sequence $0 \to I_{ip}(I) \to I_{ip}(A) \to I_{ip}(A/I_{ip}(I)) \to 0$ (see Theorem 6.3). So, if A is a *WLB* algebra (in particular, if A is an *AH* algebra or, more generally, an *LB* algebra) then $I_{ip}(A/I_{ip}(A)) = 0$ (Corollary 6.4). While we do not know whether the fact that a matrix algebra over A has the ideal property implies that A itself has the ideal property, we show (in particular) that if B is a C^*-algebra with an approximate unit of projections such that $M_n(B)$ is a *WLB* algebra for some natural number n, then $I_{ip}(M_n(B)) = M_n(I_{ip}(B))$ and also that $I_{ip}(B)$ has the weak projection property (see Definition 4.8 and Theorem 6.9). (This clearly shows that if in addition $M_n(B)$ has the ideal property, then B has the ideal property.)

If A is a C^*-algebra and $E \subseteq A$, then $\langle E\rangle_A$ stands for the ideal of A generated by E. When there is no ambiguity, we may simply write $\langle E\rangle$.

Let A be a C^*-algebra, E and F subsets of A, and $\epsilon > 0$. We say that F is ϵ-included in E, written $F \subseteq_\epsilon E$, if for each element $f \in F$ there is an element $e \in E$ such that $\|e - f\| < \epsilon$. For any set E, $\mathcal{F}(E)$ is the set of all finite subsets of E.

Let A be a C^*-algebra. An *approximate unit* of A is a net $(a_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of A such that $\|a_\lambda a - a\| \to 0$ for every $a \in A$. We do not require approximate units to be increasing.

If A is a C^*-algebra, then $\mathcal{P}(A)$ is the set of all the projections of A, i.e., $\mathcal{P}(A) = \{p \in A\colon\ p = p^* = p^2\}$.

If A and B are C^*-algebras, then $A \otimes B$ will denote their minimal tensor product.

For a Hilbert space H, $\mathcal{K}(H)$ will denote the C^*-algebra of compact operators on H. If $H = \ell^2(\mathbb{N})$, then we will write $\mathcal{K} = \mathcal{K}(H)$.

2. The ideal property in tensor products of *LS* algebras

Definition 2.1. Suppose $(A_\lambda)_{\lambda\in\Lambda}$ is a family of sub-C^*-algebras of the C^*-algebra A. If for each finite subset F of A and every $\epsilon > 0$ there is $\lambda_0 \in \Lambda$ such that $F \subseteq_\epsilon A_{\lambda_0}$, we say that A is *locally approximated by the family of sub-C^*-algebras* $(A_\lambda)_{\lambda\in\Lambda}$, written $A \approx (A_\lambda)_{\lambda\in\Lambda}$.

Definition 2.2. Let A be a C^*-algebra and $(A_\lambda)_{\lambda\in\Lambda}$ be a family of C^*-algebras. Suppose that for each λ there is a $*$-homomorphism $\varphi_\lambda : A_\lambda \to A$. We say that A is *locally approximated by* $(A_\lambda)_{\lambda\in\Lambda}$ *via* (φ_λ), written $A \approx (A_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$, if for each finite subset F of A and each $\epsilon > 0$, there is $\lambda_0 \in \Lambda$ such that

$$F \subseteq_\epsilon \varphi_{\lambda_0}(A_{\lambda_0}) \quad \text{and} \quad \mathcal{P}(A) \cap F \subseteq_\epsilon \varphi_{\lambda_0}\bigl(\mathcal{P}(A_{\lambda_0})\bigr),$$

i.e., that projections in F can be approximated by projections in the image of A_{λ_0}.

When there is no ambiguity, we may omit the index set and write $A \approx (A_\lambda, \varphi_\lambda)$.

Note that if $A = \varinjlim A_\lambda$ and if $\varphi_\lambda : A_\lambda \to A$ are the canonical $*$-homomorphisms, then $A \approx (A_\lambda, \varphi_\lambda)$.

If in Definition 2.1 we define $\iota_\lambda : A_\lambda \hookrightarrow A$ to be the corresponding inclusion map, then $A \approx (A_\lambda, \iota_\lambda)$ as in Definition 2.2. Indeed, the only thing to see is that projections may be approximated by projections, but this follows from a standard approximation argument, see, e.g., [35, Chapter 2].

It is also useful to observe that $A \approx (A_\lambda, \varphi_\lambda)$ implies that $A \approx (\varphi_\lambda(A_\lambda))$.

Proposition 2.3. *If a C^*-algebra A is locally approximated by a family of C^*-algebras each with an approximate unit of projections, then A has an approximate unit of projections.*

Proof. Suppose $A \approx (A_\lambda, \varphi_\lambda)$ and that each A_λ has an approximate unit of projections. Let $F = \{a_1, \dots, a_m\}$ be a finite subset of A and n a natural number. Then for some λ_0 and some elements $a'_1, \dots, a'_m$ of A_{λ_0}, we have $\|a_i - \varphi_{\lambda_0}(a'_i)\| < 1/3n$ for each $i = 1, \dots, m$. Choose a projection e in A_{λ_0} such that $\|ea'_i - a'_i\| < 1/3n$ for each i; such a projection exists since A_{λ_0} has an approximate unit of projections. Then for each i we have the estimate

$$\bigl\|a_i - \varphi_{\lambda_0}(e)a_i\bigr\| \leqslant \bigl\|a_i - \varphi_{\lambda_0}\bigl(a'_i\bigr)\bigr\| + \bigl\|\varphi_{\lambda_0}\bigl(a'_i\bigr) - \varphi_{\lambda_0}\bigl(ea'_i\bigr)\bigr\| + \bigl\|\varphi_{\lambda_0}\bigl(ea'_i\bigr) - \varphi_{\lambda_0}(e)a_i\bigr\| < 1/n.$$

For $\mu = (F, n)$, define $p_\mu = \varphi_{\lambda_0}(e)$, a projection in A.

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Let M be the set $\mathcal{F}(A) \times \mathbb{N}$, ordered as follows: $(F, n) \leqslant (F', n')$ if $F \subseteq F'$ and $n \leqslant n'$. Then M is a directed set, and $(p_\mu)_{\mu \in M}$ is a net of projections in A. It is easy to see that $(p_\mu)_{\mu \in M}$ is an approximate unit of projections of A. $\square$

The proof of the following proposition is adapted from that of [18, Theorem 6.2.6], where it is shown that an ideal of an AF algebra is an AF algebra (a result of Bratteli [1]).

Proposition 2.4. *Suppose A is a C^*-algebra and that $A \approx (A_\lambda, \varphi_\lambda)$. If I is an ideal of A, then I is generated (as an ideal) by the family $\{I \cap \varphi_\lambda(A_\lambda)\colon \lambda \in \Lambda\}$.*

Proof. Let $J = \langle \{I \cap \varphi_\lambda(A_\lambda)\colon \lambda \in \Lambda\} \rangle$. Then J is an ideal of A contained in I, so the map

$$\psi : A/J \to A/I, \quad \psi(a + J) = a + I$$

is a well-defined $*$-homomorphism. It is also surjective. If we show that ψ is injective, then we will have shown that $I = J$, since we would have $x + J \in \ker \psi = \{0_{A/J}\} = \{J\}$ for each $x \in I$.

We will prove that ψ is injective by proving that it is isometric. This will be accomplished by proving that ψ is isometric on the sub-C^*-algebras $(\varphi_\lambda(A_\lambda) + J)/J$ of A/J, since the union of all of these subalgebras is dense in A/J. Now, $\varphi_\lambda(A_\lambda) \cap J$ is clearly equal to $\varphi_\lambda(A_\lambda) \cap I$, and we have the sequence

$$\big(\varphi_\lambda(A_\lambda) + J\big)/J \xrightarrow{\alpha} \varphi_\lambda(A_\lambda)/\big(\varphi_\lambda(A_\lambda) \cap J\big) = \varphi_\lambda(A_\lambda)/\big(\varphi_\lambda(A_\lambda) \cap I\big) \xrightarrow{\beta} \big(\varphi_\lambda(A_\lambda) + I\big)/I \hookrightarrow A/I$$

where α and β are canonical $*$-isomorphisms. An element $a + J$ of $(\varphi_\lambda(A_\lambda) + J)/J$ is moved by this (norm-preserving) sequence to $a + I = \psi(a + J)$, so $\|a + J\| = \|a + I\| = \|\psi(a + J)\|$. $\square$

Corollary 2.5. *If a C^*-algebra A is locally approximated by a family of C^*-algebras with the ideal property, then A has the ideal property.*

One would like to conclude in the proposition above that the ideal I is locally approximated by the family of all $I \cap \varphi_\lambda(A_\lambda)$, analogous to the inductive limit case; we proved instead a weaker result. However, if we require that the ideal be generated by its projections, we are rewarded with an even better approximation.

Lemma 2.6. *Suppose A is a C^*-algebra and that $A \approx (A_\lambda, \varphi_\lambda)$. Let I be an ideal of A. The following are equivalent*:

(1) *I is generated by its projections*;
(2) *for each λ there is an ideal J_λ of A_λ, generated by its projections, such that $\varphi_\lambda(J_\lambda) \subseteq I$ and $I \approx (J_\lambda, \varphi_\lambda|_{J_\lambda})$*;
(3) *$I \approx (I_\lambda)$, where I_λ is the ideal generated by $\varphi_\lambda(\mathcal{P}(A_\lambda)) \cap I$*;
(4) *I is the ideal generated by the family $\{I_\lambda\colon \lambda \in \Lambda\}$, where the ideals I_λ are as in* (3);
(5) *$I \approx (I'_\lambda)$, where I'_λ is the ideal generated by $\mathcal{P}(\varphi_\lambda(A_\lambda)) \cap I$*;
(6) *I is the ideal generated by the family $\{I'_\lambda\colon \lambda \in \Lambda\}$, where the ideals I'_λ are as in* (5).

Proof. (1) $\Rightarrow$ (2). Let $F = \{x_1, \dots, x_m\}$ be a finite subset of I and let $\epsilon > 0$. Using the fact that I is generated by its projections, we have the existence of natural numbers n_k, of elements $a_{i,k}$ and $b_{i,k}$ of A, and of projections $p_{i,k}$ in I ($k = 1, \dots, m$, $i = 1, \dots, n_k$) satisfying

$$\left\| x_k - \sum_{i=1}^{n_k} a_{i,k} p_{i,k} b_{i,k} \right\| < \epsilon/2. \tag{1}$$

Let $0 < \delta < 1$ (δ depends on n_k, $a_{i,k}$, $b_{i,k}$, and ϵ) be a number that will be specified later. Using the fact that $A \approx (A_\lambda, \varphi_\lambda)$, we have the existence of $\lambda_0 \in \Lambda$, of elements $a'_{i,k}$ and $b'_{i,k}$ in A_{λ_0}, and of projections $p'_{i,k}$ in A_{λ_0} satisfying

$$\begin{aligned} &\big\| a_{i,k} - \varphi_{\lambda_0}\big(a'_{i,k}\big) \big\| < \delta, \\ &\big\| b_{i,k} - \varphi_{\lambda_0}\big(b'_{i,k}\big) \big\| < \delta, \\ &\big\| p_{i,k} - \varphi_{\lambda_0}\big(p'_{i,k}\big) \big\| < \delta \end{aligned} \tag{2}$$

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for all i and k. By the continuity of multiplication, if δ is small enough, then for each k we have

$$\left\| \sum_{i=1}^{n_k} a_{i,k} p_{i,k} b_{i,k} - \sum_{i=1}^{n_k} \varphi_{\lambda_0}\left(a'_{i,k} p'_{i,k} b'_{i,k}\right) \right\| < \epsilon/2. \tag{3}$$

But (2) and the fact that $\delta < 1$ imply that $p_{i,k}$ and $\varphi_{\lambda_0}(p'_{i,k})$ are Murray–von Neumann equivalent, so $\varphi_{\lambda_0}(p'_{i,k}) \in I$ (since $p_{i,k} \in I$) for all i and k. Then by (1) and (3), we have

$$\left\| x_k - \sum_{i=1}^{n_k} \varphi_{\lambda_0}\left(a'_{i,k} p'_{i,k} b'_{i,k}\right) \right\| < \epsilon,$$

where for each k we have $\sum_{i=1}^{n_k} a'_{i,k} p'_{i,k} b'_{i,k}$ is an element of the ideal $J_{\lambda_0} := \langle \mathcal{P}(A_{\lambda_0}) \cap \varphi_{\lambda_0}^{-1}(I) \rangle$ of A_{λ_0}. In conclusion, we proved that $F \subseteq_\epsilon \varphi_{\lambda_0}(J_{\lambda_0})$. Condition (2) follows.

(2) $\Rightarrow$ (3). Since $I \approx (J_\lambda, \varphi_\lambda|_{J_\lambda})$, we have $I \approx (\varphi_\lambda(J_\lambda))$. But J_λ is generated by its projections by hypothesis, so $J_\lambda \subseteq \langle \mathcal{P}(A_\lambda) \rangle$. Therefore, $\varphi_\lambda(J_\lambda)$ is included in the ideal of A generated by $\varphi_\lambda(\mathcal{P}(A_\lambda))$. Since also $\varphi_\lambda(J_\lambda) \subseteq I$ by hypothesis, we are done.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1). Immediate.

To see that (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6), note that $A \approx (\varphi_\lambda(A_\lambda), \iota_\lambda)$ where $\iota_\lambda : \varphi_\lambda(A_\lambda) \hookrightarrow A$ is the corresponding inclusion map. The equivalences now follow from (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). □

Standard C^*-algebras generalize finite direct sums of e-blocks, the building blocks (in the exact case) of the so-called exceptional *GAH* algebras, considered in [29]. We will see that the minimal tensor product of C^*-algebras that are locally approximated by standard sub-C^*-algebras behaves well with respect to the ideal property (Theorem 2.15).

Definition 2.7. A C^*-algebra A is *standard* if it is unital and satisfies the following condition: if B is a simple and unital C^*-algebra and $I \lhd A \otimes B$ is generated by its projections, then there is $J \lhd A$, generated by its projections, such that $I = J \otimes B$.

Let us recall the following definitions from [29].

Definition 2.8. A C^*-algebra is an *e-block* if it is of the form $pC(X, A)p$, where X is a compact, Hausdorff and connected space, A is a simple and unital C^*-algebra, and $p \in \mathcal{P}(C(X, A))$.

An *exceptional GAH algebra* is a countable inductive limit of finite direct sums of *exact* e-blocks.

Note that every *AH* algebra is an exceptional *GAH* algebra.

Remarks 2.9. We make the following observations:

(1) If A is standard, then $A \otimes B$ is standard for every simple and unital C^*-algebra B.
(2) If A is the finite direct sum of e-blocks, then A is standard.

Proof. (1) Since A and B are both unital, we see that $A \otimes B$ is unital. Let C be a simple and unital C^*-algebra. Note that $B \otimes C$ is simple (by a well-known result of Takesaki [36]) and unital. If $I \lhd (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ is generated by its projections, then $I = J \otimes (B \otimes C)$ for some $J \lhd A$ that is generated by its projections (since A is standard). Setting $J' = J \otimes B$, we have $I = J' \otimes C$ where $J' \lhd A \otimes B$ is generated by its projections. Then $A \otimes B$ is standard.

(2) Let $A = \bigoplus_{k=1}^n p_k C(X_k, A_k) p_k$, where each space X_k is compact, Hausdorff and connected, each C^*-algebra A_k is simple and unital and each p_k is a projection in $C(X_k, A_k)$. Since A is the direct sum of unital C^*-algebras, it is unital.

Now let B be a simple and unital C^*-algebra, and suppose that $I \lhd A \otimes B$ is an ideal that is generated by its projections. Then I is the direct sum of ideals $I_k \lhd (p_k C(X_k, A_k) p_k) \otimes B$, $k = 1, 2, \ldots, n$. Observe that

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$$\begin{aligned} p_k C(X_k, A_k) p_k \otimes B &= p_k C(X_k, A_k) p_k \otimes 1_B B 1_B \\ &= (p_k \otimes 1_B)\big(C(X_k, A_k) \otimes B\big)(p_k \otimes 1_B) \\ &= (p_k \otimes 1_B)\big(C(X_k) \otimes A_k \otimes B\big)(p_k \otimes 1_B) \\ &= (p_k \otimes 1_B) C(X_k, A_k \otimes B)(p_k \otimes 1_B). \end{aligned}$$

Letting $C_k = C(X_k, A_k \otimes B)$ and $q_k = p_k \otimes 1_B$, we see that $I_k \lhd q_k C_k q_k$, which is a hereditary sub-C^*-algebra of C_k. Hence $I_k = J_k \cap q_k C_k q_k$, for some $J_k \lhd C_k$ $(k = 1, \dots, n)$.

By a well-known result of Takesaki, $A_k \otimes B$ is simple, since both A_k and B are. We claim that every ideal of $C_k = C(X_k, A_k \otimes B)$ is of the form

$$\big\{ f \in C(X_k, A_k \otimes B):\ f|_F = 0 \big\}$$

for some closed subset F of X_k. Indeed, we have $C(X_k, A_k \otimes B) \cong C(X_k) \otimes (A_k \otimes B)$ and that $C(X_k)$ is exact (being nuclear), so by a result of Kirchberg [17, Proposition 2.3], any ideal J of this tensor product is generated by the family of rectangular ideals $\{K_\alpha \otimes (A_k \otimes B)\}$ contained in J. Therefore, $J = K \otimes (A_k \otimes B)$ for some ideal K of $C(X_k)$. The claim follows.

Let F_k be the closed subset of X_k corresponding (as above) to the ideal J_k $(k = 1, \dots, n)$. Assume that $F_k \neq \emptyset$. Then $\mathcal{P}(J_k) = 0$ (since X_k is connected), so, since $I = \bigoplus_{k=1}^n I_k$ is generated by its projections, we have

$$I_k = \big\langle \mathcal{P}(I_k) \big\rangle_{I_k} \subseteq \big\langle \mathcal{P}(I_k) \big\rangle_{C_k} \subseteq \big\langle \mathcal{P}(J_k) \big\rangle_{C_k} = 0$$

(the first inclusion above is actually an equality). Then $I = \bigoplus_{k=1}^n I_k$, where each I_k is either 0 or the whole C^*-algebra $p_k C(X_k, A_k) p_k \otimes B$. Hence I is of the form $J \otimes B$ for an ideal $J \lhd A$ where J is unital (and therefore generated by its projections). □

Remarks 2.10. Let A and B be C^*-algebras. The following observations will be used to prove the next lemma:

(1) If $A \approx (A_\lambda)_{\lambda \in \Lambda}$ and $B \approx (B_\mu)_{\mu \in M}$, then $A \otimes B \approx (A_\lambda \otimes B_\mu)$.
(2) If A_1 and A_2 are sub-C^*-algebras of A, B_1 and B_2 sub-C^*-algebras of B, and $A_1 \otimes B_1 \subseteq A_2 \otimes B_2$, then $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$.

The proof of (1) is routine, while (2) may be proved using slice maps, for example.

Lemma 2.11. *Let A and B be C^*-algebras. Suppose that A is locally approximated by a family of standard sub-C^*-algebras and that B is nonzero, simple, and has an approximate unit of projections. If $A \otimes B$ has the ideal property, then A has the ideal property.*

Proof. Let $(e_\mu)_{\mu \in M}$ be an approximate unit of projections of B. Let $B_\mu = e_\mu B e_\mu$ $(\mu \in M)$. It follows that $B \approx (B_\mu)$. Moreover, each B_μ is unital (with unit e_μ) and simple, since it is a hereditary C^*-subalgebra of the simple C^*-algebra B. Also, we have that $A \otimes B \approx (A_\lambda \otimes B_\mu)$ by Remark 2.10(1).

Let $I \lhd A$. Then $I \otimes B$ is an ideal of $A \otimes B$, which has the ideal property. Then $I \otimes B$ is generated by its projections, or equivalently (by Lemma 2.6)

$$I \otimes B \approx \Big(\big\langle \mathcal{P}\big((I \otimes B) \cap (A_\lambda \otimes B_\mu) \big) \big\rangle \Big)_{\lambda, \mu}. \tag{4}$$

Define $I_{\lambda,\mu} := \langle \mathcal{P}((I \otimes B) \cap (A_\lambda \otimes B_\mu)) \rangle$. Since A_λ is standard, and $I_{\lambda,\mu}$ is an ideal in $A_\lambda \otimes B_\mu$ we have that

$$I_{\lambda,\mu} = J_{\lambda,\mu} \otimes B_\mu \tag{5}$$

for some ideal $J_{\lambda,\mu}$ of A_λ that is generated by its projections. Also, we have the inclusion $J_{\lambda,\mu} \subseteq I$, using Remark 2.10(2) and the following inclusions:

$$J_{\lambda,\mu} \otimes B_\mu = I_{\lambda,\mu} = \big\langle \mathcal{P}\big((I \otimes B) \cap (A_\lambda \otimes B_\mu) \big) \big\rangle \subseteq \big\langle \mathcal{P}(I \otimes B) \big\rangle \subseteq I \otimes B.$$

Hence $J_{\lambda,\mu} \subseteq I \cap A_\lambda$.

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We deduce from (4), (5), and the inclusion $J_{\lambda,\mu} \subseteq I \cap A_\lambda$, that

$$I \otimes B \approx \{J \otimes B_\mu\colon\ J \lhd A_\lambda \cap I \text{ with } J = \langle \mathcal{P}(J)\rangle,\ \lambda \in \Lambda,\ \mu \in M\}.$$

This implies in turn that $I \otimes B \subseteq J_0 \otimes B$, where

$$J_0 = \langle \{J \lhd I \cap A_\lambda\colon\ J = \langle \mathcal{P}(J)\rangle \text{ and } \lambda \in \Lambda\}\rangle.$$

We conclude that

$$I \subseteq J_0 \subseteq \langle \mathcal{P}(I)\rangle \subseteq I. \qquad \square$$

Remark 2.12. The conclusion of the above lemma still holds if we substitute the hypothesis "B is simple and has an approximate unit of projections" with "B is locally approximated by a family of unital and simple sub-C^*-algebras." The statement as given in the lemma, however, has the advantage of being more "natural." Also note that the fact that standard C^*-algebras are unital was not used. This fact will be used soon, however.

Definition 2.13. A C^*-algebra A is an *LS algebra* ("locally standard") if $A \approx (A_\lambda)$ where each A_λ is a standard sub-C^*-algebra of A.

Note that any inductive limit of standard C^*-algebras with injective connecting $*$-homomorphisms is an *LS* algebra.

Remark 2.14. Since standard C^*-algebras are unital, it follows from Proposition 2.3 that every *LS* algebra has an approximate unit of projections.

Now we state the main theorem of this section.

Theorem 2.15. *Let A and B be nonzero* LS *algebras. Consider the conditions*

(1) *A and B have the ideal property*;
(2) *$A \otimes B$ has the ideal property.*

Then

(a) (2) $\Rightarrow$ (1);
(b) *if at least one of A and B is exact, then* (1) $\Rightarrow$ (2).

The proof (given below) uses Lemma 2.11 and two general results about the ideal property. The first is based on a result of Kirchberg in [17].

Proposition 2.16. *(Cf. [31, Corollary 1.3].) Let A and B be C^*-algebras with the ideal property and suppose that at least one of A and B is exact. Then $A \otimes B$ has the ideal property.*

The second result is a simple observation (found, e.g., in [20]).

Proposition 2.17. *If A is a C^*-algebra with the ideal property and $I \lhd A$, then both I and A/I have the ideal property.*

Proof of Theorem 2.15. (a) We will apply Lemma 2.11. Let I be a maximal ideal of B. Then B/I is simple, nonzero, and has an approximate unit of projections (since B has one). Then $A \otimes (B/I)$ is a quotient of $A \otimes B$, which has the ideal property, so it has the ideal property itself (by Proposition 2.17). Applying Lemma 2.11 to the tensor product $A \otimes (B/I)$, we see that A has the ideal property. That B has the ideal property follows by symmetry.

(b) This implication follows directly from Proposition 2.16. $\square$

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Remark 2.18. Theorem 2.15 extends [29, Corollary 2.17], which says that if A and B are nonzero *AH* algebras, then $A \otimes B$ has the ideal property if and only if both A and B have the ideal property. Recall that an *AH algebra* is the inductive limit of a sequence of C^*-algebras A_n, where each A_n is a finite direct sum of C^*-algebras of the form $pC(X, M_m)p$, where X is a finite simplicial complex and p is a projection in $C(X, M_m)$. Using a recent result of Elliott, Gong, and Li [10], we may suppose that the connecting morphisms in the inductive sequence of A_n's are injective. Then A is locally approximated by the family (A_n) of sub-C^*-algebras of A. It follows from Remark 2.9(2) that each A_n is standard, and this shows that Theorem 2.15 is a generalization of [29, Corollary 2.17], as asserted.

Also, Theorem 2.15 generalizes the main result in [29] (i.e., [29, Theorem 2.11]) in the case when the sequences of finite direct sums of e-blocks defining the nonzero exceptional *GAH* algebras A and B in [29, Theorem 2.11] have injective connecting $*$-homomorphisms.

3. Quasidiagonal extensions

Definition 3.1. Let A be a C^*-algebra and $I \lhd A$. The canonical extension

$$0 \to I \to A \to A/I \to 0 \tag{6}$$

is *quasidiagonal* if there is an approximate unit of projections $(p_\lambda)_{\lambda\in\Lambda}$ of I that is *quasicentral* in A, that is, for every $a \in A$ we have

$$\lim_{\lambda\in\Lambda} \|p_\lambda a - a p_\lambda\| = 0.$$

We may also say that A *is quasidiagonal relative to* I to mean that the extension (6) is quasidiagonal.

Our interest in quasidiagonal extensions lies in their applications to the study of the ideal property. They have proven a useful tool in this respect, see, e.g., [20,24,26,28]. We will see, for example, that if (6) is quasidiagonal and both I and A/I have the ideal property, then so does A (Theorem 3.7). We start with the following useful proposition.

Proposition 3.2. *If* (6) *is quasidiagonal, then any projection in A/I lifts to a projection in A.*

We will need a few preparatory results to prove this proposition. Our line of reasoning follows closely the one in [20], where a version of Proposition 3.2 is proved for quasidiagonal extensions where the quasicentral approximate unit is assumed to be countable. The assumption that the approximate unit in Definition 3.1 be countable is commonly made, and the following claims, originally stated with this assumption, follow mostly with only slight modifications to their original proofs.

The proof of the following lemma is the same as the analogous version for quasidiagonal extensions with countable approximate units. We present it for the convenience of the reader.

Lemma 3.3. *(Cf. [20, Lemma 3.7].) Suppose the extension* (6) *is quasidiagonal. If p is a projection in A and q is its image in A/I, then the extension*

$$0 \to pIp \to pAp \to q(A/I)q \to 0 \tag{7}$$

is quasidiagonal.

Proof. Let $(p_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of projections of I that is quasicentral in A. We first show that $(pp_\lambda p)$ is an approximate unit of pIp. Indeed, if $x \in I$, then

$$pp_\lambda p \cdot pxp - pxp = p \cdot p_\lambda pxp - pxp \to p \cdot pxp - pxp = 0,$$

since $pxp \in I$. Moreover, $(pp_\lambda p)_{\lambda\in\Lambda}$ is quasicentral in pAp. For if $a \in A$, then

$$\begin{aligned} pp_\lambda p \cdot pap - pap \cdot pp_\lambda p &= pp_\lambda p \cdot pap - pap \cdot p_\lambda + pap \cdot p_\lambda - p_\lambda \cdot pap + p_\lambda \cdot pap - pap \cdot pp_\lambda p \\ &= p(p_\lambda \cdot pap - pap \cdot p_\lambda) + pap \cdot p_\lambda - p_\lambda \cdot pap + (p_\lambda \cdot pap - pap \cdot p_\lambda)p \\ &\to 0 \end{aligned}$$

since $(p_\lambda)_{\lambda\in\Lambda}$ is quasicentral in A.

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Again using the fact that (p_λ) is quasicentral in A, we have

$$(pp_\lambda p)^2 - pp_\lambda p = pp_\lambda pp_\lambda p - pp_\lambda^2 p = p(p_\lambda p - pp_\lambda)p_\lambda p \to 0.$$

Then, there is $\lambda_0 \in \Lambda$ such that for every $\lambda \in \Lambda$, $\lambda \geqslant \lambda_0$, we have $\|(pp_\lambda p)^2 - pp_\lambda p\| < 1/4$, so there is a projection $r_\lambda \in C^*(pp_\lambda p)$ such that

$$\|r_\lambda - pp_\lambda p\| \leqslant 2\big\|(pp_\lambda p)^2 - pp_\lambda p\big\|$$

(see, e.g., [35, Lemma 6.3.1]). Since $\|(pp_\lambda p)^2 - pp_\lambda p\| \to 0$ it follows that $(r_\lambda)_{\lambda\in\Lambda,\lambda\geqslant\lambda_0}$ is an approximate unit of projections of pIp that is quasicentral in pAp. Then the reduced extension (7) is quasidiagonal, as was to be proved. □

The following lemma is similar to Lemma 3.13 of Brown and Pedersen's paper [3] (see also Zhang [39]), which assumed that the ideal I had real rank zero. It was originally observed in [20, Lemma 3.8] that the conclusion of said lemma still holds if one assumes instead the quasidiagonality of the canonical extension but involving countable approximate units.

Lemma 3.4. *(Cf. [20, Lemma 3.8].) If the extension* (6) *is quasidiagonal and B is a hereditary sub-C^*-algebra of A, then every projection in $B/B \cap I \cong (B + I)/I$ that lifts to a projection in A can be lifted to a projection in B.*

Proof. Use Lemma 3.3 and proceed as in the proof of [20, Lemma 3.8]. □

We will need one more theorem, a slight modification of a result due to L.G. Brown and Dădărlat.

Theorem 3.5. *(Cf. [2, Theorem 8].) Suppose the extension* (6) *is quasidiagonal. Then the index maps $\delta_i : K_i(A/I) \to K_{i+1}(I)$ are zero* $(i = 0, 1)$. *In particular, if the extension* (6) *is quasidiagonal, then the induced homomorphism from $K_0(A)$ to $K_0(A/I)$ is surjective.*

Finally, we prove Proposition 3.2.

Proof of Proposition 3.2. Let q be a projection in A/I. Let us suppose for a moment that $q \oplus 1_s \oplus 0_t$ lifts to a projection in $M_{1+s+t}(\tilde{A})$ for some $s, t \in \mathbb{N}$. We may apply Lemma 3.4 to see that the projection $q \oplus 1_s$ lifts to a projection p in $M_{1+s}(\tilde{A})$, since $q \oplus 1_s \oplus 0_t$ is in the hereditary sub-C^*-algebra $M_{1+s}(\tilde{A}/I) \oplus 0_t$ of $M_{1+s+t}(\tilde{A}/I)$. Then $(1 - q) \oplus 0_s = 1_{1+s} - q \oplus 1_s$ lifts to $1_{1+s} - p$, and again an application of Lemma 3.4 shows that the projection $1 - q$ lifts to a projection $p' \in \tilde{A}$. But then $q = 1 - (1 - q)$, a projection in A/I, lifts to $1 - p' \in \tilde{A}$. Since A is a hereditary sub-C^*-algebra of $\tilde{A}$, one more application of Lemma 3.4 gives that q lifts to a projection in A.

That $q \oplus 1_s \oplus 0_t$ lifts to a projection in $M_{1+s+t}(\tilde{A})$ for some $s, t \in \mathbb{N}$ follows from Theorem 3.5 by a standard argument, for example as in the proof of [6, Lemma 9.6]. □

For the proof of our next result, we will use the following remark of Dădărlat (see [24, Lemma 2.10]).

Lemma 3.6. *Suppose $0 \to I \to A \to B \to 0$ is an exact sequence of C^*-algebras such that every projection in B lifts to a projection in A. If both I and B are generated as ideals* (*of themselves*) *by their projections, then A is generated as an ideal* (*of itself*) *by its projections.*

Theorem 3.7. *Suppose the extension* (6) *is quasidiagonal. If I and A/I have the ideal property, then so does A.*

Proof. Let J be an ideal of A. Then J is a hereditary sub-C^*-algebra of A, so every projection in $J/(I \cap J)$ that lifts to a projection in A also lifts to a projection in J (by Lemma 3.4). But every projection in A/I lifts to a projection in A by Proposition 3.2.

We have the exact sequence

$$0 \to I \cap J \to J \to J/(I \cap J) \to 0$$

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where both $I \cap J$ ($\lhd I$) and $J/(I \cap J) \cong (J+I)/I$ ($\lhd A/I$) are generated, as ideals of themselves, by their projections (in fact, they have the ideal property). Moreover, every projection in $(J+I)/J$ lifts to a projection in J, by the argument in the preceding paragraph. By Lemma 3.6, J is generated as an ideal of itself—and hence as an ideal of A—by its projections. It follows that A has the ideal property. □

4. *WLB* algebras

Definition 4.1. A C^*-algebra A is a *WB algebra* if for any ideal I of A that is generated by its projections one has that A is quasidiagonal relative to I.

Remark 4.2. Note that if A is a *WB* algebra and I is an ideal of A that is generated by its projections, then both I and A/I are *WB* algebras. That I is a *WB* algebra is easy to see. But using Proposition 3.2 we have every projection in A/I lifts to a projection in A, and a straightforward argument shows that A/I is a *WB* algebra as well.

Definition 4.3. A C^*-algebra A is a *WLB algebra* ("weakly locally basic") if it has an approximate unit of projections and if it is locally approximated by *WB* algebras (in the sense of Definition 2.2).

Recall that a unital C^*-algebra A is *basic* if any ideal of A that is generated by its projections is a direct summand of A [28], or, equivalently, is unital. A C^*-algebra is an *LB algebra* ("locally basic") if it is locally approximated (in the sense of Definition 2.2) by a family of basic C^*-algebras. This class, and several important subclasses of it, were introduced and studied in [28]. The class of *LB* algebras contains the *GAH*—and thus the *AH*—algebras. Since basic C^*-algebras are clearly *WB* algebras, every *LB* algebra is a *WLB* algebra.

The class of *WLB* algebras is closed under local approximations. Indeed, Proposition 2.3 and a straightforward approximation argument prove this fact. The class is not closed under extensions (Proposition 4.7), but quotients do remain in the class under certain conditions (Remark 4.6).

The theorem below adds to the list of equivalences found in Lemma 2.6, for the class of *WLB* algebras, and it also generalizes results in [21,24,28].

Theorem 4.4. *Suppose A is a* WLB *algebra and I is an ideal of A. The following are equivalent*:

(1) *I is generated by its projections*;
(2) *A is quasidiagonal relative to I*;
(3) *I has an approximate unit of projections*;
(4) *I is a* WLB *algebra*;
(5) *suppose $A \approx (A_\lambda, \varphi_\lambda)$ where each A_λ is a WB algebra. Then for each λ there is an ideal I_λ of A_λ such that A_λ is quasidiagonal relative to I_λ, $\varphi_\lambda(I_\lambda) \subseteq I$ and $I \approx (I_\lambda, \varphi_\lambda|_{I_\lambda})$.*

Proof. (1) ⇒ (2). By hypothesis we have $A \approx (A_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ where each A_λ is a *WB* algebra. We are to construct an approximate unit of projections of I that is quasicentral in A. The approximate unit we will construct will be indexed by $\mathcal{F}(A) \times \mathcal{F}(I) \times \mathbb{N}$ (where for a any set E, $\mathcal{F}(E)$ is the set of all finite subsets of E).

Let $F = \{x_1, \ldots, x_m\}$ be a finite subset of I, G be a finite subset of A, and n a natural number. Since I is generated by its projections, for each $i = 1, \ldots, m$ we have

$$\left\| x_i - \sum_{k=1}^{m_i} a_{i,k} p_{i,k} b_{i,k} \right\| < \frac{1}{3n} \tag{8}$$

for some positive integers m_i, some $a_{i,k}, b_{i,k} \in A$, and some $p_{i,k} \in \mathcal{P}(I)$, $k = 1, \ldots, m_i$.

Let $\epsilon > 0$ be such that if $a'_{i,k}, b'_{i,k}, p'_{i,k} \in A$ satisfy

$$\begin{aligned} &\|a_{i,k} - a'_{i,k}\| < \epsilon, \\ &\|b_{i,k} - b'_{i,k}\| < \epsilon, \\ &\|p_{i,k} - p'_{i,k}\| < \epsilon \end{aligned} \tag{9}$$

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for each i and k, then

$$\left\| \sum_{k=1}^{m_i} a_{i,k} p_{i,k} b_{i,k} - \sum_{k=1}^{m_i} a'_{i,k} p'_{i,k} b'_{i,k} \right\| < \frac{1}{9n} \tag{10}$$

for each i. We also require that $\epsilon < 1/3n$.

Since $A \approx (A_\lambda, \varphi_\lambda)$ and

$$\{a_{i,k}, p_{i,k}, b_{i,k}\colon\ i = 1, \dots, m,\ k = 1, \dots, m_i\} \cup G$$

is a finite subset of A, there is λ_0 such that this set is ϵ-included in $\varphi_{\lambda_0}(A_{\lambda_0})$. Then there are $a''_{i,k}, b''_{i,k} \in A_{\lambda_0}$ and $p''_{i,k} \in \mathcal{P}(A_{\lambda_0})$ such that (9) is satisfied with $a'_{i,k} = \varphi_{\lambda_0}(a''_{i,k})$, $b'_{i,k} = \varphi_{\lambda_0}(b''_{i,k})$ and $p'_{i,k} = \varphi_{\lambda_0}(p''_{i,k})$. Note that the projection $p'_{i,k}$ is Murray–von Neumann equivalent to $p_{i,k}$, since $\epsilon < 1$. Since each $p_{i,k}$ belongs to the ideal I, so does each $p'_{i,k}$. That is, every $p''_{i,k}$ is a projection in $\varphi_{\lambda_0}^{-1}(I)$.

Each $\sum_{k=1}^{m_i} a''_{i,k} p''_{i,k} b''_{i,k}$ belongs to the ideal I_{λ_0} of A_{λ_0} that is generated by $\mathcal{P}(\varphi_{\lambda_0}^{-1}(I))$. The definition of *WB* algebras implies that the extension

$$0 \to I_{\lambda_0} \to A_{\lambda_0} \to A_{\lambda_0}/I_{\lambda_0} \to 0$$

is quasidiagonal. Using this quasidiagonality and taking into account that $G \subseteq_{1/3n} \varphi_{\lambda_0}(A_{\lambda_0})$, we have the existence of a projection $e \in I_{\lambda_0}$ such that, with $e' := \varphi_{\lambda_0}(e)$,

$$\|e'g - ge'\| < \frac{1}{n}$$

for all $g \in G$, and

$$\left\| e \cdot \sum_{k=1}^{m_i} a''_{i,k} p''_{i,k} b''_{i,k} - \sum_{k=1}^{m_i} a''_{i,k} p''_{i,k} b''_{i,k} \right\| < \frac{1}{9n}$$

for each i. This, (8) and (10) imply that

$$\|e'g - ge'\| < 1/n \quad \text{for all } g \in G, \quad \text{and} \quad \|e'x - x\| < 1/n \quad \text{for all } x \in F. \tag{11}$$

For $\mu = (G, F, n) \in \mathcal{F}(A) \times \mathcal{F}(I) \times \mathbb{N}$, let $e_\mu := e'$ as defined above. Note that e_μ is a projection in I.

The set $M := \mathcal{F}(A) \times \mathcal{F}(I) \times \mathbb{N}$ is a directed set if we define $(G_1, F_1, n_1) \leqslant (G_2, F_2, n_2)$ if $G_1 \subseteq G_2$, $F_1 \subseteq F_2$, and $n_1 \leqslant n_2$. Then $(e_\mu)_{\mu \in M}$ is a net of projections in I. Moreover, if $x \in I$, $a \in A$, and $n \in \mathbb{N}$ are given, then for all $\mu \geqslant (\{a\}, \{x\}, n)$, we have by (11) that

$$\|e_\mu x - x\| < 1/n$$

and

$$\|e_\mu a - a e_\mu\| < 1/n.$$

Therefore, $(e_\mu)_{\mu \in M}$ is an approximate unit of projections of I that is quasicentral in A. We conclude that A is quasidiagonal relative to I.

(2) ⇒ (3). Obvious.

(3) ⇒ (4). Since any ideal of I is an ideal of A, if $J \lhd I$ is generated by its projections, then A is quasidiagonal relative to J. Evidently this implies that I is quasidiagonal relative to J. Then I is a *WB* algebra and has an approximate unit of projections, so it is a *WLB* algebra.

The implication (4) ⇒ (1) follows immediately from the fact that *WLB* algebras have an approximate unit of projections.

That (5) ⇔ (1) follows from Lemma 2.6 and the definition of a *WB* algebra. □

Corollary 4.5. *A C^*-algebra A is a* WLB *algebra if and only if it is a* WB *algebra with an approximate unit of projections.*

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Remark 4.6. Corollary 4.5 combined with Remark 4.2 shows that if I is an ideal of a *WLB* algebra A that is generated by its projections, then A/I is a *WLB* algebra.

This next proposition shows that the class of *WLB* algebras is not closed under taking extensions (even in the separable case). In fact, it shows that an extension of an *AH* algebra by an *AH* algebra need not be a *WLB* algebra.

Proposition 4.7. *Let H be an infinite-dimensional, separable Hilbert space. Then the C^*-algebras $B(H)$ and $\mathcal{T}$ (the Toeplitz algebra) are not* WLB *algebras.*

Proof. Since $\mathcal{K}$ is simple and has an approximate unit of projections, it is a *WLB* algebra. That $\mathcal{Q}(H)$ (the Calkin algebra $B(H)/\mathcal{K}$) is a *WLB* algebra follows from the fact that it is simple and unital. However, the index map $\delta_1 : K_1(\mathcal{Q}(H)) \to K_0(\mathcal{K})$ is not zero (see, e.g., [35, Example 9.4.3]), so the extension

$$0 \to \mathcal{K} \to B(H) \to \mathcal{Q}(H) \to 0$$

cannot be quasidiagonal by Theorem 3.5. But $\mathcal{K}$ is an ideal of $B(H)$ that is generated by its projections, so $B(H)$ cannot be a *WLB* algebra.

None of the proper ideals of $C(\mathbb{T})$ contain any nonzero projections, so—being unital—$C(\mathbb{T})$ is clearly a *WLB* algebra. The index map $\delta_1 : K_1(C(\mathbb{T})) \to K_0(\mathcal{K})$ is not zero (see, e.g., [35, Example 9.4.4]), so the same argument as in the preceding paragraph applies. □

Definition 4.8. A C^*-algebra A has the *weak projection property* if every ideal of A has an approximate unit of projections (not necessarily increasing).

This definition extends the notion of the projection property, introduced in [26], which requires that every ideal have an increasing approximate unit of projections. From Theorem 4.4 we see that the weak projection property and the ideal property coincide for *WLB* algebras. The projection property is considered in some detail in [26], where it is shown that the projection property and the ideal property do not coincide in general (even in the separable case). In particular, the ideal property and the weak projection property do not coincide in general (even in the separable case).

Proposition 4.9. *Suppose A is a* WLB *algebra and that p is a projection of A. If A has the ideal property, then the C^*-algebra pAp is quasidiagonal relative to any of its ideals. In particular, pAp has the weak projection property, and therefore pAp has the ideal property.*

Proof. Let I be an ideal of pAp. Since pAp is a hereditary sub-C^*-algebra of A, $I = J \cap pAp$ for some ideal J of A, which has the ideal property. Then J is generated by its projections, so A is quasidiagonal relative to J, since A is a *WLB* algebra. By Lemma 3.3, pAp is quasidiagonal relative to pJp. We claim that $pJp = I$, which would end the proof. The inclusion $pJp \subseteq J \cap pAp = I$ is obvious. But if x is an element of J of the form pap for some $a \in A$, then $x = pap = p(pap)p = pxp \in pJp$. Thus $J \cap pAp \subseteq pJp$. □

It is known that the ideal property and the real rank zero are not preserved under extensions (see [3,22]). However, we have:

Theorem 4.10. *Let A be a* WLB *algebra, I an ideal of A, and B a sub-C^*-algebra of A that contains I. The following are equivalent*:

(1) *B has the ideal property* (*respectively B has real rank zero*);
(2) *I and B/I have the ideal property* (*respectively I and B/I have real rank zero*).

Proof. The implication (1) ⇒ (2) follows directly from Proposition 2.17 (respectively [3, Theorem 3.6], see also [39]).

(2) ⇒ (1). First we dispatch the real rank zero case. If I has real rank zero, then it has an approximate unit of projections (see [3, Theorem 2.6]). Therefore, A is quasidiagonal relative to I, A being a *WLB* algebra. Hence

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B (which contains I) is quasidiagonal relative to I, whence any projection in B/I lifts to a projection in B, by Proposition 3.2. Then [3, Theorem 3.6] shows that B has real rank zero.

Now we deal with the ideal property case. Since by Theorem 4.4 A is quasidiagonal relative to I and $I \subseteq B$, it follows that B is quasidiagonal relative to I. Now, since I and B/I have the ideal property, Theorem 3.7 implies that B also has the ideal property. □

Remark 4.11. The "ideal property part" of Theorem 4.10 generalizes [20, Theorem 3.1], [24, Theorem 2.6] and [28, Theorem 2.11].

Let us recall a few definitions and provide some context before stating our next theorem. Let A be a C^*-algebra, H a subgroup of $K_0(A)$, and set $H^+ = H \cap K_0(A)^+$. We say that H is an *ideal* of $K_0(A)$ if $H = H^+ - H^+$ and H^+ is hereditary, that is,

$$\text{if} \quad 0 \leqslant g \leqslant h \quad \text{for some } g \in K_0(A) \text{ and } h \in H^+, \quad \text{then } g \in H.$$

A C^*-algebra A is *stably finite* if there are no projections $p, q \in M_\infty(A)$ such that $p \oplus q \sim q$ with $p \neq 0$. Recall that $(K_0(A), K_0(A)^+)$ is an ordered abelian group if A is stably finite and unital (see, e.g., [35, Chapter 5]).

The ideal structure of some interesting classes of C^*-algebras has been related to the ideal structure of the corresponding (ordered) K_0 groups, or more generally, the ideal structure of the local semigroup $D(-)$ (see below). For example, one has the nice result that if A is a stably finite C^*-algebra with the ideal property and $A \otimes \mathcal{K}$ has an increasing approximate unit of projections, the ideal lattice of A is order isomorphic to the ideal lattice of $K_0(A)$ (cf. [34, Proposition I.5.3]). Also, Goodearl [16] has shown that the ideal lattice of the K_0 group of a C^*-algebra of real rank zero is order isomorphic to the lattice of stably cofinite ideals (defined below) of the C^*-algebra. In proving Theorem 4.13 below, we borrow from some of the methods used to prove this last result (see [16, Theorem 10.9]). Other results of the above type appeared in [22,24,31].

Let A be a C^*-algebra. We will use the standard notation $D(A)$ for the abelian local semigroup of Murray–von Neumann equivalence classes of projections in A. (The addition of two elements $[p]$ and $[q]$ in $D(A)$ is defined whenever there are representatives p' and q' of $[p]$ and $[q]$, respectively, such that $p' \perp q'$.) There is a natural order on $D(A)$ given by defining $[p] \leqslant [q]$ if p is Murray–von Neumann equivalent to a subprojection of q. By an *ideal* in $D(A)$ we mean a nonempty hereditary subset of $D(A)$ that is closed under addition, where defined.

The ideal structure of *AH* algebras (in particular *AH* algebras with the ideal property) was studied in [22]. One of the main results obtained there is the following:

Theorem 4.12. *(See [22, Theorem 4.1].) Let A be an* AH *algebra. Then there is a lattice isomorphism*

$$\{I \lhd A\colon\ I \text{ is generated by its projections}\} \xrightarrow{\sim} \{J\colon\ J \text{ is an ideal of } D(A \otimes \mathcal{K})\}.$$

If in addition the projections in $M_\infty(A)$ satisfy the Riesz decomposition property, then there are lattice isomorphisms

$$\begin{aligned}\{I \lhd A\colon\ I \text{ is generated by its projections}\} &\cong \{J\colon\ J \text{ is an ideal of } D(A)\}\\ &\cong \{L\colon\ L \text{ is an ideal of } K_0(A)\}.\end{aligned}$$

(The projections in $M_\infty(A)$ satisfy the *Riesz decomposition property* if whenever p, q_1 and q_2 are projections in $M_\infty(A)$ such that $p \precsim q_1 \oplus q_2$, then $p = p_1 \oplus p_2$ for some projections $p_i \in M_\infty(A)$ such that $p_i \precsim q_i$ for $i = 1, 2$.) Zhang [38] had previously proved a result along these lines for C^*-algebras of real rank zero. Specifically, if A is a C^*-algebra of real rank zero, then the ideal lattice of $\mathcal{M}(A)$ is isomorphic to the ideal lattice of $D(\mathcal{M}(A))$ (where $\mathcal{M}(A)$ is the multiplier algebra of A). This had already been obtained by Elliott [7] for *AF* algebras.

Our next theorem generalizes [22, Lemma 4.10], one of the key ingredients in the proof of [22, Theorem 4.1]. We need one last definition: an ideal I in a C^*-algebra A is *stably cofinite* if the C^*-algebra A/I is stably finite.

Theorem 4.13. *Let A be a separable* WLB *algebra and suppose that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. Then there is an order isomorphism*

$$\mathcal{I} := \{I \lhd A\colon\ I \text{ is stably cofinite and generated by its projections}\} \cong \{H\colon\ H \text{ is an ideal of } K_0(A)\} =: \mathcal{H}$$

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More precisely, there are order isomorphisms sending each element I of $\mathcal{I}$ to the kernel of $K_0(\pi)$, where $\pi : A \to A/I$ is the canonical quotient map, and sending each element H of $\mathcal{H}$ to the ideal generated by those projections $p \in A$ such that $[p] \in H$.

It follows from [22, Remark 4.3] *that $\mathcal{I}$ and $\mathcal{H}$ are lattices and the map above is a lattice isomorphism.*

The above theorem was proved in [28] for so-called special *LB* algebras. For the proof, we will use the following lemma.

Lemma 4.14. *(Cf. [28, Lemma 3.15].) Let A be a* WLB *algebra and let I be an ideal of A that is generated by its projections. Then, for each $n \in \mathbb{N}$, all the projections in $M_n(A/I)$ lift to projections in $M_n(A)$.*

Proof. Let n be a natural number. Since I is generated by its projections, A is quasidiagonal relative to I. That is, I has an approximate unit of projections $(p_\lambda)_{\lambda\in\Lambda}$ that is quasicentral in A. Then $(1_n \otimes p_\lambda)_{\lambda\in\Lambda}$ is an approximate unit of projections of $M_n \otimes I = M_n(I)$ that is quasicentral in $M_n(A)$. Then $M_n(A)$ is quasidiagonal relative to $M_n(I)$. By Proposition 3.2, every projection in $M_n(A/I)$ lifts to a projection in $M_n(A)$. □

Proof of Theorem 4.13. The proof is similar to that of [28, Theorem 3.14]. We will sketch the proof for the convenience of the reader.

Let $I \in \mathcal{I}$. Consider the canonical extension

$$0 \to I \xrightarrow{i} A \xrightarrow{\pi} A/I \to 0.$$

Let $H \subseteq K_0(A)$ be the kernel of $K_0(\pi)$. Since the functor K_0 is half exact,

$$H = K_0(i)\big(K_0(I)\big) = \ker K_0(\pi).$$

One then proves the following claim using the fact that A is separable (so I has a countable, increasing approximate unit of projections) and that I is stable cofinite.

Claim. *$H \in \mathcal{H}$ and I is the ideal generated by the projections $p \in A$ satisfying $[p] \in H$.*

Next, let $H \in \mathcal{H}$ and let I be the ideal generated by the projections $p \in A$ satisfying $[p] \in H$. One proves the following claim using the fact that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property, Lemma 4.14, and [16, 10.7].

Claim. *$I \in \mathcal{I}$ and $H = \ker K_0(\pi)$, where $\pi : A \to A/I$ is the canonical quotient map.*

To finish the proof of the theorem, we observe that the two maps defined by the two claims above are order-preserving bijections. □

5. *SLB* algebras

Definition 5.1. A C^*-algebra A is an *SB algebra* if for every natural number n and every ideal I of $M_n(A)$ that is generated by its projections, $I = M_n(J)$ where J is an ideal of A that is generated by its projections.

Note that every stably basic C^*-algebra (see [28]) is an *SB* algebra. In particular, the C^*-algebra defined by a continuous field of simple and unital C^*-algebras over a compact and Hausdorff space with finitely many connected components is an *SB* algebra. Also, every standard C^*-algebra is an *SB* algebra.

Obviously $M_n(A)$ is an *SB* algebra if A is.

Definition 5.2. A C^*-algebra A is an *SLB algebra* ("stably locally basic") if for any natural number n we have

$$M_n(A) \approx \big(M_n(A_\lambda), \mathrm{id}_{M_n} \otimes \varphi_\lambda\big)_{\lambda\in\Lambda},$$

where each A_λ is an *SB* algebra and $\varphi_\lambda : A_\lambda \to A$ is a $*$-homomorphism.

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Note that every special *LB* algebra (see [28]) is an *SLB* algebra.

Observe that if $A = \varinjlim (A_\lambda, \varphi_{\lambda\mu})$ and each A_λ is an *SB* algebra, then

$$\begin{aligned} M_n(A) &= M_n \otimes A = M_n \otimes \varinjlim (A_\lambda, \varphi_{\lambda\mu}) \\ &= \varinjlim (M_n \otimes A_\lambda, \mathrm{id}_{M_n} \otimes \varphi_{\lambda\mu}) = \varinjlim \big(M_n(A_\lambda), \mathrm{id}_{M_n} \otimes \varphi_{\lambda\mu}\big), \end{aligned}$$

so A is an *SLB* algebra. The next proposition gives another example of an *SLB* algebra; it follows immediately from Proposition 6.6.

Proposition 5.3. *Let A be a C^*-algebra with an approximate unit of projections. If for some $n \in \mathbb{N}$ we have that $M_n(A)$ is quasidiagonal relative to every one of its ideals that is generated by its projections, then we have that $M_n(J) \lhd M_n(A)$ is generated by its projections if and only if $J \lhd A$ is.*

In other words, if $M_n(A)$ is a WLB *algebra for each n, then A is an* SLB *algebra.*

The result below was proved in [28, Lemma 3.10] for the so-called special *LB* algebras.

Lemma 5.4. *Let A be an* SLB *algebra, let $\mathcal{I}$ be the ideal lattice of A, and let $\mathcal{J}$ be the ideal lattice of $A \otimes \mathcal{K}$. Then the map*

$$\Theta : \mathcal{I} \to \mathcal{J}, \quad \Theta(I) = I \otimes \mathcal{K}$$

is a lattice isomorphism taking the set $\mathcal{I}_p$ of all ideals of A that are generated by their projections to the set $\mathcal{J}_p$ of all ideals of $A \otimes \mathcal{K}$ that are generated by their projections. It follows that these last two sets are isomorphic sublattices of $\mathcal{I}$ and $\mathcal{J}$, respectively. In particular,

A has the ideal property $\quad\Leftrightarrow\quad$ $A \otimes \mathcal{K}$ has the ideal property.

Proof. The proof is similar to that of [28, Lemma 3.10]. There are some differences, however, and therefore we give the full argument.

The correspondence between $\mathcal{I}$ and $\mathcal{J}$ is well known (e.g., since $\mathcal{K}$ is simple and exact). Also, if we prove that (the restriction of) Θ is an order isomorphism of $\mathcal{I}_p$ and $\mathcal{J}_p$, these are lattices by [22, Remark 4.3] (see also the end of the proof of [22, Lemma 4.5]). Thus we are left to prove that if J is an ideal of $A \otimes \mathcal{K}$ that is generated by projections, then I is generated by its projections, where I is that ideal of A such that $J = I \otimes \mathcal{K}$.

For each natural number n, let $\psi_n : M_n(A) \to M_{n+1}(A)$ be the $*$-homomorphism sending $a \in M_n(A)$ to $a \oplus 0 \in M_{n+1}(A)$. Then $A \otimes \mathcal{K} = \varinjlim (M_k(A), \psi_k)$. If $n \leqslant m$ are natural numbers, we will abuse notation and identify $M_n(A)$ with its canonical image $\psi_{n,m}(M_n(A))$ in $M_m(A)$ and also with its canonical image $\psi_{n,\infty}(M_n(A))$ in $A \otimes \mathcal{K}$.

Let $\epsilon > 0$ and let $x \in I$. Since

$$I \subseteq I \otimes \mathcal{K} = \varinjlim \big(M_n(I), \psi_n|_{M_n(I)}\big)$$

and $I \otimes \mathcal{K} = J$ is generated by its projections, there are elements a_i and b_i of $M_m(A)$ and projections p_i of $M_m(I)$, for some natural number m, (where $i = 1, \dots, l$ for some $l \in \mathbb{N}$) such that

$$\left\| x - \sum_{i=1}^{l} a_i p_i b_i \right\| < \epsilon/2 \tag{12}$$

(consult, e.g., [37] to see that we can take the elements p_i to be projections). Since A is an *SLB* algebra, given any $0 < \delta < 1$ (to be specified later) there are an *SB* algebra B, a $*$-homomorphism $\varphi : B \to A$, elements $a'_i, b'_i \in M_m(B)$, and projections p'_i of $M_m(B)$ such that, with $\Phi = \varphi \otimes \mathrm{id}_{M_n}$,

$$\big\|\Phi\big(p'_i\big) - p_i\big\| < \delta, \qquad \big\|\Phi\big(a'_i\big) - a_i\big\| < \delta, \quad \text{and} \quad \big\|\Phi\big(b'_i\big) - b_i\big\| < \delta, \tag{13}$$

where $i = 1, \dots, l$.

Let L be the ideal of $M_m(B)$ generated by the projections $p'_1, \dots, p'_l$. Since B is an *SB* algebra, $L = M_m(K)$ for some ideal K of B that is generated by its projections. We will prove that

$$\varphi(K) \subseteq I. \tag{14}$$

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First we claim that $\Phi(M_m(K)) \subseteq M_m(I)$. Indeed, since $\delta < 1$, Eq. (13) implies that the projections p_i and $\Phi(p_i')$ are Murray–von Neumann equivalent in $M_m(A)$ for each i. Since p_i belongs to the ideal $M_m(I)$, so does $\Phi(p_i')$. Hence $\Phi(M_m(K)) \subseteq M_m(I)$, as claimed. The inclusion (14) follows.

Let $\Phi_0 : M_m(K) \to M_m(I)$ be given by $\Phi_0(a) = \Phi(a) = (\varphi \otimes \mathrm{id}_{M_n})(a)$ for $a \in M_m(K)$. This is well defined by the argument in the preceding paragraph.

Now, if δ is small enough, it follows from (13) that

$$\left\| \sum_{i=1}^{l} a_i p_i b_i - \Phi_0\left(\sum_{i=1}^{l} a_i' p_i' b_i' \right) \right\| < \epsilon/2.$$

Combining this and the estimate (12) we obtain

$$\left\| x - \Phi_0\left(\sum_{i=1}^{l} a_i' p_i' b_i' \right) \right\| < \epsilon. \tag{15}$$

Let $\{e_{ij}\}_{i,j=1}^{m}$ be the canonical matrix units of M_m. Since $p_i' \in L$ for each i,

$$\sum_{i=1}^{l} a_i' p_i' b_i' \in L = M_m(K),$$

and so there are $y_{ij} \in K$ such that

$$\Phi_0\left(\sum_{i=1}^{l} a_i' p_i' b_i' \right) = \sum_{i,j=1}^{m} \varphi(y_{ij}) \otimes e_{ij}.$$

We may rewrite (15) as

$$\left\| x - \sum_{i,j=1}^{m} \varphi(y_{ij}) \otimes e_{ij} \right\| < \epsilon.$$

But (recalling our abuse of notation) this implies

$$\left\| x - \varphi(y_{11}) \right\| < \epsilon \tag{16}$$

since the norm of each entry of a matrix (over A) is at most the norm of the matrix itself.

Finally, since K is generated by its projections, $y_{11} \in K$, and $\varphi(K) \subseteq I$, we are finished. □

Remark 5.5. Although in Lemma 5.4 we used $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$, the same result holds for an arbitrary Hilbert space H instead of $\ell^2(\mathbb{N})$. Indeed, to prove Lemma 5.4 with $\mathcal{K}(H)$ instead of $\mathcal{K}$ one may use the following two observations:

(1) If J is an ideal of $A \otimes \mathcal{K}(H)$, then there is a unique ideal I of A such that $J = I \otimes \mathcal{K}(H)$. This follows from a result of Kirchberg [17, Proposition 2.13], since $\mathcal{K}(H)$ is both simple and exact.

(2) Writing $H = \bigoplus_{i \in \mathbb{I}} \mathbb{C}$, we have

$$\mathcal{K}(H) = \overline{\bigcup_{F \subseteq \mathbb{I},\, |F| < \infty} P_F B(H) P_F},$$

where P_F is the orthogonal projection of H onto the subspace $\bigoplus_{i \in F} \mathbb{C}$ of H. Also, $P_F B(H) P_F \cong M_{|F|}$, so $\mathcal{K}(H)$ is the inductive limit of a net of matrix algebras.

The next result generalizes [28, Theorem 3.9]:

Theorem 5.6. *Let A be an* SLB *algebra. Then there is a lattice isomorphism*

$$\{I \lhd A\colon\ I \text{ is generated by its projections}\} \cong \big\{J\colon\ J \text{ is an ideal of } D(A \otimes \mathcal{K})\big\}. \tag{17}$$

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Proof. [22, Lemma 4.2] implies that there is a lattice isomorphism between

$$\{I \lhd A \otimes \mathcal{K}\colon\ I \text{ is generated by its projections}\}$$

and the right-hand side of (17). The result now follows from the previous lemma. □

Recall that a C^*-algebra is of *type I* if the image of every nonzero irreducible representation contains a nonzero compact operator. Any quotient of a type I C^*-algebra is of type I (see, e.g., [18, Theorem 5.6.2]), and a simple type I C^*-algebra is $*$-isomorphic to $\mathcal{K}(H)$ for some Hilbert space H (see, e.g., [18, Theorem 2.4.9]).

Proposition 5.7. *(Cf. [29, Theorem 2.24].) Let A be an* SLB *algebra and B a C^*-algebra. If $A \otimes B$ has the ideal property and $B \neq 0$ is of type I, then A has the ideal property.*

Proof. Let I be a maximal ideal of B. Then B/I is a simple type I C^*-algebra, so it is isomorphic to $\mathcal{K}(H)$ for some Hilbert space H. Moreover, since $A \otimes (B/I)$ is a quotient of $A \otimes B$, it also has the ideal property, by Proposition 2.17. Hence $A \otimes \mathcal{K}(H)$ has the ideal property. By Remark 5.5, A has the ideal property. □

The next result generalizes [29, Theorem 2.25]:

Theorem 5.8. *Let A and B be nonzero* SLB *algebras of type I. The following are equivalent*:

(1) *$A \otimes B$ has the ideal property*;
(2) *A and B have the ideal property.*

Proof. This follows using the preceding proposition and Proposition 2.16, since every type I C^*-algebra is nuclear [36], and therefore exact. □

6. On the largest ideal with the ideal property

The following result is perhaps known, but because we were unable to find a reference for it, we include a proof:

Lemma 6.1. *Let A be a C^*-algebra and I, J, and K be ideals of A. Then $(I + J) \cap K = I \cap K + J \cap K$.*

Proof. The difficulty is with the inclusion

$$(I + J) \cap K \subseteq I \cap K + J \cap K.$$

Let $x \in ((I + J) \cap K)^+$. Then $x = y^*y$ for some $y \in (I + J) \cap K$, so $x \in (I + J)^+ \cap K^+$. By [33, Proposition 1.5.9], $(I + J)^+ = I^+ + J^+$, and therefore $x = a + b$ for some $a \in I^+$, $b \in J^+$. Hence

$$0 \leqslant a \leqslant a + b = x \in K^+ \quad \text{and} \quad 0 \leqslant b \leqslant a + b = x \in K^+,$$

so $a, b \in K$, since K is a hereditary sub-C^*-algebra of A. Then $x = a + b \in I^+ \cap K^+ + J^+ \cap K^+$. It follows that $((I + J) \cap K)^+ \subseteq I \cap K + J \cap K$, and so $(I + J) \cap K \subseteq I \cap K + J \cap K$. □

Proposition 6.2. *Every C^*-algebra has a largest ideal with the ideal property.*

Proof. Let A be a C^*-algebra and let $\mathcal{I}$ be the family of all ideals of A with the ideal property. If $I, J \in \mathcal{I}$, let $K \lhd I + J$. Then $K = K' \cap (I + J)$ for some ideal K' of A, since $I + J$ is a hereditary sub-C^*-algebra of A. But then, by Lemma 6.1, $K = K' \cap I + K' \cap J$, and these last two are ideals of I and J (respectively), and are therefore generated by their projections. This clearly implies that K is generated by its projections. Thus $I + J$ has the ideal property. Since the ideal property is preserved by inductive limits [22, Proposition 2.3],

$$\overline{\bigcup_{I \in \mathcal{I}} I} = \varinjlim_{J \in \mathcal{I}} J$$

is an ideal of A with the ideal property, and it clearly contains every ideal of A that has the ideal property. □

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Notation. We denote by $I_{ip}(A)$ the largest ideal with the ideal property of a C^*-algebra A.

Theorem 6.3. *Let I be an ideal of a C^*-algebra A. If A is a* WLB *algebra, then the sequence*

$$0 \to I_{ip}(I) \xrightarrow{\iota'} I_{ip}(A) \xrightarrow{\pi'} I_{ip}\big(A/I_{ip}(I)\big) \to 0 \tag{18}$$

is exact (where ι' and π' are the appropriate restrictions of the canonical inclusion and quotient maps $\iota: I_{ip}(I) \to A$ and $\pi: A \to A/I_{ip}(I)$).

Proof. Let $B = A/I_{ip}(I)$. First we should see that the sequence (18) makes sense. It is clear that $\iota(I_{ip}(I)) \subseteq I_{ip}(A)$. Let us prove that $\pi(I_{ip}(A)) \subseteq I_{ip}(B)$. Note that $\pi(I_{ip}(A)) \cong I_{ip}(A)/I_{ip}(I)$ has the ideal property since $I_{ip}(A)$ does (the ideal property passes to quotients, see Proposition 2.17). Hence $\pi(I_{ip}(A)) \subseteq I_{ip}(B)$ (since $\pi(I_{ip}(A)) \lhd B$, π being surjective).

Let $\iota': I_{ip}(I) \to I_{ip}(A)$ be the appropriate co-restriction of ι. Clearly ι' is injective. Let $\pi': I_{ip}(A) \to I_{ip}(B)$ be the appropriate restriction (and co-restriction) of π. Clearly $\ker \pi' = \iota'(I_{ip}(I))$. We are left to prove that π' is surjective. For this, we show that $I_{ip}(B) \subseteq \pi(I_{ip}(A))$. Define $J := I_{ip}(B)$, and let $J_0 = \pi^{-1}(J)$. If we prove that the ideal J_0 has the ideal property, we are finished, since $J = \pi(J_0)$ and $J_0 \lhd A$. Indeed, since $J \cong J_0/(I_{ip}(I) \cap J_0)$, we have an exact sequence

$$0 \to I_{ip}(I) \cap J_0 \to J_0 \to J \to 0. \tag{19}$$

But $I_{ip}(I) \cap J_0 \lhd I_{ip}(I)$ and $I_{ip}(I)$ has the ideal property, so $I_{ip}(I) \cap J_0$ has the ideal property. Since

$$0 \to I_{ip}(I) \cap J_0 \to A \to A/I_{ip}(I) \cap J_0 \to 0$$

is quasidiagonal by Theorem 4.4 (we are assuming that A is a *WLB* algebra), the extension (19) is quasidiagonal as well. It follows from Theorem 3.7 that J_0 has the ideal property (note that J has the ideal property). □

Note that if in the theorem above we assume in addition that I has the ideal property, we obtain that the sequence $0 \to I_{ip}(I) \to I_{ip}(A) \to I_{ip}(A/I) \to 0$ is exact. In general, this is not true if I does not have the ideal property, even if we assume that A is a *WLB* algebra.

Corollary 6.4. *If A is a* WLB *algebra (in particular, if A is an* AH *algebra or, more generally, an* LB *algebra), then*

$$I_{ip}\big(A/I_{ip}(A)\big) = 0.$$

Proof. The sequence $0 \to I_{ip}(A) \xrightarrow{\iota} A \xrightarrow{\pi} A/I_{ip}(A) \to 0$ is exact (where ι is the inclusion and π is the canonical quotient map). By Theorem 6.3, the induced sequence

$$0 \to I_{ip}\big(I_{ip}(A)\big) \to I_{ip}(A) \to I_{ip}\big(A/I_{ip}\big(I_{ip}(A)\big)\big) \to 0$$

is exact. But $I_{ip}(I_{ip}(A)) = I_{ip}(A)$. □

Lemma 6.5. *Let A be a unital C^*-algebra, n be a natural number, and $M_n(J)$ be an ideal of $M_n(A)$ (where J is an ideal of A). If $M_n(A)$ is quasidiagonal relative to $M_n(J)$, then A is quasidiagonal relative to J.*

Proof. Let $(p_\lambda)_{\lambda \in \Lambda}$ be an approximate unit of projections of $M_n(J)$ that is quasicentral in $M_n(A)$, and let (e_{ij}) be the canonical system of matrix units of M_n. Write each p_λ as a matrix $[p^\lambda_{ij}]^n_{i,j=1}$. If $a \in A$ and $i \in \{1, \dots, n\}$, then

$$p_\lambda(a \otimes e_{ii}) - (a \otimes e_{ii})p_\lambda = \begin{pmatrix} 0 & \cdots & p^\lambda_{1i}a & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -ap^\lambda_{i1} & \cdots & p^\lambda_{ii}a - ap^\lambda_{ii} & -ap^\lambda_{i,i+1} & \cdots & -ap^\lambda_{in} \\ 0 & \cdots & p^\lambda_{i+1,i}a & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & p^\lambda_{ni}a & 0 & \cdots & 0 \end{pmatrix}.$$

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Since $p_\lambda(a \otimes e_{ii}) - (a \otimes e_{ii})p_\lambda \to 0$, we obtain that $(p_{ii}^\lambda)_{\lambda\in\Lambda}$ is quasicentral in A and that $ap_{ij}^\lambda \to 0$ if $i \neq j$. Then $p_{ij}^\lambda \to 0$ if $i \neq j$ (putting $a = 1_A$). An entirely similar argument, computing $p_\lambda(a \otimes e_{ii}) - a \otimes e_{ii}$, shows that $(p_{ii}^\lambda)_{\lambda\in\Lambda}$ is an approximate unit (not necessarily increasing) of J.

Now, $p_\lambda - p_\lambda^2 = 0$ for each λ. Looking at the (i,i)th entry of this equation, we have $0 = p_{ii}^\lambda - (p_{ii}^\lambda)^2 - S^\lambda$, where

$$S^\lambda = p_{i1}^\lambda p_{1i}^\lambda + \cdots + p_{i,i-1}^\lambda p_{i-1,i}^\lambda + p_{i,i+1}^\lambda p_{i+1,i}^\lambda + \cdots + p_{in}^\lambda p_{ni}^\lambda,$$

so $S^\lambda \to 0$ (since $p_{ij}^\lambda \to 0$ if $i \neq j$). Then $p_{ii}^\lambda - (p_{ii}^\lambda)^2 \to 0$. A similar argument as the one found at the end of the proof of Lemma 3.3 shows that we may suppose that for each i there is a net of projections $(r_i^\lambda)_{\lambda\in\Lambda}$ in J such that $p_{ii}^\lambda - r_i^\lambda \to 0$. Evidently $(r_i^\lambda)_{\lambda\in\Lambda}$ is an approximate unit of projections of J that is quasicentral in A. □

Proposition 6.6. *Let A be a C^*-algebra with an approximate unit of projections, let n be a natural number, and let $M_n(J)$ be an ideal of $M_n(A)$ (where J is an ideal of A). If $M_n(A)$ is quasidiagonal relative to $M_n(J)$, then A is quasidiagonal relative to J.*

Proof. Let $(e_\mu)_{\mu\in M}$ be an approximate unit of projections of A. Let $A_\mu := e_\mu A e_\mu$ and $J_\mu := e_\mu J e_\mu$. Note that each A_μ is unital. Let $\hat{e}_\mu = 1_n \otimes e_\mu$, where 1_n is the unit of M_n. Note that each $\hat{e}_\mu$ is a projection of $M_n(A)$.

Since $M_n(A)$ is quasidiagonal relative to $M_n(J)$, by Lemma 3.3 we see that $\hat{e}_\mu M_n(A)\hat{e}_\mu$ is quasidiagonal relative to $\hat{e}_\mu M_n(J)\hat{e}_\mu$. But $\hat{e}_\mu M_n(A)\hat{e}_\mu = M_n(e_\mu A e_\mu) = M_n(A_\mu)$, and similarly $\hat{e}_\mu M_n(J)\hat{e}_\mu = M_n(J_\mu)$. That is, $M_n(A_\mu)$ is quasidiagonal relative to $M_n(J_\mu)$. Since A_μ is unital, by Lemma 6.5 we see that A_μ is quasidiagonal relative to J_μ, for all $\mu \in M$.

Let $F = \{a_1, \ldots, a_k\} \subseteq A$, $G = \{b_1, \ldots, b_l\} \subseteq J$, and m be a natural number. Since (e_μ) is an approximate unit of A, it follows that there is $\mu \in M$ such that $F \subseteq_{1/3m} A_\mu$ and $G \subseteq_{1/3m} J_\mu$. Let $F' = \{a'_1, \ldots, a'_k\} \subseteq A_\mu$ and $G = \{b'_1, \ldots, b'_l\} \subseteq J_\mu$ be such that

$$\|a_i - a'_i\| < \frac{1}{3m} \quad \text{and} \quad \|b_j - b'_j\| < \frac{1}{3m} \tag{20}$$

for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$. Since A_μ is quasidiagonal relative to J_μ, there is a projection $p \in J_\mu$ such that

$$\|a'_i p - pa'_i\| < \frac{1}{3m} \quad \text{and} \quad \|b'_j - pb'_j\| < \frac{1}{3m} \tag{21}$$

for all i and j. Thus

$$\|a_i p - pa_i\| < \frac{1}{m} \quad \text{and} \quad \|b_j - pb_j\| < \frac{1}{m} \tag{22}$$

for all i and j, using (20) and (21). Let $\Lambda = \mathcal{F}(A) \times \mathcal{F}(J) \times \mathbb{N}$. For $\lambda = (F, G, m) \in \Lambda$, let $p_\lambda = p \in J$ as in (22).

Defining an order on Λ by $(F, G, m) \leqslant (F', G', m')$ if $F \subseteq F'$, $G \subseteq G'$, and $m \leqslant m'$, Λ becomes a directed set, and $(p_\lambda)_{\lambda\in\Lambda}$ becomes a net of projections in J. It follows from (22) that (p_λ) is an approximate unit of projections in J that is quasicentral in A. □

The following proposition follows immediately from the previous one.

Proposition 6.7. *Let A be a C^*-algebra with an approximate unit of projections. If $M_n(A)$ is a* WLB *algebra with the ideal property for some $n \in \mathbb{N}$, then A has the weak projection property. In particular, A has the ideal property.*

Question 6.8. *If A is a C^*-algebra such that $M_n(A)$ has the ideal property for some $n \in \mathbb{N}$, does it follow that A has the ideal property?*

Theorem 6.9. *Let A be a C^*-algebra with an approximate unit of projections. If $M_n(A)$ is a* WLB *algebra for some $n \in \mathbb{N}$, then $I_{ip}(M_n(A)) = M_n(I_{ip}(A))$, and $I_{ip}(A)$ has the weak projection property.*

Proof. Let $I = I_{ip}(M_n(A))$. Then $I = M_n(J)$ for some ideal J of A. Since I is generated by its projections, $M_n(A)$ is quasidiagonal relative to $I = M_n(J)$; hence A is quasidiagonal relative to J by Proposition 6.6. Thus J has an approximate unit of projections.

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By Theorem 4.4, I is a *WLB* algebra. By Proposition 6.7, J has the weak projection property, i.e., every one of its ideals has an approximate unit of projections. In particular, J has the ideal property. Then

$$I_{ip}\big(M_n(A)\big) = I = M_n(J) \subseteq M_n\big(I_{ip}(A)\big)$$

and the last term has the ideal property. Hence the above inclusion must be an equality. □

Remark 6.10. Note that if A is a C^*-algebra such that A does not have the ideal property but $A \otimes \mathcal{K}$ has the ideal property (see, e.g., [29]), then

$$I_{ip}(A \otimes \mathcal{K}) = A \otimes \mathcal{K} \supsetneqq I_{ip}(A) \otimes I_{ip}(\mathcal{K}) = I_{ip}(A) \otimes \mathcal{K}.$$

However, if A is an *SLB* algebra, then using a result of Kirchberg [17] and Remark 5.5, one can easily get that $I_{ip}(A \otimes \mathcal{K}(H)) = I_{ip}(A) \otimes \mathcal{K}(H)$ for every Hilbert space H.

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