CLASSIFYING *-HOMOMORPHISMS I: UNITAL SIMPLE NUCLEAR C^*-ALGEBRAS

JOSÉ R. CARRIÓN, JAMES GABE, CHRISTOPHER SCHAFHAUSER, AARON TIKUISIS, AND STUART WHITE

Abstract. We classify the unital embeddings of a unital separable nuclear C^*-algebra satisfying the universal coefficient theorem into a unital simple separable nuclear C^*-algebra that tensorially absorbs the Jiang–Su algebra.

This gives a new and essentially self-contained proof of the stably finite case of the unital classification theorem: unital simple separable nuclear C^*-algebras that absorb the Jiang–Su algebra tensorially and satisfy the universal coefficient theorem are classified by Elliott's invariant of K-theory and traces.

Overview of results

This is the first of a series of papers in which we give an abstract approach to classification results for simple separable nuclear C^*-algebras. This line of research, known as the Elliott classification program, has been a large-scale endeavor since the 1990s and seeks to obtain results for C^*-algebras analogous to the celebrated structure and classification theorems for amenable von Neumann algebras.

This paper handles the case of unital C^*-algebras. Our overarching objective is to give a new and comparatively self-contained proof of the unital classification theorem (Theorem A). This says that a suitable class of unital simple separable nuclear C^*-algebras is classified by operator algebraic K-theory (the non-commutative version of Atiyah and Hirzebruch's topological K-theory) and its pairing with traces (thought of as non-commutative measures on C^*-algebras). We put the hypotheses of the theorem, especially $\mathcal{Z}$-stability and the universal coefficient theorem (UCT) of Rosenberg and Schochet, into context in Section 1.1. At this point, we emphasize that these two conditions are both necessary: the classifiable class is maximal for such a result. Moreover, powerful tools exist to verify the hypotheses in a wide range of concrete situations (see Section 1.1.4). This classification theorem may be regarded as the C^*-algebraic analog of the Connes–Haagerup classification of injective factors ([46, 130]).

Theorem A (The unital classification theorem; see Theorem 9.9)**.** *Unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebras satisfying Rosenberg and Schochet's universal coefficient theorem are classified by Elliott's invariant consisting of K-theory and traces.*

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The unital classification theorem was first obtained in 2015 by combining [125, 126, 98, 269, 293, 292, 291] (and the large body of work on which these papers rely) with the Kirchberg–Phillips theorem [161, 221] from the 1990s.[1] It is a capstone result for the Elliott classification program, involving decades of work by large numbers of researchers, and so is often attributed to "many hands". Our approach follows an entirely different strategy — made possible by the recent developments of [252, 253, 32] — which, in our opinion, is both shorter and more conceptual.

Beginning with Elliott's classification of approximately finite dimensional C^*-algebras and continuing to the present day, classification results for C^*-algebras have been built on classification results for morphisms. Our strategy in this paper fits the version of this framework pioneered by Rørdam in [238]: the main goal of the paper is the following classification of unital embeddings, from which Theorem A is quickly deduced. We discuss the need for the total invariant $\underline{K}T_u$ further below (in Sections 1.1, 1.2, and 1.3), and describe its components in detail in Sections 2.2, 2.3, 3.1, and 3.2.

Theorem B (Classification of unital embeddings; see Theorem 9.3)**.** *Let A be a unital separable nuclear C^*-algebra satisfying the UCT and B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra. Up to approximate unitary equivalence, the unital embeddings $A \hookrightarrow B$ correspond bijectively with those morphisms of the total invariant $\underline{K}T_u$ — consisting of enriched K-theoretic and tracial data — mapping traces on B to faithful traces on A.*

Both Theorem A and Theorem B admit a dichotomy thanks to a special case of a beautiful theorem of Kirchberg: a C^*-algebra B as in Theorem B is either *purely infinite* or *stably finite*. Moreover, the tracial component of the invariants used in Theorems A and B detects these conditions: purely infinite simple C^*-algebras are traceless, while a deep theorem of Haagerup ([131]) combines with work of Blackadar and Handelman ([131, 12]) to give traces on unital simple nuclear stably finite C^*-algebras. Accordingly, both theorems and their proofs split into these two cases. In the purely infinite setting, Theorem A was established independently by Kirchberg and by Phillips in the 1990s ([161, 164, 221]; see also Rørdam's expository account in [240]). Subsequent work to reach Theorem A was then devoted to the stably finite case. The same is true for our proof. In this paper we only handle the stably finite aspect of Theorem A, and one obtains the entire unital classification theorem by combining this with the Kirchberg–Phillips theorem. Likewise, when B is purely infinite, Kirchberg obtained a classification of embeddings; our contribution is the stably finite counterpart, allowing for the unified statement of Theorem B.[2]

The generality of the stably finite case of Theorem B is the headline new result of the paper. It simultaneously unifies and generalizes earlier classifications of embeddings of nuclear C^*-algebras into simple nuclear stably finite C^*-algebras by

[1] Note that [125, 126] were originally made available as [123]. Our formulation of Theorem A differs from the 2015 version, which had the hypothesis of finite nuclear dimension in place of $\mathcal{Z}$-stability. These two conditions are now known to be equivalent ([291, 35]).

[2] In fact, Kirchberg's embedding theorem is more general, transferring some of the hypotheses from the domain and codomain to hypotheses on the morphisms. The second paper in this series, devoted to non-unital classification, will also extend the stably finite aspect of Theorem B to Kirchberg's framework. With the recent passing of Eberhard Kirchberg, his long-term project [159] describing these, and many more results, will sadly remain unfinished. JG gives a new approach to Kirchberg's classification theorems in [113, 110].

K-theoretic data. The key point is the abstract nature of the hypotheses: the theorem allows all separable nuclear C^*-algebras as domains of embeddings, whereas prior results such as [182, 186, 197] rely on internal structure for the domains (either commutativity, or explicit inductive limit or tracial approximation structure), and often also for the codomains. While this paper was in preparation, Gong, Lin, and Niu independently obtained Theorem B when A is additionally simple ([124], building on [190]). Their work uses the power of the unital classification theorem to obtain very precise internal structure on UHF-stabilizations of A.

The passage back from Theorem B to Theorem A is via an Elliott intertwining argument, for which it is vital that Theorem B has both existence and uniqueness components. That is, Theorem B specifies exactly which maps between total invariants are realized by injective *-homomorphisms, and then shows uniqueness of these. Moreover, although the total invariant $\underline{K}T_u$ contains more information than K-theory and traces, any map between K-theory and traces can be extended to $\underline{K}T_u$. In this way, the existence aspect of Theorem B yields the following:

Corollary C. *Let A be a unital separable nuclear C^*-algebra satisfying the UCT and B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra. Any morphism from the Elliott invariant of A to that of B that maps traces on B to faithful traces on A is realized by a unital injective *-homomorphism $A \to B$.*

We will put the key ingredients in these theorems into context in Section 1.1 of the introduction, making the connection with von Neumann algebras and describing the range of examples covered by Theorem A. We follow this with a brief history of the unital classification theorem to set the scene for the outline of our methods in Section 1.3.

We first announced the results contained in this paper at NCGOA in Münster in 2018, and then in Oberwolfach ([268]). A summary of these results is also included in the survey [286].

Contents

1. Introduction

1.1. **Context and examples.** From the outset, the classification of simple nuclear C^*-algebras has been intimately tied to developments in von Neumann algebras. Indeed, the first classification theorem for simple C^*-algebras is Glimm's classification of uniformly hyperfinite C^*-algebras ([119]), which is a direct analog of Murray and von Neumann's uniqueness theorem for the hyperfinite II_1 factor ([205]) taking into account the more refined invariants needed for C^*-algebras.

Later developments in C^*-algebra classification were similarly influenced by the spectacular successes on the von Neumann algebra side. Connes' groundbreaking work on injective von Neumann algebras ([46]) gives a clean, testable, and abstract characterization of hyperfiniteness and is a landmark of 20th century mathematics. Connes' theorem provides a game-changing enlargement of the scope of Murray and von Neumann's uniqueness theorem and led to a complete classification of injective infinite factors.[3] Beyond its impact in von Neumann algebras — which continues to the present day, for example in subfactor theory ([145, 227, 146]) and Popa's deformation/rigidity theory ([229, 280]) — Connes' theorem inspired classification results for amenable measurable dynamical systems ([47]) and is vital in developing approximation properties for C^*-algebras.

The unital classification theorem is the counterpart for unital C^*-algebras of the von Neumann algebra classification theorems. The hypotheses of unitality, simplicity, separability, and nuclearity in Theorem A directly correspond to the von Neumann algebra framework. All von Neumann algebras are unital, and factors are the simple von Neumann algebras. Separability for C^*-algebras corresponds to von Neumann algebras with separable predual, and these hypotheses play essentially the same role in the two settings.[4] Finally, nuclearity corresponds to injectivity in Connes' theorem: a C^*-algebra A is nuclear if and only if its bidual A^{**} is injective ([37, 38]).

The other two hypotheses in the unital classification theorem — tensorial absorption of the Jiang–Su algebra $\mathcal{Z}$ and belonging to the UCT class — are of a more subtle nature. We describe these next.

1.1.1. *$\mathcal{Z}$-stability.* A key step in Connes' work is that every separably acting injective II_1 factor $\mathcal{M}$ absorbs the hyperfinite II_1 factor $\mathcal{R}$ tensorially, i.e., $\mathcal{M} \cong \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{R}$. This property was first extensively studied in [202], and such II_1 factors are said to be *McDuff*. In contrast, not all infinite dimensional unital simple separable nuclear C^*-algebras factorize as a non-trivial tensor product.[5] Yet tensorial absorption has been crucial in C^*-classification, coming into prominence in Kirchberg's Geneva theorems, one of which shows that for simple separable nuclear C^*-algebras, pure infiniteness is characterized by tensorial absorption of the Cuntz algebra $\mathcal{O}_\infty$. Moreover, "$\mathcal{O}_\infty$-stability" is a key ingredient in the Kirchberg–Phillips theorem.

The Jiang–Su algebra $\mathcal{Z}$, introduced in [144], is the stably finite counterpart of $\mathcal{O}_\infty$, and a C^*-algebra A is said to be $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$. Both $\mathcal{O}_\infty$ and $\mathcal{Z}$ are infinite dimensional, unital, simple, separable, nuclear, and have the same K-theory as the complex numbers. However, $\mathcal{Z}$ has a unique trace, so it is stably finite, whereas $\mathcal{O}_\infty$ is purely infinite. Both are (strongly) tensorially self-absorbing. Moreover, in a sense that can be made precise using [274] and [290], the property of $\mathcal{Z}$-stability is the mildest C^*-tensorial absorption condition akin to the McDuff property for II_1 factors. Thus, from this point of view, $\mathcal{Z}$ is the most natural analog of $\mathcal{R}$.

Just as $\mathcal{O}_\infty$-stability is essential for the Kirchberg–Phillips classification, we use $\mathcal{Z}$-stability as the fundamental regularity property to separate classifiable C^*-algebras from the higher dimensional examples of [281, 282, 241, 271, 276, 118]

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[3]This classification combines Connes' theorem with Connes' and Krieger's prior work ([42, 45, 44, 171]) and Haagerup's subsequent result on the uniqueness of the hyperfinite III_1 factor ([130]).

[4]Separability enters through an intertwining argument and is necessary for both von Neumann and C^*-classification results, as shown by the examples of [108].

[5]Rørdam's famous example of a unital simple separable nuclear C^*-algebra with both an infinite and finite projection ([241]) has no such factorization.

pioneered by Villadsen. The use of $\mathcal{Z}$-stability in classification was explicitly predicted by Toms in 2005 ([271]): "Optimistically $\mathcal{Z}$-stability is an abstraction of slow dimension growth, and the Elliott conjecture will be confirmed for all simple separable nuclear C^*-algebras having this property." See also [105]. Not only does $\mathcal{Z}$ have the same K-theory and traces as $\mathbb{C}$, but for any C^*-algebra A, the K-theory and traces of A and of $A \otimes \mathcal{Z}$ are isomorphic. For this reason, $\mathcal{Z}$-stability is for all intents and purposes necessary for classification by K-theory and traces.[6] We describe the Jiang–Su algebra and collect the key consequences of $\mathcal{Z}$-stability used throughout the paper in Section 4.

1.1.2. *The UCT.* Kasparov's KK-theory is a bivariant theory unifying K-theory and K-homology ([152, 153]). It has proved indispensable in both C^*-algebra classification and index theory (see for example [154, 136]). The Kirchberg–Phillips classification of stable simple separable nuclear purely infinite C^*-algebras is expressed directly in terms of KK-theory: two such algebras A and B are isomorphic if and only if they are KK-equivalent. Thinking of KK-equivalence as a very loose notion of homotopy between C^*-algebras,[7] this has the spirit of Mostow's rigidity theorem in geometric topology.

Inspired by earlier work of Brown ([26]), Rosenberg and Schochet established their universal coefficient theorem (UCT), which computes KK-theory in terms of K-theory for a large class of C^*-algebras ([245]) with generous permanence properties. This class is precisely the C^*-algebras that are KK-equivalent to some abelian C^*-algebra. Such algebras are said to *satisfy the UCT*.

In contrast to $\mathcal{Z}$-stability, where examples lacking the property are known, it remains a crucial open question whether all (simple) separable nuclear C^*-algebras satisfy the UCT. For naturally occurring, concrete examples, this is less problematic as there are very general tools for verifying the UCT. Notably, all amenable groupoid C^*-algebras satisfy the UCT ([278]). In the setting of group actions, the UCT is preserved by crossed products of nuclear C^*-algebras by countable torsion-free amenable groups ([203]). Whether the same holds for finite groups is open (in fact, equivalent to the UCT problem). See Section 8.1 and Remark 8.2 for more details.

Among the classification hypotheses, the UCT is the only one that does not appear to have a von Neumann algebraic counterpart, due to its inherently topological nature.

1.1.3. *The invariant.* The data comprising the *Elliott invariant* of a unital stably finite C^*-algebra evolved over the early stages of the classification program, from the dimension groups appearing in Elliott's classification of approximately finite dimensional C^*-algebras ([85]), to its present form ([88, 265]; see [240]). This consists of ordered K_0, K_1, the class of the unit, traces, and the pairing map. As alluded to earlier, the order on K_0 is redundant under the hypothesis of $\mathcal{Z}$-stability (see [242, Corollary 4.10]). While we have stated Theorem A and Corollary C in terms of Elliott's invariant, in the main body of the paper we use the (formally weaker) invariant KT_u consisting of K-theory, the class of the unit, traces, and the pairing map, but not the order on K_0. See the discussion in Section 2.1.

[6] In language we introduce in Section 2.1, $KT_u(A) \cong KT_u(A \otimes \mathcal{Z})$. However, the order structure on the K_0-groups, and therefore their Elliott invariants, may differ between these two algebras.

[7] Indeed, C^*-algebras that are homotopy equivalent are KK-equivalent.

The unital classification theorem is complemented by an important range of invariant result: all pairings of K-theory (without order) and traces that can occur for unital separable C^*-algebras are realized by the C^*-algebras of Theorem A. The range of the invariant can be characterized abstractly as follows. In the purely infinite case (when $T(A) = \emptyset$), $K_0(A)$ and $K_1(A)$ can be arbitrary countable abelian groups, $[1_A]_0$ can be any element of $K_0(A)$, and the pairing map can be disregarded since there are no traces. In the stably finite case (when $T(A) \neq \emptyset$), $K_1(A)$ can be an arbitrary countable abelian group, $K_0(A)$ is any countable abelian group with a specified element $[1_A]_0$ of infinite order, $T(A)$ can be an arbitrary metrizable Choquet simplex, and the pairing map $\rho_A \colon K_0(A) \to \operatorname{Aff} T(A)$ can be any unit-preserving group homomorphism.

These range of invariant theorems are established via crossed product constructions ([238]) in the purely infinite setting and inductive limit constructions ([90]) in the stably finite one. Understanding the range of the invariant leads to a wealth of other automatic structural results such as the existence of Cartan subalgebras inside the C^*-algebras covered by Theorem A ([174, 40]); equivalently via [231], classifiable C^*-algebras all arise from twisted groupoids.

Turning to Theorem B, examples in the 1990s show that K-theory and traces alone are not enough to classify *-homomorphisms (this was made explicit in [209]). This forces enlargement of the invariant, formalized in the *total invariant*, written $\underline{K}T_u$. This is obtained by adjoining a Hausdorffized unitary algebraic K_1-group developed by Thomsen in [266], which we denote $\overline{K}_1^{\mathrm{alg}}(A)$, and Schochet's K-theory with $\mathbb{Z}/n\mathbb{Z}$-coefficients ([255]), known as *total K-theory*, to KT_u.

While our approach does not need a range of invariant result (i.e., existence of objects) to classify C^*-algebras in the unital classification theorem, it is vital in the deduction of Theorem A from Theorem B that the classification of unital embeddings does include a range of invariant result. Any morphism at the level of the invariant $\underline{K}T_u$ in Theorem B that is faithful on traces is realized by a unital *-monomorphism. This is more subtle than the previous sentence might initially suggest. Just as any morphism between the Elliott invariants of C^*-algebras is required to intertwine the pairing maps between K-theory and traces, morphisms between the enlarged invariant $\underline{K}T_u$ must intertwine the natural maps between its components. Most of these are familiar to experts: the Bockstein operations connecting the various K-groups with coefficients, the de la Harpe–Skandalis determinant from [68] relating $\overline{K}_1^{\mathrm{alg}}$ and traces, and the canonical map from $\overline{K}_1^{\mathrm{alg}}$ to K_1. We review these in Section 2.2 and collect properties of the Hausdorffized algebraic K_1-group and total K-theory there as well.

There is one more natural family of maps

$$\zeta_A^{(n)} \colon K_0(A; \mathbb{Z}/n\mathbb{Z}) \to \overline{K}_1^{\mathrm{alg}}(A) \tag{1.1}$$

through which the Bockstein maps $K_0(A; \mathbb{Z}/n\mathbb{Z}) \to K_1(A)$ factorize. When $K_1(A)$ is torsion-free, the maps $\zeta_A^{(n)}$ are redundant, but in general, compatibility with these maps is an additional requirement in order to construct a morphism between C^*-algebras realizing specified behavior on K-theory, traces, total K-theory, and $\overline{K}_1^{\mathrm{alg}}$. In the setting of the classification of unital embeddings (Theorem B) this is the only additional compatibility needed.

We give an explicit construction of the maps (1.1) in Section 3.1, formalizing the total invariant in Section 3.2. During the (lengthy) gestation of this paper, Gong, Lin, and Niu also discovered the need for additional compatibility between total and algebraic K-theory to produce morphisms ([124]). They took a different, more abstract approach. See Remark 3.4.

1.1.4. *Examples.* It is important to be able to recognize $\mathcal{Z}$-stability in examples in order to apply the classification theorem. To start, we note that $\mathcal{Z}$-stability can be described in terms of approximately central sequences, without reference to the Jiang–Su algebra itself ([289, Proposition 2.3], using [244], together with [163] or [274]). This formulation may be regarded as a suitable softening to positive elements of McDuff's criterion for absorption of $\mathcal{R}$ in terms of approximately central matrix embeddings from [202].

Over the last 15 years, a major program of work on the Toms–Winter conjecture (see [105, 275] and [294, Section 5]) has provided both a topological characterization of $\mathcal{Z}$-stability and Matui and Sato's powerful von Neumann techniques for establishing $\mathcal{Z}$-stability ([200]). *Nuclear dimension* is a noncommutative generalization of topological covering dimension introduced by Winter and Zacharias ([295]). For unital simple separable nuclear non-elementary C^*-algebras, $\mathcal{Z}$-stability is equivalent to finiteness of the nuclear dimension ([291, 35, 31], building on [289, 201, 251, 21]). This gives two paths to classifiability: estimating the nuclear dimension or directly obtaining $\mathcal{Z}$-stability. For examples built from objects of bounded topological dimension, it has typically been possible to estimate the nuclear dimension. However, for many examples outside this setting, direct estimates on the nuclear dimension seem inaccessible, and von Neumann techniques provide a route to $\mathcal{Z}$-stability.

For example, a unital simple separable approximately subhomogeneous C^*-algebra A is always in the UCT class. The nuclear dimension of subhomogeneous algebras is controlled by the dimension of the spectrum ([287]), so that when A has no dimension growth it also has finite nuclear dimension. When the dimension grows, a direct estimate on the nuclear dimension fails, yet, using [291], Toms proved that A is $\mathcal{Z}$-stable precisely when it has slow dimension growth ([272]).

Given a finitely generated amenable group G, every irreducible unitary representation π generates a simple C^*-algebra precisely when G is virtually nilpotent (this goes back to [204, 225]; see also [74]). In this case, $C^*_\pi(G)$ is $\mathcal{Z}$-stable whenever π is infinite dimensional ([76], building on [77], an argument using C^*-classification and established parts of the Toms–Winter conjecture). For nilpotent G, the UCT for $C^*_\pi(G)$ was established in [75]; this remains open in the virtually nilpotent case.

A vast class of examples arise from free minimal actions $G \curvearrowright X$ of a countable discrete amenable group on a compact metrizable space X. Here, the associated crossed product C^*-algebra is unital, simple (by [79, Corollary 5.16]), separable, nuclear, and satisfies the UCT (by [278]). It is increasingly clear that $\mathcal{Z}$-stability is linked to the dimension of X, or more generally, of the action of G on X.

When X has finite covering dimension, direct estimates on the nuclear dimension are possible when G is virtually nilpotent ([262], going back to [140, 261]). For more general groups, Kerr made a major breakthrough in [156], generalizing Matui's notion of almost finiteness from [198].[8] This has been used to show $\mathcal{Z}$-stability (hence classifiability) of such crossed products when G has locally subexponential growth ([158, 72]), when G is elementary amenable ([157, 206]), or for generic actions of any amenable group G on the Cantor set ([41]). It is conceivable that

[8] Kerr's $\mathcal{Z}$-stability theorem makes crucial use of Matui and Sato's von Neumann algebra transfer technique ([200]) in the form given in [138].

free minimal actions of countable discrete amenable groups on finite dimensional spaces give classifiable crossed products.

Outside the setting of finite dimensional base spaces, examples of non-$\mathcal{Z}$-stable free minimal crossed products by $\mathbb{Z}$ were shown to exist by Giol and Kerr ([118]); these all have strictly positive mean dimension in the sense of Gromov, Lindenstrauss, and Weiss ([129, 193]). In the positive direction, Elliott and Niu showed that minimal $\mathbb{Z}$-actions of mean dimension zero give rise to a $\mathcal{Z}$-stable, and hence classifiable, crossed product ([101]). This has been extended by Niu to $\mathbb{Z}^d$-actions in [210]. Conjecturally, a crossed product associated to a free minimal action of an amenable group on a compact metrizable space is classifiable precisely when the mean dimension vanishes.[9]

Another very natural class of examples arises from non-commutative dynamics: crossed products from outer actions $G \curvearrowright A$ of amenable groups on classifiable C^*-algebras. When A has a unique trace, Sato showed that the crossed product $A \rtimes G$ is $\mathcal{Z}$-stable ([250]), so is classifiable when it satisfies the UCT (see Remark 8.2). This result extends outside the monotracial setting ([250, 115, 114]), although at present there are restrictions both on the size of the tracial state space, and on the structure of the action of G on traces.

1.2. **A brief history of the unital classification theorem.** In the previous section, we took advantage of hindsight to set out the modern context for the unital classification theorem. Here we give a brief (and necessarily selective) account of how the subject got to this point. We proceed thematically (rather than strictly chronologically) and focus on classification rather than regularity. Therefore, we do not describe the history of the Toms–Winter conjecture (for which, see [105] and [294, Section 5]) and only bring in those parts of this huge body of work that directly relate to the proof of the unital classification theorem.

1.2.1. *Early days.* The first classification results for simple C^*-algebras were for inductive limits of finite dimensional C^*-algebras (AF algebras) from the late 1950s to the mid-1970s ([119, 71, 22, 84]), inspired by Murray and von Neumann's uniqueness of the hyperfinite II_1 factor. The range of the invariant (ordered K_0) for AF algebras was described abstractly at the end of the 1970s in [86, 80].

Also at the end of the 1970s came Cuntz's work on his now eponymous C^*-algebras $\mathcal{O}_n$ ([50, 52], building on an example given by Dixmier [70]). In this work, Cuntz identified the class of simple purely infinite C^*-algebras; these provide analogues of type III factors. Other important examples include simple Cuntz–Krieger algebras ([57]).

In 1980, a three-week meeting on operator algebras was held in Kingston, Ontario, which is widely regarded as a landmark event in the field. Cuntz's survey article from this conference gives a nice account of early progress and examples of simple C^*-algebras ([53]). A talk by Effros laid out a number of open problems ([78]) related to the structure and classification of (simple) nuclear C^*-algebras (particularly Problems 5–10). As we set out below, these proved influential over the next phase of the subject.

[9]This conjecture would be a consequence of the Phillips–Toms conjecture (mentioned in the introduction to [139]), relating mean dimension to comparison properties of the crossed product, and a still-open component of the Toms–Winter conjecture.

1.2.2. *Purely infinite C^*-algebras.* In [55], Cuntz asked whether the Cuntz–Krieger algebras were classified by their K_0-groups (perhaps along with other invariants), following related classification results in symbolic dynamics ([109]). Interest in such questions led to early results including classification theorems for inductive limits of building blocks constructed from Cuntz algebras ([236, 191]), followed by breakthrough tensorial absorption results: $\mathcal{O}_2 \otimes M_{2^\infty} \cong \mathcal{O}_2$ (a problem posed by Cuntz in [53]) was obtained using dynamical methods in [24], and Elliott proved that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.[10] In [238], Rørdam introduced an important strategy for classifying C^*-algebras by classifying embeddings, which has been crucial to subsequent classification results. Using this strategy, Rørdam classified a collection of C^*-algebras[11] that contains all simple Cuntz–Krieger algebras ([239]) and exhausts all possible combinations of K-groups ([104]). Accordingly, it is the largest possible class of simple separable nuclear purely infinite C^*-algebras satisfying the UCT that can be classified by K-theory.

Rørdam's work reduced the classification problem (in the purely infinite setting) to classifying embeddings by KL. Checking this condition in concrete settings would be a herculean task — an abstract approach was needed. The stage for this had been set by decades of advances, including Arveson's extension theorem ([5]), the Choi–Effros lifting theorem ([36]), Voiculescu's theorem ([283]), and Kasparov's development of KK-theory ([152, 153]).

Building on his deep work on tensor products and exactness ([160, 162]), Kirchberg announced his breakthrough "Geneva theorems" at the ICM satellite meeting in 1994: every separable exact C^*-algebra embeds into $\mathcal{O}_2$; $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ for any unital simple separable nuclear C^*-algebra A; and such an A is purely infinite if and only if $A \otimes \mathcal{O}_\infty \cong A$.[12] Using these results, Kirchberg ([159]) and Phillips ([221]) independently showed that simple separable nuclear purely infinite C^*-algebras — now known as *Kirchberg algebras* — satisfying the UCT are classified by K-theory. Consequently, Rørdam's class consists exactly of the UCT Kirchberg algebras, and [104] is a range of invariant result for the Kirchberg–Phillips classification.

1.2.3. *Approximately type I C^*-algebras.* Stepping back a bit, the late 1980s saw two of Effros' problems from [78] solved: Blackadar showed that the fixed-point algebra of a $\mathbb{Z}/2\mathbb{Z}$ action on an AF algebra need not be AF ([9]), and Evans and Kishimoto showed that removing an AF tensor factor from an AF algebra need not give an AF algebra ([106]). These and related developments brought renewed interest in classifying inductive limits of general type I building blocks — again one of Effros' problems — particularly AI and A$\mathbb{T}$ algebras,[13] and A$\mathbb{T}$ structure was found in important examples coming from dynamics ([230, 93], the latter of which was also roughly conjectured in Effros' [78]). In his pioneering classification of A$\mathbb{T}$ algebras of real rank zero ([88]), Elliott enlarged his previous invariant, used to

[10] Following Rørdam's short argument in [237], Elliott's original proof remains unpublished.

[11] Rørdam's class comprises those unital simple separable nuclear purely infinite C^*-algebras satisfying the UCT, all of whose embeddings into simple purely infinite C^*-algebras are classified by KL-theory.

[12] See [161]. The first published proof was in collaboration with Phillips in [164].

[13] That is, inductive limits of C^*-algebras of the form $C([0,1], F)$ and $C(\mathbb{T}, F)$ respectively, where F is finite dimensional. In particular, Blackadar's example in [9] relies on presenting the CAR algebra M_{2^∞} not as an inductive limit of finite dimensional C^*-algebras, but as an A$\mathbb{T}$ algebra.

classify AF algebras, to include the K_1-group.[14] This led to his slogan "K-theory suffices", and arguably, the Elliott classification program began in earnest at this point.

It soon turned out that, outside the real rank zero setting, topological K-theory alone is not sufficient. In the first of many important contributions, Thomsen observed this in [265],[15] leading to the addition of traces (and their pairing with K_0) to the invariant. With this larger invariant, Elliott managed to classify simple AI algebras in [87], building on [15, 265], and then extended this to the A$\mathbb{T}$ case in [91]. Thus the Elliott invariant reached its modern form, and Elliott stated his classification conjecture in [89].

The need for additional data beyond the Elliott invariant — total K-theory and the Hausdorffized unitary algebraic K_1-group — to establish uniqueness theorems for *-homomorphisms was first seen in the setting of inductive limits of building blocks with topological dimension one. The role of total K-theory emerged in [65, 82] and was used by Dadarlat and Loring ([66]) to classify morphisms between certain (not necessarily simple) real rank zero inductive limits previously considered by Elliott. The new ingredient introduced in [66] was the universal multicoefficient theorem (UMCT), which was used to relate maps at the level of total K-theory to KK-classes. This has become a crucial tool in classification.

Subsequently, Nielsen and Thomsen gave a new approach to the classification of simple A$\mathbb{T}$ algebras ([209, 208]). They give examples to show that the Elliott invariant alone does not classify the morphisms and prove a uniqueness theorem for *-homomorphisms by adjoining the unitary group modulo the closure of the commutators. As A$\mathbb{T}$ algebras have stable rank one, this additional data is precisely the Hausdorffized unitary algebraic K_1-group previously studied by Thomsen ([266]) using the de la Harpe–Skandalis determinant of [68].

Moving to higher dimensional building blocks, large-scale work culminated in far-reaching classification results for approximately homogeneous (AH) C^*-algebras by Elliott and Gong ([94]) and then by Elliott, Gong, and Li ([120, 96]; preprint versions were circulated in 1998). At this point, the key hypothesis entailing classifiability was given in terms of relative upper bounds on the topological dimension of the spectrum of the building blocks of the inductive limit,[16] which was used in a careful analysis to reduce to the case of inductive limits of very specific building blocks.

At the Fields Institute in 1995, Villadsen announced his work on exotic AH algebras with perforated K-theory ([281, 282]). None of these examples can absorb $\mathcal{Z}$ tensorially, as shown by Gong, Jiang, and Su ([122]), who initiated the abstract study of $\mathcal{Z}$-stable C^*-algebras, continued in [242]. Through clever adaptations of

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[14] In talks at the time, Elliott noted inspiration from Pasnicu's use of K_1 in the classification of tensor products of Bunce–Deddens algebras ([216]) and Brown's use of K_1 to show that extension of AF algebras are AF [25].

[15] Thomsen showed that that any Choquet simplex can arise as the trace space of an AI algebra whose K_0-group is $\mathbb{Q}$ with the usual order. Examples which can be obtained from the work of Goodearl in [128] (see [240, Proposition 3.1.8]) show that it is also necessary to adjoin the pairing to the invariant. The need for traces was a big revelation at the time.

[16] A strong condition of *no dimension growth*, meaning uniform bound on the dimension of the building block's spectrum, is required in [96]. Gong's decomposition theorem [120] relaxes this to *very slow dimension growth* relative to the size of the matrix algebras appearing in an inductive limit. See [240, Chapter 3] for a discussion and definitions. An early dimension-reduction result (of a different flavor) for simple C^*-algebras can be found in [67].

Villadsen's constructions, Rørdam ([241]) and Toms ([271]) produced strong counterexamples to Elliott's classification conjecture in 2002 and 2003. It is around this time that $\mathcal{Z}$-stability was first suggested as a potential abstract hypothesis for classification; see [271, Section 5] and [242, Page 64].

Inductive limits of subhomogeneous C^*-algebras (ASH algebras)[17] provided further challenges, needing the techniques described in Sections 1.2.4 and 1.2.5, and it was not until the unital classification theorem in 2015 when unital simple ASH algebras with slow dimension growth (or, equivalently by [272, 291], $\mathcal{Z}$-stable ASH algebras) were classified. Particularly important examples came much earlier: Elliott exhausted the range of the invariant of stably finite classifiable C^*-algebras in [90], using inductive limits of certain subhomogeneous C^*-algebras with at most 2-dimensional spectrum, and Jiang and Su's classification of simple inductive limits of sums of dimension drop C^*-algebras underpins their work on $\mathcal{Z}$ and strong self-absorption ([144], building on [95]). It remains an open problem whether every stably finite simple separable nuclear C^*-algebra is ASH.

1.2.4. *Tracially AF C^*-algebras.* Returning to the setting of stably finite simple nuclear C^*-algebras, the major task around the turn of the millennium, articulated by Lin in his foundational paper [176], was to "...establish a classification result for C^*-algebras that are not assumed to be direct limits of some special form."

The impetus for the next phase of developments was given by breakthrough results for quasidiagonal C^*-algebras: [13], and especially Popa's approximation property ([228]). This allows one to obtain finite dimensional subalgebras from quasidiagonality in the presence of sufficiently many projections. In particular, if A is a unital simple quasidiagonal C^*-algebra of real rank zero, then for any finite subset $\mathcal{F}$ of A, one can find a non-zero approximately central projection p such that the corner pAp contains a finite dimensional unital subalgebra approximately containing the compression $p\mathcal{F}p$. Popa's work was motivated in part by Effros' problem to abstractly characterise AF algebras ([78]); the approximation property is based on Popa's earlier 2-norm version of the approximation property for von Neumann algebras, where maximality allows one to take $p = 1$, and recover Connes' Theorem that injective II_1 factors are hyperfinite ([226]). Likewise, if one can always take $p = 1$ in Popa's condition for a C^*-algebra A, then A must be AF. Lin's innovation was to ask for p to be large in a suitable sense[18] using this concept to introduce the class of *tracially AF* (TAF) C^*-algebras ([176]).

Lin obtained a classification theorem for unital simple separable nuclear tracially AF algebras satisfying the UCT in [175], and using a range of the invariant result, showed that this class coincides with unital simple AH algebras of real rank zero and slow dimension growth ([179]).

The classification of TAF algebras instigated the systematic use of absorption in the stably finite setting, prompting the "stable uniqueness" and "stable existence" theorems of Dadarlat–Eilers ([63]) and Lin ([177]) together with a novel use of the UCT as a way of controlling the complexity of the stabilizations in KK-theory. Morally, Lin's TAF uniqueness theorem is obtained from stable uniqueness, hiding the stabilization in tracially small corners via the UCT. The corresponding

[17] More restrictive definitions of ASH algebras are sometimes used, especially in early literature.

[18] The precise condition requires $1 - p$ to be small in the Cuntz semigroup. Assuming strict comparison, this is equivalent to p being uniformly large in all traces, so one tends to say that p is "tracially large".

existence theorem is even more difficult and proceeds to build approximately multiplicative maps $A \to B$ by factoring through explicitly constructed model algebras C with extremely precise internal structure. Lin uses Blackadar and Kirchberg's work on simple nuclear quasidiagonal C^*-algebras ([13, 14]) to produce approximately multiplicative maps $A \to C$ ([180, Theorem 4.2]) and the classification of simple real rank zero AH algebras for approximately multiplicative maps $C \to B$ ([175, Corollary 4.4 and Definition 3.1]).[19]

To use Lin's theorem one also needs tools for obtaining tracial approximations more abstractly. A number of dynamical examples were brought within the scope of this classification by various ad hoc means (for example [178, Corollary 3.13, Theorem 3.6], [192]). Winter was the first to provide a systematic approach in 2005, via the *decomposition rank* ([288]). The decomposition rank, introduced by Kirchberg and Winter in [168], is another important non-commutative covering dimension, which preceded Winter and Zacharias' nuclear dimension. The two notions are subtly related, but having finite decomposition rank is a strong condition, entailing quasidiagonality, and can therefore be challenging to verify directly, although it does cover ASH algebras with no dimension growth ([287]).

For a unital simple separable C^*-algebra with real rank zero and finite decomposition rank,[20] Winter obtains a "Popa approximation" where the projection p is bounded below in all traces in terms of the decomposition rank. Repeating the argument inductively in the complementary corner leads to well-approximated corners of arbitrarily large trace, proving such algebras are tracially AF. In particular (together with the later [289]), Winter's result gave the first classification of simple ASH algebras of real rank zero and no dimension growth.

1.2.5. *Localization and rational tracial approximation.* The potential absence of projections in simple C^*-algebras is a serious obstruction to obtaining general classification results by means of tracial approximations. Winter's "localization technique" ([292], which was first circulated as a preprint in 2007) overcomes this difficulty via an ingenious use of the Jiang–Su algebra $\mathcal{Z}$. By viewing $\mathcal{Z}$ as an inductive limit of generalized dimension drop algebras with UHF fibers as in [244], Winter gives a strategy to pass from a classification of UHF-stable algebras to a classification of $\mathcal{Z}$-stable algebras. This localization technique comes at a heavy price: at the UHF-stable level, one must classify *-isomorphisms up to "strong asymptotic unitary equivalence" instead of approximate unitary equivalence — that is, sequences of unitaries are replaced with one-parameter families of unitaries starting at the unit. There are considerable extra K-theoretic obstructions to uniqueness in this setting,[21] and to account for this, correspondingly stronger existence results are also required.

Strong asymptotic classification results were nevertheless proved for C^*-algebras with tracial approximations: Lin and Niu achieved this for simple separable nuclear TAF C^*-algebras with the UCT in [183, 189]. There is a real quantum leap in difficulty over approximate classification. The fact that KK-theory (in contrast to KL)

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[19]A streamlined approach to this classification (which also obtains stronger results for embeddings) was developed by Dadarlat in [60]. This more general existence result also factors through models in a similar way.

[20]At the time, $\mathcal{Z}$-stability was also needed but was later shown to be redundant in [289]; unital simple separable infinite dimensional C^*-algebras of finite decomposition rank are $\mathcal{Z}$-stable.

[21]Among other things, one must consider KK instead of its Hausdorffization KL.

fails to preserve inductive limits calls for highly intricate analysis involving careful use of so-called "basic homotopy lemmas" ([184]), allowing approximately central unitaries to be connected by approximately central homotopies. This concept goes back to [23].

In 2013, Matui and Sato used the Lin–Niu–Winter results together with breakthrough work on the Toms–Winter conjecture to give the first truly abstract definitive stably finite classification theorem: unital simple separable nuclear $\mathcal{Z}$-stable quasidiagonal monotracial C^*-algebras satisfying the UCT are classified by Elliott's invariant ([201]).[22] They show directly that the UHF-stabilizations of such a C^*-algebra A (possibly without the UCT) are TAF (A is said to be *rationally TAF*).[23] This class covers C^*-algebras arising from a minimal uniquely ergodic homeomorphism of a finite dimensional compact metrizable space with infinitely many points, using [276] and [222] to prove the regularity and quasidiagonality hypotheses, respectively.

1.2.6. *The Gong–Lin–Niu class and classification by embeddings.* Stepping back a bit, following Lin's classification of tracially AF C^*-algebras, there was rapid progress in the classification of C^*-algebras with tracial approximations by more complicated type I building blocks such as interval algebras ([182]). The new target of strong asymptotic classification promoted by localization was first obtained for TAI algebras in [185]. This sparked the line of research culminating in the Gong–Lin–Niu strong asymptotic classification ([125, 126]) of unital simple separable nuclear UHF-stable C^*-algebras which satisfy the UCT and are tracially approximated by non-commutative 1-dimensional CW-complexes also known as Elliott–Thomsen building blocks.[24] The relevance of the Gong–Lin–Niu class is that it exhausts the values of the Elliott invariant on unital simple separable nuclear stably finite UHF-stable C^*-algebras. No larger class of building blocks would be needed.

Gong, Lin, and Niu's argument is exceptionally technically demanding and involves detailed analysis and many layers of quantification and approximation. Just as in the setting of real rank zero inductive limits of no dimension growth, where the ASH case proved an order of magnitude more challenging than the AH case (and came a decade later through tracial approximations), the subhomogeneous nature of the Elliott–Thomsen building blocks present huge technical hurdles that must be overcome. Highly bespoke homotopy lemmas and existence and uniqueness results between the building blocks are needed throughout the paper. The arguments also depend on elaborately constructed explicit models, through which to factor their existence and uniqueness results. It is an incredible achievement to put all this together and overcome all these difficulties, yet the daunting length and extreme technicality of [125, 126] have made it very difficult for researchers to fully absorb the entire argument.

[22]Via their earlier breakthrough [200], one could replace $\mathcal{Z}$-stability by strict comparison here.

[23]Their paper is focused around the Toms–Winter conjecture and shows that such a C^*-algebra A has finite decomposition rank, which enables them to obtain the tracial approximations for UHF-stabilizations from [288, 289]. Matui and Sato also give a direct argument which parallels the last part of Connes' proof of injectivity implies hyperfiniteness for II_1 factors; see the introduction to [201].

[24]The Gong–Lin–Niu theorem was announced in Oberwolfach in 2012 ([121, 187]) and was circulated as a preprint in early 2015 ([123]).

Given the scope of Gong, Lin, and Niu's work, research on the stably finite case of the unital classification theorem pivoted to the task of determining abstractly which UHF-stable C^*-algebras fall into the Gong–Lin–Niu class. Shortly after Matui and Sato's abstract approach to rational TAF structure, Winter introduced a general technique of *classification by embeddings* ([293]), and applied it (together with [233, 259]) to bring C^*-algebras associated to minimal diffeomorphisms on odd spheres within the scope of the Elliott program by means of rational TAI classification. Winter's result shows that a significantly weaker form of tracial approximation implies rational (Lin-style) tracial approximations in the presence of finite nuclear dimension. Combined with a classification result by Robert ([233]), Winter's technique only requires appropriate external approximations in order to get (internal) rational tracial approximations. As a proof of concept, these external approximations arise immediately for quasidiagonal monotracial C^*-algebras.[25]

Classification by embeddings was to become the route to the Gong–Lin–Niu class. For example, as precursors to the full classification theorem, it was used by Lin to classify simple $\mathbb{Z}$-crossed products of finite dimensional spaces ([188]) and by Elliott, Gong, Lin, and Niu to classify simple $\mathcal{Z}$-stable ASH algebras ([97]).

1.2.7. *Stable uniqueness across the interval and quasidiagonality.* Beyond these precursor results, major breakthroughs quickly followed the Gong–Lin–Niu preprint ([123]) throughout 2015. Demonstrating a new technique for obtaining Winter-style external approximations, Niu announced joint work with Elliott ([100]) in April 2015, that C^*-algebras with finite decomposition rank and $K_0(A) \otimes \mathbb{Q} \cong \mathbb{Q}$ are rationally TAI. This was quickly followed by Elliott, Gong, Lin, and Niu's general result ([98]; circulated as a preprint in July 2015) which removed the K_0 hypothesis, and therefore showed all unital simple separable nuclear C^*-algebras with finite decomposition rank satisfying the UCT, are rationally in the Gong–Lin–Niu class, and so classified by [125, 126]. The $\mathcal{Z}$-stability needed for classification is obtained from Winter's first $\mathcal{Z}$-stability theorem ([289]).

The fundamental new idea in [100, 98] is a *stable uniqueness across the interval* argument to connect two maps ϕ and ψ into the universal UHF algebra (agreeing suitably on invariants) by a homotopy of approximately multiplicative maps with approximately constant tracial behavior. The UCT enters to control the matrix size required in the stable uniqueness theorem. This is then applied to maps arising from quasidiagonality and a range of invariant result to produce Winter-style approximations into (UHF-stabilizations of) Elliott–Thomsen building blocks.

As noted in Section 1.2.4, finite decomposition rank is much harder to verify in examples than finite nuclear dimension. In part, this is because finite decomposition rank entails quasidiagonality. Right back in his foundational TAF paper ([176]), Lin noted that "recent developments suggest one may further assume simple (nuclear stably finite) C^*-algebras are quasidiagonal", and indeed Elliott's examples exhausting the range of the invariant ([90]) amongst stably finite C^*-algebras are all quasidiagonal, and moreover, all their traces are quasidiagonal.[26]

While there are many deep theorems giving quasidiagonality in examples, in the abstract setting this is a major challenge: Blackadar and Kirchberg's famous question ([13, Question 7.3.1]) of whether all nuclear stably finite C^*-algebras are

[25]See [293, Corollary 2.4], which at the time was a special case of [201, Theorem 6.1].

[26]Brown was the first to notice the importance of quasidiagonality of traces in C^*-classification in his memoir [28].

quasidiagonal remains open. In fact, the Elliott–Gong–Lin–Niu theorem isolates quasidiagonality of traces as a key hypothesis: their theorem holds for unital simple separable finite C^*-algebras of finite nuclear dimension,[27] satisfying the UCT, such that all traces are quasidiagonal.[28]

A little earlier in 2014, Ozawa, Rørdam, and Sato showed that C^*-classification results could give an unexpected route to quasidiagonality — even for algebras that are far from simple — by using Matui and Sato's [201] to show that elementary amenable groups have quasidiagonal C^*-algebras ([214]). Inspired by this, the Kirchberg–Phillips theorem, and talks on [100, 98], AT, SW and Winter used a stable uniqueness across the interval argument to prove their quasidiagonality theorem ([269]) over the summer of 2015: faithful traces on a separable nuclear C^*-algebra satisfying the UCT are quasidiagonal. This established Rosenberg's conjecture — C^*-algebras associated to discrete amenable groups are quasidiagonal — and removed the quasidiagonality hypothesis from [98]. In this way, the combination of [125, 126, 98, 269, 293, 291, 292] and the large body of work these papers rely on comes together to prove the unital stably finite classification theorem (with the hypothesis of finite nuclear dimension replacing that of $\mathcal{Z}$-stability).

1.2.8. *Post-2015: $\mathcal{Z}$-stability and abstract classification.* Developments continued apace post 2015 in numerous directions. These included classification results in the non-unital setting, a vast body of work bringing examples within the scope of the unital classification theorem as discussed in Section 1.1.4, and the continued large-scale study of fine structure in the Cuntz semigroup. We do not describe these here.

The present paper has its origins in two results announced at BIRS in 2017 ([102]). Firstly Castillejos, Evington, AT, SW, and Winter showed that $\mathcal{Z}$-stable unital simple separable nuclear C^*-algebras have nuclear dimension at most one ([35]). This allows the unital classification theorem to be accessed from $\mathcal{Z}$-stability.[29] Secondly, building on his abstract approach to the quasidiagonality theorem ([252]), CS announced a precursor of what would become his AF-embedding theorem ([253]): a classification of unital full nuclear embeddings of unital separable nuclear UCT C^*-algebras into the ultrapower of the universal UHF algebra. The techniques provided an abstract classification of unital simple separable nuclear UHF-stable monotracial C^*-algebras satisfying the UCT which does not rely on obtaining any form of tracial approximation.[30] This was the first abstract stably finite classification theorem which does not pass through some kind of internal C^*-algebra structure in the proof.

[27] This uses Winter's second $\mathcal{Z}$-stability theorem ([291]) in place of [289].

[28] Later, through [21] and [35, 31], it was shown, without reference to the UCT, that a simple separable C^*-algebra A of finite nuclear dimension has finite decomposition rank precisely when both A and all its traces are quasidiagonal.

[29] The methods of [35] can also be combined with Matui and Sato's strategy for obtaining rational TAF structure ([201]) to prove Winter's classification by embeddings theorem without assuming finite nuclear dimension ([33]). This bypasses the only use of nuclear dimension in the 2015 proof of the unital classification theorem.

[30] The version published in [253] assumes $\mathcal{Q}$-stability and trivial K_1-group as this allows one to bypass some technicalities and is sufficient to obtain the headline AF-embedding result.

1.3. **Discussion of the methods behind Theorem B.** The overarching strategy of our paper is to combine the ideas in [253] and [35] together with a new KK-uniqueness theorem (Theorem 1.4) which allows us to avoid Winter's localization technique and directly classify $\mathcal{Z}$-stable C^*-algebras without first classifying UHF-stable algebras.

1.3.1. *Classifying full approximate embeddings.* The existence portion of Theorem B requires producing *-homomorphisms between abstract C^*-algebras. Constructing these directly is hard. Instead, the usual strategy is to classify approximate *-homomorphisms — these are sequences of maps that become *-homomorphisms in the limit. These are often easier to produce. Indeed, for example, quasidiagonality provides a rich supply of approximate *-homomorphisms into matrix algebras.

Let A and B be as in Theorem B. To avoid the need to carefully keep track of quantifiers, we work with the sequence algebra $B_\infty := \ell^\infty(B)/c_0(B)$, and *-homomorphisms $A \to B_\infty$; these encode approximate *-homomorphisms. Unitary equivalence of *-homomorphisms into B_∞ corresponds to a quantified approximate unitary equivalence of approximate *-homomorphisms into B. Although B is simple, B_∞ is not, so we impose a simplicity condition on the allowed morphisms $\phi\colon A \to B_\infty$: we require that $\phi(a)$ generates B_∞ as an ideal for each non-zero $a \in A$. Such maps are called *full*, and (under our hypotheses on B) fullness is readily tested using traces via Cuntz comparison.

With the setup above, our main objective in this paper is to prove the following theorem.

Theorem 1.1 (Classification of unital full approximate embeddings; see Theorem 9.1)**.** *Let A be a unital separable nuclear C^*-algebra satisfying the UCT and B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra. Up to unitary equivalence, the unital full embeddings $A \to B_\infty$ are classified by faithful morphisms on $\underline{K}T_u$.*

The deduction of Theorem B from Theorem 1.1 is a standard intertwining argument. For this, it is vital that the classification in Theorem 1.1 contains both uniqueness and existence statements. This is set out in Section 9.

As with Theorem B, the purely infinite case of Theorem 1.1 is a known consequence of the methods behind the Kirchberg–Phillips classification theorem, while our proof handles the finite (i.e., tracial) situation. We proceed by dividing B_∞ into two pieces: a "tracially small" part,

$$J_B := \{b \in B_\infty : \tau(b^*b) = 0 \text{ for all } \tau \in T(B_\infty)\} \lhd B_\infty, \tag{1.2}$$

and a "tracially large" part, $B^\infty := B_\infty/J_B$. These are known as the *trace-kernel ideal* and the *trace-kernel quotient*, respectively. In this way, B_∞ fits into the *trace-kernel extension*

$$0 \longrightarrow J_B \xrightarrow{j_B} B_\infty \xrightarrow{q_B} B^\infty \longrightarrow 0, \tag{1.3}$$

an ultrapower version of which first came to the fore in Matui and Sato's work on the Toms–Winter conjecture ([200, 201]; see also [167]).

Just as with CS's $\mathcal{Q}$-stable monotracial classification theorem ([253]), the trace-kernel extension allows us to organize the proof of Theorem 1.1 into two conceptual steps: classify morphisms into B^∞, and then classify lifts of these morphisms to B_∞:

$$\begin{tikzcd}
 & & & A \arrow[dl, dashed, "\substack{\text{lifts are classified by}\\ K\text{-theoretic data}}"'] \arrow[d, "{\theta,\ \text{classified by traces}}"] & \\
0 \arrow[r] & J_B \arrow[r, "j_B"] & B_\infty \arrow[r, "q_B"] & B^\infty \arrow[r] & 0.
\end{tikzcd} \tag{1.4}$$

The first step only uses tracial data and follows from [35]. The major work we undertake here is the classification of lifts. The precise data needed for the existence of a lift lives in $KK(A, B_\infty)$. On the other hand, the data needed for uniqueness resides in a different location, the group $KL(A, J_B)$ (a Hausdorffization of $KK(A, J_B)$).

Theorem 1.2 (Classification of unital lifts along the trace-kernel extension; see Theorem 7.10)**.** *Let A be a unital separable nuclear C^*-algebra and B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Suppose that $\theta\colon A \to B^\infty$ is a unital full embedding.*

(i) There is a unital full embedding $A \to B_\infty$ lifting θ if and only if there is an element in $KK(A, B_\infty)$ lifting $[\theta]_{KK(A,B^\infty)}$ and preserving the unit of K_0.

(ii) Any choice of a unital full embedding $A \to B_\infty$ lifting θ determines a bijection between $KL(A, J_B)$ and the unitary equivalence classes of unital full lifts of θ.

The invariants in Theorem 1.2 are somewhat unwieldy. Once Theorem 1.2 is proved, our last major task will be to compute $KL(A, J_B)$ using the UCT, allowing the classification of lifts to be reinterpreted in terms of the total invariant.

Next, we provide a bit more detail regarding these components and how they come together to prove Theorem 1.1. We follow this by a discussion of the role of the UCT and $\mathcal{Z}$-stability, and a brief comparison of our methods with the techniques in the tracial approximation based approach.

1.3.2. *Classification into the trace-kernel quotient.* It is an important consequence of Connes' theorem that maps from separable nuclear C^*-algebras A into finite von Neumann algebras are classified by traces. When B has a unique trace, this provides a classification of *-homomorphisms into the trace-kernel quotient.[31] In general, neither B^∞ nor B^ω is a von Neumann algebra, but the techniques introduced by Castillejos, Evington, AT, SW, and Winter in [35] implicitly show that when B is $\mathcal{Z}$-stable, B^∞ behaves enough like an ultraproduct of von Neumann algebras for our purposes. In particular, one still has a classification of maps into B^∞ by traces. The precise classification result we need was recorded in the short sequel [33] to [35] for use in this paper.

Theorem 1.3 (Classification of unital morphisms into the trace-kernel quotient; see Theorem 6.5)**.** *Let A be a unital separable nuclear C^*-algebra and B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Up to unitary equivalence, the unital *-homomorphisms $A \to B^\infty$ are classified by traces.*

We describe the trace-kernel extension in Section 6, giving a little more detail about how one thinks of B^∞ as being "von Neumann-like".

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[31] This is more often stated with the tracial ultrapower B^ω associated to a free ultrafilter ω on $\mathbb{N}$ — which is a II_1 factor — in place of B^∞, but it is routine to convert between these classifications.

1.3.3. *Classification of lifts along the trace-kernel extension (Theorem 1.2).* The strategy for the proof of Theorem 1.2 builds on CS's work on the non-stable KK-theory of the trace-kernel ideal J_B from [252, 253] which essentially proves Theorem 1.2 under the additional hypotheses that B is $\mathcal{Q}$-stable and monotracial with $K_1(B) = 0$ (see Remark 7.4). Absorption — in the spirit of Voiculescu's Weyl–von Neumann theorem — is the fundamental tool both in [252, 253] and our arguments.

For a unital full embedding $\theta\colon A \to B^\infty$, we may take the pullback of the trace-kernel extension along θ to form a commutative diagram

$$\begin{tikzcd} \mathsf{e}\colon 0 \arrow[r] & J_B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\theta"] & 0 \\ 0 \arrow[r] & J_B \arrow[r, "j_B"] & B_\infty \arrow[r, "q_B"] & B^\infty \arrow[r] & 0. \end{tikzcd} \tag{1.5}$$

The extension e in the top row of (1.5) defines a class in $\mathrm{Ext}(A, J_B)$. Let us pretend that J_B is separable and stable.[32] Given a KK-lift of θ in $KK(A, B_\infty)$ as in the statement of Theorem 1.2(i), it follows that the Ext-class of e vanishes, so e stably splits, i.e., there exists a trivial (i.e., split) extension t such that $\mathsf{e} \oplus \mathsf{t}$ splits. This is where absorption enters in the existence part of Theorem 1.2: if we can show e is absorbing, then $\mathsf{e} \cong \mathsf{e} \oplus \mathsf{t}$ so that e itself splits, providing the required lift of θ. This overall strategy is an abstract version of CS's proof in [252] of the quasidiagonality theorem of [269].

Absorption is equally crucial in the second part of Theorem 1.2, but appears in a slightly different guise. A pair of lifts $\phi, \psi\colon A \to B_\infty$ of θ gives rise to a class $[\phi, \psi]_{KK(A,J_B)}$ in $KK(A, J_B)$ via the Cuntz–Thomsen description of KK-theory.[33]

$$\begin{tikzcd} & & & A \arrow[dl, shift right, "\phi"'] \arrow[dl, shift left, "\psi"] \arrow[d, "\theta"] & \\ 0 \arrow[r] & J_B \arrow[r, "j_B"'] \arrow[d, equal] & B_\infty \arrow[r, "q_B"'] \arrow[d, "\mu"] & B^\infty \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J_B \arrow[r] & \mathcal{M}(J_B) \arrow[r] & \mathcal{Q}(J_B) \arrow[r] & 0 \end{tikzcd} \tag{1.6}$$

Our objective is to show that ϕ and ψ are unitarily equivalent if and only if the associated class $[\phi, \psi]_{KL(A,J_B)}$ vanishes. This is achieved through a "$\mathcal{Z}$-stable KL-uniqueness theorem".

Uniqueness theorems for KK and KL are subtle and intimately involve absorption. For example, via the *stable uniqueness theorems* of Dadarlat–Eilers and Lin from [63] and [177] (and continuing to pretend that J_B is separable and stable), if

[32] The trace-kernel J_B is massive. It is neither separable, nor even σ-unital, which requires us to take some care with KK-theory. Moreover J_B is not stable, but as we discuss below, it is *separably stable* (every separable subalgebra of J_B is contained in a stable separable subalgebra). The analysis that follows is really done on a suitable separable and stable subextension of the top row of (1.5).

[33] With $\mathcal{M}(J_B)$ as the multiplier algebra of J_B, the trace-kernel extension induces a canonical map $\mu\colon B_\infty \to \mathcal{M}(J_B)$, and then the maps $(\mu \circ \phi, \mu \circ \psi)\colon A \rightrightarrows \mathcal{M}(J_B)$ form an (A, J_B)-Cuntz pair, i.e., $\mu(\phi(a)) - \mu(\psi(a)) \in J_B$ for $a \in A$.

$[\phi,\psi]_{KL(A,J_B)}$ vanishes, then $\mu\circ\phi$ and $\mu\circ\psi$ become properly approximately unitarily equivalent after adjoining any absorbing representation $\eta\colon A\to\mathcal{M}(J_B)$.[34] If $\mu\circ\phi$ and $\mu\circ\psi$ are already known to be absorbing, then we can use these for η in two applications of stable uniqueness to obtain proper approximate unitary equivalence of $(\mu\circ\phi)\oplus(\mu\circ\phi)$ and $(\mu\circ\psi)\oplus(\mu\circ\psi)$. Assuming further that B_∞ and J_B absorb the UHF algebra M_{2^∞} tensorially,[35] a key observation in [253] is that one can remove the two-fold amplification and lift back to B_∞ to obtain unitary equivalence of ϕ and ψ. Importantly, the unitaries are in the minimal unitization of J_B, and hence in B_∞, not just in $\mathcal{M}(J_B)$. We view this result as a "UHF-stable KL-uniqueness theorem," and it can be used to prove our objective whenever B absorbs a UHF algebra of infinite type.

In this paper, some major new ingredients are KK- and KL-uniqueness theorems that work directly with $\mathcal{Z}$-stability in place of UHF-stability. We state the KK-version below; the KL-version is similar, instead characterizing proper *approximate* unitary equivalence. The strategy behind our KK-uniqueness theorem goes back to ideas in Dadarlat and Eilers' stable uniqueness theorem ([63]), as discussed further in Section 5.4.

Theorem 1.4 ($\mathcal{Z}$-stable KK-uniqueness theorem; see Theorem 5.15(i))**.** *Let A be a separable C^*-algebra and I be a σ-unital and stable C^*-algebra. Suppose that $\phi,\psi\colon A\to\mathcal{M}(I)$ are absorbing and $\phi(a)-\psi(a)\in I$ for all $a\in A$. Then the following are equivalent:*

(i) $[\phi,\psi]_{KK(A,I)}=0$

(ii) $\phi\otimes 1_{\mathcal{Z}},\psi\otimes 1_{\mathcal{Z}}\colon A\to\mathcal{M}(I\otimes\mathcal{Z})$ *are properly asymptotically unitarily equivalent.*

The Dadarlat–Eilers stable uniqueness theorem obtains the equivalence of (i) and

(iii) $\phi\oplus\eta,\psi\oplus\eta$ *are properly asymptotically unitarily equivalent for all absorbing* $\eta\colon A\to\mathcal{M}(I)$.

Using $\mathcal{Z}$-stability of B to remove the tensor factor of $1_{\mathcal{Z}}$ (and taking due care with issues around separability and tensorial absorption), lifting the outcome of the $\mathcal{Z}$-stable KL-uniqueness theorem back to B_∞, is exactly what we need to prove our objective — provided we can show that the maps $\mu\circ\phi$ and $\mu\circ\psi$ from (1.6) are suitably absorbing.

1.3.4. *Obtaining absorption.* Elliott and Kucerovsky's vast generalization ([99]) of Voiculescu's theorem provides a criterion — "pure largeness" — that abstractly characterizes absorption under nuclearity constraints. The precise definition of pure largeness is not so important to our paper, as there are now tools enabling it to be accessed through regularity properties such as $\mathcal{Z}$-stability ([172, 212]). This gives rise to an absorption theorem stating that an extension of a separable nuclear C^*-algebra by a σ-unital stable $\mathcal{Z}$-stable ideal is absorbing if and only if the (forced) unitization of the extension is full.

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[34] That is, $(\mu\circ\phi)\oplus\eta$ and $(\mu\circ\psi)\oplus\eta$ are approximately unitarily equivalent via unitaries in the minimal unitization of J_B.

[35] Even when B is M_{2^∞}-absorbing, B_∞ and J_B are not, for similar reasons that J_B is not stable. However, in this case, they will be "separably M_{2^∞}-stable", which is enough to perform the argument working in a suitable separable subextension of (1.3).

Using the version of the Elliott–Kucerovsky theorem described above requires separable stability and $\mathcal{Z}$-stability of J_B. The separable $\mathcal{Z}$-stability of J_B follows directly from the $\mathcal{Z}$-stability of B. Hjelmborg and Rørdam characterized stability for σ-unital C^*-algebras in [141]; this also characterizes separable stability. CS introduced the ad hoc notion of an *admissible kernel* for the purpose of obtaining separable stability of J_B when $B = \mathcal{Q}$ ([252]) (and more generally, when B is a unital simple $\mathcal{Q}$-stable AF algebra with unique trace ([253])). Both of these reduce to the Hjelmborg–Rørdam characterization. Here we use Cuntz semigroup methods to verify the Hjelmborg–Rørdam conditions for J_B from strict comparison (and hence from $\mathcal{Z}$-stability).

The final detail in obtaining absorption is a "de-unitization" trick. The unital extensions that we use can never be absorbing, as they never absorb a non-unital extension. As in [252, 253], we resolve this by taking the direct sum with the zero extension — see Section 7.4.

It is tempting to try and replace the trace-kernel quotient B^∞ with a von Neumann algebra $\mathcal{M}$, such as an ultrapower of the finite part of the bidual of B. In this case, $\mathcal{M}$ would encode all the traces on B and one has the classification of maps $A \to \mathcal{M}$ by traces directly from Connes' theorem, and Theorem 1.3 (and the work of [35]) would not be needed. However, the ideal in the ultrapower B_ω corresponding to the quotient onto $\mathcal{M}$ is not separably stable, obstructing the route to absorption (see Remark 6.12). This prevents us from lifting the classification back from $\mathcal{M}$ to B_ω. Loosely speaking, we must work with 2-norm estimates that are uniform over the traces in order to take advantage of strict comparison in B.

1.3.5. *Computing KL-theory of the trace-kernel ideal.* The major remaining task is to compute the group $KL(A, J_B)$ by combining the universal multicoefficient theorem with total K-theory computations for the trace-kernel extension.

For any C^*-algebra D, the Kasparov product induces a natural map

$$KL(A, D) \longrightarrow \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(D)), \tag{1.7}$$

where $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(D))$ is the group of morphisms in total K-theory. When A satisfies the UCT, Dadarlat and Loring's universal multicoefficient theorem shows that (1.7) is an isomorphism. Applying this to the trace-kernel extension allows $KL(A, J_B)$ to be computed using the resulting exact sequence:

$$\begin{aligned} 0 \longrightarrow \ker \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(j_B)\big) \longrightarrow KL(A, J_B) &\longrightarrow \\ \longrightarrow \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B_\infty)\big) &\longrightarrow \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B^\infty)\big). \end{aligned} \tag{1.8}$$

The first and fourth terms above can be simplified using the von Neumann algebra-like behavior of B^∞: $K_1(B^\infty) = 0$ and the pairing map $\rho_{B^\infty} \colon K_0(B^\infty) \to \mathrm{Aff}\, T(B_\infty)$ is an isomorphism,[36] just as the K_0-group of a II_1 factor is identified with $\mathbb{R}$ using the trace. Consequently, $\underline{K}(B^\infty)$ and $\ker \underline{K}(j_B)$ are concentrated in their K_0 and K_1 components respectively, so the last entry in (1.8) is isomorphic to $\mathrm{Hom}(K_0(A), \mathrm{Aff}\, T(B_\infty))$ (see Proposition 8.6 and (8.19)). This allows us to determine $KL(A, J_B)$ by computing the maps in (1.8) explicitly.

[36] The regularity hypotheses on B ensure that the map $q_B \colon B_\infty \to B^\infty$ induces an isomorphism of traces, so we can and do identify $\mathrm{Aff}\, T(B^\infty)$ with $\mathrm{Aff}\, T(B_\infty)$.

These computations can be cleanly interpreted in terms of Hausdorffized unitary algebraic K_1, as developed by Thomsen. For a C^*-algebra D, $\overline{K}_1^{\mathrm{alg}}(D)$ is defined as the quotient of the infinite unitary group of D by the closure (in the inductive limit topology) of its commutator subgroup and there is a natural surjection $\not{a}_D\colon \overline{K}_1^{\mathrm{alg}}(D) \to K_1(D)$. Thomsen used the de la Harpe–Skandalis determinant to provide a natural sequence

$$K_0(D) \xrightarrow{\rho_D} \operatorname{Aff} T(D) \xrightarrow{\mathrm{Th}_D} \overline{K}_1^{\mathrm{alg}}(D) \xrightarrow{\not{a}_D} K_1(D) \longrightarrow 0 \tag{1.9}$$

that is exact at $\overline{K}_1^{\mathrm{alg}}(D)$. Here, ρ_D is the pairing between K_0 and traces, and the *Thomsen map* Th_D can be constructed from an inverse of the de la Harpe–Skandalis determinant (see Section 2.2, particularly (2.23)). When we take $D = B_\infty$, a fragment of the six-term exact sequence of the trace-kernel extension corresponds to Thomsen's sequence as in the following theorem.

Theorem 1.5 (Calculating $K_1(J_B)$; see Theorem 6.8)**.** *Let B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Then there is a natural isomorphism $\omega_B\colon K_1(J_B) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$ that (together with the computations $K_0(B^\infty) \cong \operatorname{Aff} T(B_\infty)$ and $K_1(B^\infty) = 0$) gives the commutative diagram*

$$\begin{tikzcd}
K_0(B_\infty) \arrow[r, "K_0(q_B)"] \arrow[d, equal] & K_0(B^\infty) \arrow[r, "\partial"] \arrow[d, "\cong"] & K_1(J_B) \arrow[r, "K_1(j_B)"] \arrow[d, "\omega_B"] & K_1(B_\infty) \arrow[r] \arrow[d, equal] & 0 \\
K_0(B_\infty) \arrow[r, "\rho_{B_\infty}"] & \operatorname{Aff} T(B_\infty) \arrow[r, "\mathrm{Th}_{B_\infty}"] & \overline{K}_1^{\mathrm{alg}}(B_\infty) \arrow[r, "\not{a}_{B_\infty}"] & K_1(B_\infty) \arrow[r] & 0.
\end{tikzcd} \tag{1.10}$$

Putting this all together, and keeping track of the maps explicitly, allows us to compute $KL(A, J_B)$ in terms of the enriched invariant $\underline{K}T_u(\,\cdot\,)$.

Theorem 1.6 (Calculating $KL(A, J_B)$; see Theorem 8.9)**.** *Let A be a unital separable C^*-algebra satisfying the UCT and let B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Then the sequence in* (1.8) *is exact. Moreover:*

(i) The third arrow in (1.8) *is a group homomorphism from $KL(A, J_B)$ onto the set of $\underline{K}T_u$-morphisms $\underline{K}T_u(A) \to \underline{K}T_u(B_\infty)$ that vanish on the trace.*
(ii) Given a full *-homomorphism $\psi\colon A \to B_\infty$, the kernel of the above homomorphism identifies with the set of $\underline{K}T_u$-morphisms $\underline{K}T_u(A) \to \underline{K}T_u(B_\infty)$ that agree with ψ on total K-theory and traces.

Compatibility with $\zeta^{(n)}$ is essential in establishing surjectivity in Theorem 1.6(ii). In slightly more detail, given a $\underline{K}T_u$-morphism $(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B_\infty)$ that agrees with ψ on total K-theory and traces, $\beta - \overline{K}_1^{\mathrm{alg}}(\psi)$ provides a group morphism $r\colon K_1(A) \to \ker K_1(j_B)$ via (1.10). In order to extend this map to total K-theory, as in the first term of (1.8), r must vanish on torsion. This is equivalent to compatibility with $\zeta^{(n)}$. See Sections 3.1 and 8.3.

Theorem 1.1 follows by combining the existence and uniqueness parts of Theorems 1.3, 1.2, and 1.6.

1.3.6. *The role of the UCT and $\mathcal{Z}$-stability.* The UCT appears twice in the proof. First, the UCT is needed to compute the group $KL(A, J_B)$ in terms of $KK(A, B_\infty)$ and traces, via the universal multicoefficient theorem, as described in Section 1.3.5

above. Second, the UCT is needed to show that the $\mathrm{Ext}(A, J_B)$ obstruction in Theorem 1.2(i) vanishes, by producing a required KK-lifting. This application of the UCT is analogous to the role it plays in CS's approach ([252]) to the quasidiagonality theorem (see Remark 7.3). It should be noted that assuming the existence of a "trace-compatible" element of $KK(A, B_\infty)$ does not appear to be sufficient for this step, as the UCT is still needed to show this element lifts a given morphism $A \to B^\infty$ in $KK(A, B_\infty)$.

In the classification of UCT Kirchberg algebras by K-theory, the UCT is only needed to lift an isomorphism at the level of K-theory to a KK-equivalence. The more subtle uses of the UCT in the stably finite setting described above mean that classification currently requires the UCT and cannot (yet) be formulated in terms of a combination of KK-theory and traces.

In contrast, regularity — i.e., $\mathcal{Z}$-stability of the codomain B — is ever-present in the argument. Jiang–Su stability is a rich hypothesis, with many ramifications. In the main body of the paper, we will typically state results with $\mathcal{Z}$-stability as the hypothesis, even if only a consequence of $\mathcal{Z}$-stability is needed. Here, we give a flavor of which aspects of $\mathcal{Z}$-stability are actually being used at distinct points in the argument.

Two consequences of $\mathcal{Z}$-stability are pertinent: unital simple finite $\mathcal{Z}$-stable C^*-algebras have *strict comparison* and *uniform property* Γ.[37] Strict comparison is a Cuntz semigroup condition, and we typically use this to obtain properties of the trace kernel ideal — in particular, the route to absorption really runs through comparison[38] — and to control the traces on B_∞.[39] Likewise, although it doesn't appear explicitly in this paper, it is really uniform property Γ rather than the full force of $\mathcal{Z}$-stability that gives B^∞ its von Neumann-like behavior. We also use that simple finite unital $\mathcal{Z}$-stable C^*-algebras have stable rank one as a technical convenience (to access cancellation) in the proof.

In contrast, our proof of the $\mathcal{Z}$-stable KK- and KL-uniqueness theorems use the full force of $\mathcal{Z}$-stability, not its consequences. We use both an explicit construction of $\mathcal{Z}$ (in the proof of Jiang's theorem; see Section 4.2) and strong-self absorption. This is the only place in the argument where strong self-absorption of $\mathcal{Z}$ is used directly.

1.3.7. *Comparison with classification via tracial approximations.* The $\mathcal{Z}$-stable KK- and KL-uniqueness theorem provides a major shortcut. It allows us to bypass Winter's localization techniques, and obtain a classification of morphisms into $\mathcal{Z}$-stable codomains up to approximate unitary equivalence directly, rather than needing to first obtain classification results for UHF-stable codomains up to (strong) asymptotic unitary equivalence. A significant amount of the work in Gong, Lin, and Niu's

[37] These both are analogs of properties of von Neumann algebras. For a unital simple separable nuclear C^*-algebra B with $T(B) \neq \emptyset$, $\mathcal{Z}$-stability is now known to be equivalent to the combination of strict comparison and uniform property Γ (using [242] and [33], which builds on [200, 249, 273, 167, 213]).

[38] In the main body of the paper, we use separable $\mathcal{Z}$-stability of J_B (inherited from $\mathcal{Z}$-stability of B) as one of the two ingredients to access absorption via Theorem 5.13 (for maps into the multiplier algebra of stable, $\mathcal{Z}$-stable separable subalgebras of J_B) for convenience. We could have worked with the weaker condition of unperforation of the Cuntz semigroup of J_B, and obtained this from strict comparison of B

[39] In the main body, we use $\mathcal{Z}$-stability directly for this, via Ozawa's theorem ([213]), but we could have used the harder [207, Theorem 1.2] to obtain this from strict comparison.

argument ([125, 126]) is obtaining the required asymptotic classification for localization. As we do not need asymptotic classification results, we do not prove any in this paper,[40] allowing us to work with simpler invariants throughout.

Another fundamental difference is that our approach does not require explicit models of classifiable C^*-algebras nor internal approximation structure such as inductive limits or tracial approximations. Indeed, the only place where internal structure arises in the argument is through Connes' theorem that injective von Neumann algebras (in this case, the bidual of the nuclear domain C^*-algebras) are hyperfinite, and so are inductive limits of finite dimensional algebras. This is crucial in the folklore result that maps from separable nuclear C^*-algebras into finite von Neumann algebras are classified (up to strong*-approximate unitary equivalence) by traces. The classification into the trace-kernel quotient (Theorem 1.3) is obtained by gluing this observation uniformly across the traces.

The absence of tracial approximations allows us to obtain our embedding results in vast generality: in this paper, a separable nuclear C^*-algebra satisfying the UCT is allowed as a domain, as opposed to the simplicity requirement found in [124] (see the discussion in Remark 9.6). In future work we will obtain embedding results for exact domains and quite general codomains, with the simplicity and nuclearity requirements at the level of the map. With our approach, internal C^*-structure — such as arising as an ASH algebra or enjoying tracial approximations — can be viewed as a consequence of C^*-classification, rather than an ingredient.

Our proof essentially contains a copy of CS's proof ([252]) of the quasidiagonality theorem, and so we no longer need this result nor UCT-powered stable uniqueness across the interval arguments used to obtain tracial approximations [98] (and in the original proof of the quasidiagonality theorem, [269]). This is replaced by the abstract lifting result Theorem 1.2(i) in combination with the UCT to verify the hypotheses. The use of the UCT in [252] is more direct than that in [269] and is similar to its use here.

Our work also has points of contact with the work of Gong, Lin, and Niu, and many others in the long history of C^*-classification. Classifying into the trace-kernel quotient is spiritually connected to classifying tracially large maps into C^*-algebras with good tracial approximations, and our use of the Elliott–Kucerovsky absorption theorem plays a role in our argument analogous to the one played by stable uniqueness theorems in [125, 126]. As with [125, 126], our existence result is proven in two stages. We first prove the existence of an approximate embedding with prescribed tracial information (through Theorem 1.2(i)), and then perturb the morphism to modify the K-theoretic behavior (in Theorem 1.2(ii)). This two-step process dates back to Lin's classification of TAF algebras ([175]); this is made explicit in Dadarlat's account ([60]).

Finally, while we do not use any of the results from the major papers establishing the 2015 classification theorem during our work, we carefully studied the invariants and rotation maps from [123]. This led us to expect that a suitable combination of total K-theory, traces, and Hausdorffized unitary algebraic K_1 would provide the classification invariant found in Theorem B.

[40] With some extra work and invariants, it is possible to do this abstractly, and we will return to this point in future work.

1.4. **Structure of the paper, notation, and references.** We hope to clarify the new ingredients of the invariants used in our classification in the next two sections and review their essential properties (with certain facts about total K-theory being collated in Appendix A for completeness). Section 3 is devoted to the maps $\zeta^{(n)}\colon \overline{K}_1^{\mathrm{alg}}(\,\cdot\,;\mathbb{Z}/n) \to \overline{K}_1^{\mathrm{alg}}(\,\cdot\,)$, and formalizes the total invariant $\underline{K}T_u$. Section 4 is concerned with the Jiang–Su algebra $\mathcal{Z}$. We give a short self-contained proof of Jiang's unpublished theorem that $\mathcal{Z}$-stable C^*-algebras are K_1-injective using the modern perspective of the Jiang–Su algebra. This result is a vital component of the $\mathcal{Z}$-stable KK- and KL-uniqueness theorems. We also collect results on strict comparison and the Cuntz semigroup in Section 4, which are used later in absorption results.

Absorption and KK-theory strike back in Section 5. Since we wish to use $KK(A, J_B)$ and $KL(A, J_B)$ and the trace-kernel quotient is not σ-unital, we will separabilize and treat these KK-groups as limits of the KK- and KL-groups corresponding to separable subalgebras J of J_B. This is explained in Section 5.1, with the proofs of facts that are standard in the σ-unital case deferred to Appendix B. The rest of Section 5 is devoted to absorption and the proof of the $\mathcal{Z}$-stable KK- and KL-uniqueness theorems.

In Section 6, we describe the trace-kernel extension, and compute the K-theory of the trace-kernel quotient. This enables us to prove Theorem 1.5. Section 6.4 shows that the trace kernel extension J_B is separably stable for use in absorption. Section 7 analyses lifts along the trace-kernel extension, proving Theorem 1.2 (which we deduce from non-unital versions).

The final act of the paper sees the return of the UCT and its cousin, the universal multicoefficient theorem. We review these results in Section 8, and compute $KL(A, J_B)$ to prove Theorem 1.6. In particular, we do not use that our domain algebras A satisfy the UCT in any results before Section 8. Finally we put everything together in Section 9, which proves the classification of full approximate *-homomorphisms (Theorem 1.1) and deduces Theorems A and B.

1.4.1. *Notation.* Throughout the paper we endeavor to consistently use A for C^*-algebras appearing as the domain of the classification of unital embeddings (Theorem B). When A is used in other results (for example, appearing as the first variable in KK-theory) it is because we envisage applying these results to the domains of Theorem B. Likewise, we use B for the codomains of the classification of embeddings, with the corresponding notation for the trace-kernel extension in (1.3). For that reason we typically write D or E for a general C^*-algebra, and I or J for a general ideal (often the second variable in KK-theory). We will write $\mathcal{M}(I)$ for the multiplier algebra of I, and $\mathcal{Q}(I)$ for its corona. The symbols α, β, γ will be reserved for the components of a morphism between invariants — see Section 3.2. We also use κ for a KK-element,[41] λ for a KL-element, and typically use ϕ, ψ, θ for *-homomorphisms between C^*-algebras. For $D \subseteq E$ an inclusion of C^*-algebras, we write $\iota_{D\subseteq E}\colon D \to E$ for the inclusion map, and $D^\dagger$ denotes the forced unitization of D (when D is already unital, a new unit is added).

1.4.2. *References underpinning the argument.* We have attempted to provide both context throughout, and also to track down the intellectual origins of the ideas we use. However, from a formal point of view, the vast majority of the (many) papers

[41] $\kappa_*^{(m,n)}$ also appears briefly as one of the Bockstein maps; no ambigiuity is possible.

we cite do not form part of our argument for the stably finite parts of Theorems A and B, which are relatively self-contained. Concretely, in addition to material in textbooks and short stand-alone lemmas which will be found straightforward by experts,[42] we only rely on the following works:

- [68, 266] for properties of algebraic K_1 and the de la Harpe–Skandalis determinant;
- [66, 255] for total K-theory;
- [144, 213, 242, 244, 274] for the Jiang–Su algebra and consequences of $\mathcal{Z}$-stability;
- [131] for the fact that quasitraces on exact C^*-algebras are traces;[43]
- [141, 212, 235] for the Cuntz semigroup, strict comparison, and its use in obtaining stability, and the corona factorization property;
- [62, 63, 99, 111, 172] for the absorption theorem; [59, 62] for ingredients in the KK- and KL-uniqueness theorems (including a convenient form of Paschke duality);
- [32, 35] for the von Neumann-like behavior[44] of B^∞ and Theorem 1.2;
- [61, 66] for the universal multicoefficient theorem, and aspects of KL;
- [113] for intertwining via reparameterizations.

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[42] In particular, every instance where we appeal to one of [21, 253, 269] for a result (rather than for context) in the main body is for a short standard fact of this nature.

[43] We only use this for the simple $\mathcal{Z}$-stable codomain algebras B; there is a short proof of this fact for C^*-algebras with finite nuclear dimension ([30]), although we don't know how to do this from $\mathcal{Z}$-stability.

[44] Note that we only need the von Neumann-like behavior, namely the complemented partition of unity technique set out in [35, Sections 2,3] and not the full force of $\mathcal{Z}$-stability implies finite nuclear dimension.

Finally, we thank the community for their collective patience. The second paper in this series on non-unital classification will follow more promptly.

2. Invariants for unital C^*-algebras

In this section, we review the invariants used for the classification of unital C^*-algebras and their *-homomorphisms.

2.1. K-theory and traces.

Elliott's invariant for the classification of simple nuclear C^*-algebras consists of ordered K-theory and traces. Write $K_* = (K_0, K_1)$ for the operator algebraic K-theory functor from C^*-algebras to pairs of abelian groups (see [243] for an overview). When A is a unital C^*-algebra, the element $[1_A]_0$ represents a distinguished element in $K_0(A)$. The positive elements $K_0(A)_+$ of $K_0(A)$ are precisely those classes that can be realized by a projection in $M_n(A)$ for some $n \geq 1$, so that $(K_0(A), K_0(A)_+)$ is an ordered group whenever A is stably finite and unital (see [10, Proposition 6.3.3], for example).

Since our focus in this paper is on unital C^*-algebras, by a *trace* we always mean a tracial state. The set of all traces on a unital C^*-algebra A is denoted by $T(A)$. This is convex, and unitality ensures that $T(A)$ is compact in the relative weak*-topology.[45] Since $A \mapsto T(A)$ is contravariant, throughout this paper, we prefer to work with the function system $\operatorname{Aff} T(A)$ of continuous real-valued affine functions on $T(A)$ rather than with $T(A)$ itself. The duality between compact convex subsets of locally convex spaces and Archimedean order unit spaces (which are concretely realized as function systems) goes back to Kadison in [147]. This works as follows: every continuous affine map $\gamma^*\colon T(B) \to T(A)$ induces a unital positive linear map $\gamma\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ given by $\gamma(f)(\tau) := f(\gamma^*(\tau))$ for $f \in \operatorname{Aff} T(A)$ and $\tau \in T(B)$. Conversely, $T(A)$ is naturally recovered as the states[46] on $\operatorname{Aff} T(A)$ ([127, Theorem 7.1]). Therefore, unital positive linear maps $\gamma\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ give rise to continuous affine maps $\gamma^*\colon T(B) \to T(A)$.

The relationship between the function system $\operatorname{Aff} T(A)$ and the space A_{sa} of self-adjoint elements of A goes back to the work of Cuntz and Pedersen ([58]). Every $a \in A_{sa}$ induces a continuous affine function $\hat{a}\colon T(A) \to \mathbb{R}$ by $\hat{a}(\tau) = \tau(a)$ for $\tau \in T(A)$. Write $[a, b] := ab - ba$ and $[A, A]$ for the span of the set $\{ab - ba : a, b \in A\}$, so a trace is precisely a state that vanishes on $[A, A]$. We need the following result, which is often attributed to [58] but does not appear there explicitly. We give a short, self-contained proof for completeness.[47]

Proposition 2.1. *Suppose A is a unital C^*-algebra.*

(i) If $f \in \operatorname{Aff} T(A)$, then there exists $a \in A_{\mathrm{sa}}$ such that $f = \hat{a}$ (i.e., $f(\tau) = \tau(a)$ for all $\tau \in T(A)$), and a is unique modulo $\overline{[A, A]}$. Moreover, given $\epsilon > 0$, we may choose a with $\|a\| < \|f\| + \epsilon$.

(ii) $[A, A] \cap A_{sa}$ is spanned by $\{[x, x^*] : x \in A\}$.

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[45] In fact, $T(A)$ is a Choquet simplex (see [247, Theorem 3.1.18], for example) that is metrizable when A is separable. This fact plays no role in our approach to classification but is certainly crucial in constructing models to realize the range of the invariant.

[46] Recall that a *state* on an ordered abelian group G with order unit e is an order-preserving group homomorphism $G \to \mathbb{R}$ that maps e to 1.

[47] This result can also be deduced from [167, Lemma 6.2(iii)] or [213, Theorem 5], both of which factor through the enveloping von Neumann algebra. Alternatively, the beginning of the proof of [266, Theorem 3.1] indicates a strategy for deducing it from [58].

Proof. Let $T_{\mathbb{C}}(A) := \{\tau \in A^* : \tau|_{[A,A]} = 0\}$ be the space of bounded tracial functionals on A, so that there is a canonical isometric isomorphism $T_{\mathbb{C}}(A) \to (A/\overline{[A,A]})^*$ preserving the weak* topology.

Given a continuous affine function $f\colon T(A) \to \mathbb{R}$, we may use the Jordan decomposition to extend f to a bounded self-adjoint linear functional $\tilde{f} \in (T_{\mathbb{C}}(A))^* \cong (A/\overline{[A,A]})^{**}$. Note that $\tilde{f}$ is also weak*-continuous: indeed, the intersection of $\ker \tilde{f}$ with any multiple of the closed unit ball in $T_{\mathbb{C}}(A)$ is weak*-closed (using weak*-compactness, replace a bounded weak* convergent net in $T_{\mathbb{C}}(A)$ with one whose Jordan decompositions also converge), so that $\ker \tilde{f}$ is weak*-closed by the Krein–Smulian Theorem ([48, Corollary V.12.6]).

As $\tilde{f}$ is weak*-continuous, there is an $a \in A$ with $\tilde{f}(\tau) = \tau(a)$ for all $\tau \in T(A)$. In fact, because $\tilde{f}$ is self-adjoint, we may choose a to be self-adjoint. Uniqueness in (i) follows from $(T_{\mathbb{C}}(A))^* \cong (A/\overline{[A,A]})^{**}$.

The claim in (ii) follows from the polarization identity applied to the sesquilinear form $(a,b) \mapsto [a,b^*]$. $\square$

The relationship between K_0 and traces is encoded in the natural *pairing map* $\rho_A\colon K_0(A) \to \operatorname{Aff} T(A)$. For $n \in \mathbb{N}$ and $\tau \in T(A)$, write τ_n for the canonical non-normalized extension[48] of τ to a tracial functional on $M_n(A)$ and define

$$\rho_A([p]_0 - [q]_0)(\tau) := \tau_n(p-q) \tag{2.1}$$

for all $\tau \in T(A)$ and all projections $p, q \in M_n(A)$.

When A is unital and exact, every state on $(K_0(A), K_0(A)_+, [1_A]_0)$ arises from a trace (i.e., is of the form $K_0(A) \ni x \mapsto \rho_A(x)(\tau)$ for some $\tau \in T(A)$), by using a celebrated result of Haagerup ([131]) and [16]. In particular, unital nuclear stably finite C^*-algebras always have traces.

The evolution of the Elliott invariant of a unital stably finite nuclear C^*-algebra A naturally led to the inclusion of the positive cone $K_0(A)_+$ in the invariant: it makes its first appearance in Elliott's classification of AF algebras (see [85, Theorem 5.1]).

Definition 2.2. If A is a unital stably finite C^*-algebra, then its *Elliott invariant,* $\operatorname{Ell}(A)$, is[49]

$$\operatorname{Ell}(A) := (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), \operatorname{Aff} T(A), \rho_A). \tag{2.2}$$

For reasons explained below, we prefer to work with a weaker invariant, KT_u, that does not explicitly keep track of the order on K_0. See Definition 2.3 below. While we are primarily interested in the case of C^*-algebras with traces, the definition makes perfect sense when $T(A) = \emptyset$. In this case, $\operatorname{Aff} T(A) = 0$, and $KT_u(A)$ reduces to $K_*(A)$ together with the position of the unit — which is precisely the information used in the classification of unital Kirchberg algebras satisfying the UCT by Kirchberg ([161]) and by Phillips ([221]).

---

[48] That is, $\tau_n(1_{M_n(A)}) = n$.

[49] It is common to formulate the Elliott invariant with $T(A)$ rather than $\operatorname{Aff} T(A)$ and to encode the pairing via the map from $T(A)$ to states on $K_0(A)$; by the duality explained above, these formulations are equivalent.

Definition 2.3. Let KT_u be the functor on the category of unital C^*-algebras with unital *-homomorphisms that assigns to a unital C^*-algebra A the quadruple

$$KT_u(A) = (K_*(A), [1_A]_0, \operatorname{Aff} T(A), \rho_A).^{50} \tag{2.3}$$

A morphism[51] $(\alpha_*, \gamma)\colon KT_u(A) \to KT_u(B)$ consists of a pair $\alpha_* = (\alpha_0, \alpha_1)$ of homomorphisms $\alpha_i\colon K_i(A) \to K_i(B)$ (for $i = 0, 1$) with $\alpha_0([1_A]_0) = [1_B]_0$, and a positive unital linear map $\gamma\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ such that

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \gamma} \\ K_0(B) & \xrightarrow{\rho_B} & \operatorname{Aff} T(B) \end{array} \tag{2.4}$$

commutes.

In the case where A is exact and has real rank zero, $T(A)$ can be recovered as states on the ordered group $(K_0(A), K_0(A)_+, [1_A]_0)$ (see [240, Theorem 1.1.11 and Proposition 1.1.12]), so $K_0(A)_+$ can be used in place of $\operatorname{Aff} T(A)$ and ρ_A. Outside this setting, traces are needed for classification, and the pairing map, which first arose explicitly in the invariant in [87], carries more data than $T(A)$ and the order structure it induces on $K_0(A)$.[52] Moreover, for unital simple nuclear finite $\mathcal{Z}$-stable C^*-algebras, the order on $K_0(A)$ is induced by the pairing map,[53] so KT_u and the Elliott invariant are equivalent invariants.[54]

In general, the order $K_0(A)$ may fail to be weakly unperforated, in which case, it cannot arise from a pairing map. This phenomenon was first exhibited for a simple nuclear stably finite C^*-algebra by Villadsen in [281]. We prefer to use the invariant KT_u for two reasons: to emphasize that our proof makes no explicit reference to the order structure on $K_0(A)$ and because for *any* unital C^*-algebra A, there is a canonical isomorphism $KT_u(A) \cong KT_u(A \otimes \mathcal{Z})$ (see Proposition 4.2 below) whereas, for a unital simple C^*-algebra A, $\operatorname{Ell}(A \otimes \mathcal{Z}) \cong \operatorname{Ell}(A)$ if and only if $(K_0(A), K_0(A)_+)$ is weakly unperforated ([122, Theorem 1]).

Recall that a trace τ on A is *faithful* if $\tau(a^*a) = 0$ implies $a = 0$. For an embedding $\theta\colon A \hookrightarrow B$ and a trace $\tau \in T(B)$, if B is simple then it follows that the induced trace $\tau \circ \theta$ is faithful on A. This gives a necessary condition for a map between invariants to arise from an embedding into a simple C^*-algebra.

---

[50] For our purposes, the target category of KT_u has as objects quadruples $((G_0, G_1), e, X, \rho)$, where G_0 and G_1 are abelian groups with $e \in G_0$, X is an Archimedean order unit space (the order unit is part of the data), and $\rho\colon G_0 \to X$ a group homomorphism mapping e to the order unit.

[51] We specify only the morphisms between $KT_u(A)$ and $KT_u(B)$; morphisms between abstract objects of the target category of KT_u are defined analogously.

[52] An example of how different pairing maps between the same collection of affine functions and abelian group G_0 can give the same order structure on G_0 can be found on page 29 of [240].

[53] This is a direct consequence of Proposition 4.13(ii).

[54] By this, we mean that there are functors F and G between the target categories for KT_u and Ell (one in each direction) such that $F \circ \operatorname{Ell}$ and KT_u are naturally isomorphic once restricted to the category of unital simple separable nuclear finite $\mathcal{Z}$-stable C^*-algebras, and similarly for $G \circ KT_u$ and Ell.

Definition 2.4. Let A and B be unital C^*-algebras. We say that $(\alpha_*, \gamma)\colon KT_u(A) \to KT_u(B)$ is *faithful* if the induced map $\gamma^*\colon T(B) \to T(A)$ (given by Kadison duality) satisfies that $\gamma^*(\tau)$ is faithful for all $\tau \in T(B)$.[55]

Remark 2.5. As our approach to classification does not require comparing C^*-algebras with concrete models, the range of the invariant KT_u does not play a role in our proof. However, it is worth noting that (in contrast to the Elliott invariant), the range of KT_u is completely understood: a quadruple $((G_0, G_1), e, X, \rho)$ in the target category of KT_u has the form $KT_u(B)$ for a separable unital C^*-algebra B if and only if G_0 and G_1 are countable and X is separable and has Riesz interpolation,[56] and in this case, there is a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra A in the UCT class with $KT_u(A) \cong KT_u(B)$.[57]

2.2. **Unitary algebraic K_1.** The role of the quotient of the infinite unitary group by the closure of the commutator subgroup in the classification program goes back to work of Nielsen and Thomsen ([209]), who showed that it is needed to determine the approximate unitary equivalence classes of maps between simple $\mathrm{A}\mathbb{T}$ algebras. Although this group and its structure is known to experts, we provide a guide for less familiar readers, both to clarify some topological points and to collect a number of details of the construction — notably, the de la Harpe–Skandalis determinant — for subsequent use.

For a unital C^*-algebra A, write $U_n(A)$ for the group of unitaries in $M_n(A)$ with the norm topology and $DU_n(A)$ for the derived subgroup of $U_n(A)$, i.e., the subgroup generated by commutators uvu^*v^* for $u, v \in U_n(A)$ (this subgroup is automatically normal). Each $U_n(A)$ is included into $U_{n+1}(A)$ via the standard connecting maps, and we write $U_\infty(A)$ for the direct limit, which is equipped with the topological inductive limit topology: $V \subseteq U_\infty(A)$ is open if and only if $V \cap U_n(A)$ is open for all $n \in \mathbb{N}$. We caution the reader that while the inverse map on $U_\infty(A)$ is continuous, multiplication is only separately continuous in general, so $U_\infty(A)$ is not a topological group with this topology.[58] Let $DU_\infty(A) \coloneqq \bigcup_n DU_n(A)$ and note that $DU_\infty(A)$ is contained in $U_\infty^{(0)}(A)$, the path component of the identity in

[55] This can be phrased directly in terms of γ as follows: if $f \in \operatorname{Aff} T(A)$ is strictly positive on faithful traces on A, then $\gamma(f)$ is strictly positive. To see this, note that if $\gamma^*(\tau)$ is not faithful for some $\tau \in T(B)$, then there is $a \in A_+ \setminus \{0\}$ with $\gamma^*(\tau)(a) = 0$. Then $\hat{a} \in \operatorname{Aff} T(A)$ is strictly positive on faithful traces but $\gamma(\hat{a}) \in \operatorname{Aff} T(B)$ is not.

[56] The relevance of Riesz interpolation is that an Archimedean order unit space has Riesz interpolation if and only if the state space of X is a Choquet simplex ([2, Corollary II.3.11]).

[57] This is a reinterpretation of the range of invariant results from [238, 90], which predate $\mathcal{Z}$-stability. While all the examples of [238, 90] are $\mathcal{Z}$-stable (for example, by $\mathcal{O}_\infty$-stability for those in [238] and by Winter's $\mathcal{Z}$-stability theorem, [291], for those in [90]), one can avoid checking this by simply replacing them with their $\mathcal{Z}$-stabilizations.

[58] If $G_1 \subset G_2 \subseteq \cdots$ is an increasing sequence of locally compact Hausdorff topological groups, then the topological inductive limit G is a topological group ([263, Theorem 2.7]). If some G_n is not locally compact, and for all n, G_n is not open in G_m for some $m > n$, then G is not a topological group by [296, Theorem 4]. Accordingly $U_\infty(A)$ is a topological group in the inductive limit topology if and only if A is finite dimensional. It may be more natural to work with the topological group inductive limit topology on $U_\infty(A)$, but this does not seem to be considered in the literature. As we note in Remark 2.13, using the topological group inductive limit topology turns out to give the same definition of the Hausdorffized unitary algebraic K_1-group.

$U_\infty(A)$, which is equal to $\bigcup_n U_n^{(0)}(A)$.[59] Moreover, any commutator in $U_\infty(A)$ is already a commutator of elements in $U_\infty^{(0)}(A)$.[60]

As multiplication is separately continuous on $U_\infty(A)$ and inversion is continuous, the closure $\overline{DU_\infty(A)}$ of $DU_\infty(A)$ in the inductive limit topology is a normal subgroup of $U_\infty(A)$. Taking the quotient gives the version of algebraic K_1 we need for the uniqueness component of the approximate classification of embeddings.[61]

Definition 2.6. Write

$$\overline{K}_1^{\mathrm{alg}}(A) \coloneqq U_\infty(A)/\overline{DU_\infty(A)} \tag{2.6}$$

and call this the *Hausdorffized unitary algebraic K_1-group* of A.[62] Write $[u]_{\mathrm{alg}}$ for the class of $u \in U_\infty(A)$ in $\overline{K}_1^{\mathrm{alg}}(A)$. For a unital *-homomorphism $\phi\colon A \to B$, we get an induced map $\overline{K}_1^{\mathrm{alg}}(\phi)\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B)$ given by $\overline{K}_1^{\mathrm{alg}}([u]_{\mathrm{alg}}) \coloneqq [\phi^{(n)}(u)]_{\mathrm{alg}}$ for $u \in U_n(A)$. In this way $\overline{K}_1^{\mathrm{alg}}$ is a functor from unital C^*-algebras to abelian groups that is invariant under approximate unitary equivalence of morphisms.

Remark 2.7. When the natural maps $\pi_0(U(A)) \to K_1(A)$ and $\pi_1(U(A)) \to K_0(A)$ are bijections, then so is $U(A)/\overline{DU(A)} \to \overline{K}_1^{\mathrm{alg}}(A)$ ([266, Corollary 3.4]). This is the case, in particular, for C^*-algebras with stable rank one ([232, Theorem 3.3]) and for $\mathcal{Z}$-stable C^*-algebras ([143, Theorem 3]). This is typically why $U(\,\cdot\,)/\overline{DU(\,\cdot\,)}$ is used in classification results (for example, in [125]). However, while our codomain algebras from Theorem B are $\mathcal{Z}$-stable, there is no such restriction on the domains, and so, in general, we cannot work with $U(A)/\overline{DU(A)}$ (or even $U_k(A)/\overline{DU_k(A)}$) in place of $\overline{K}_1^{\mathrm{alg}}(A)$.

The structure of $\overline{K}_1^{\mathrm{alg}}(A)$ was investigated by Thomsen in [266], building on de la Harpe and Skandalis' determinant map and analysis of quotients by commutator subgroups in $GL_\infty(A)$ ([68]). Importantly, it is related to $\operatorname{Aff} T(A)$ and $K_1(A)$ through natural homomorphisms of abelian groups:

$$\operatorname{Aff} T(A) \xrightarrow{\mathrm{Th}_A} \overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{\not{a}_A} K_1(A). \tag{2.7}$$

---

[59] By the argument in [132, Proposition A.1], every compact subset of $U_\infty(A)$ is contained in some $U_n(A)$; from this it follows that $U_\infty^{(0)}(A) = \bigcup_n U_n^{(0)}(A)$.

[60] For $u, v \in U_\infty(A)$, one has

$$uvu^*v^* = \begin{pmatrix} u & 0 & 0 \\ 0 & u^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^* \end{pmatrix} \begin{pmatrix} u^* & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v \end{pmatrix}. \tag{2.5}$$

[61] As pointed out to us by George Elliott, $DU_\infty(A)$ is norm-dense in $U_\infty^{(0)}(A)$. This can be seen using a trick in the proof of [134, Theorem 2.4.7] that goes back to Pearcy and Topping [218]. Thus, it is important to take the closure in an inductive limit topology in Definition 2.6.

[62] The algebraic K_1-group of a unital ring A is defined as $GL_\infty(A)/DGL_\infty(A)$. The version we use differs by using unitaries in place of invertibles, and by taking the quotient by the closure of $DU_\infty(A)$, to ensure invariance under approximate unitary equivalence of morphisms. For example, the algebraic K_1-group of $\mathbb{C}$ is $\mathbb{C}^\times$, whereas $\overline{K}_1^{\mathrm{alg}}(\mathbb{C}) = \mathbb{T}$. In the literature it is common to write $U_\infty(A)/\overline{DU_\infty(A)}$ for $\overline{K}_1^{\mathrm{alg}}(A)$. Given the fundamental role it plays in classification results we prefer to introduce the more compact notation $\overline{K}_1^{\mathrm{alg}}(A)$. The Hausdorffized algebraic K_1-group (using invertibles) is naturally recovered from $\underline{K}T_u$; this fact and further study of the relationship between the different versions of algebraic K_1 can be found in [92, 248].

As $\overline{DU_\infty(A)} \subseteq U_\infty^{(0)}(A)$, the second of these maps is readily defined.

Definition 2.8. Let A be a unital C^*-algebra. The map $\not\mathrm{a}_A\colon \overline{K}_1^{\mathrm{alg}}(A) \to K_1(A)$ is the canonical surjection given by $\not\mathrm{a}_A([u]_{\mathrm{alg}}) \coloneqq [u]_1$ for $u \in U_\infty(A)$.

The map Th_A, which we call the *Thomsen map*, is a little harder to set up. We outline the construction after Proposition 2.10, and collect the following key properties, due to Thomsen ([266]), here.

Properties 2.9. *For a unital C^*-algebra A,*

- *(i)* $\mathrm{Th}_A(\hat{a}) = [e^{2\pi i a}]_{\mathrm{alg}}$ *for $a \in A_{\mathrm{sa}}$, where $\hat{a}$ is the continuous affine function $\hat{a}(\tau) = \tau(a)$ for $\tau \in T(A)$;*
- *(ii)* $\operatorname{im}\mathrm{Th}_A = \ker \not\mathrm{a}_A$*;*
- *(iii)* $\ker \mathrm{Th}_A = \overline{\operatorname{im}\rho_A}$.

Condition (iii) gives a group isomorphism

$$\overline{\mathrm{Th}}_A\colon \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A} \xrightarrow{\cong} U_\infty^{(0)}(A)/\overline{DU_\infty(A)} \subset \overline{K}_1^{\mathrm{alg}}(A), \tag{2.8}$$

and so (2.7) induces the short exact sequence

$$0 \longrightarrow \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A} \xrightarrow{\overline{\mathrm{Th}}_A} \overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{\not\mathrm{a}_A} K_1(A) \longrightarrow 0. \tag{2.9}$$

Since $\operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A}$ is a divisible group, (2.9) splits unnaturally,[63] identifying $\overline{K}_1^{\mathrm{alg}}(A)$ as

$$\overline{K}_1^{\mathrm{alg}}(A) \cong \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A} \oplus K_1(A) \tag{2.10}$$

(see [266, Corollary 3.3]). Consequently, any morphism between K-theory and traces can be extended to a map between the $\overline{K}_1^{\mathrm{alg}}$-groups (as in the following proposition), albeit in a non-canonical way. Therefore, the additional data encoded by $\overline{K}_1^{\mathrm{alg}}$, beyond that held in KT_u, is found not in $\overline{K}_1^{\mathrm{alg}}(A)$ but in the behavior of $\overline{K}_1^{\mathrm{alg}}$ on morphisms. To our knowledge, the first explicit result extending KT_u-morphisms to $\overline{K}_1^{\mathrm{alg}}$ was set out by Nielsen in [208, Section 3] in the context of simple $\mathrm{A}\mathbb{T}$ algebras (see also [209, Lemma 3.2]), although the argument works generally. We give a short proof of this from the above facts for completeness, as this forms an essential point in deducing the unital classification theorem (Theorem A) from the classification of embeddings (Theorem B).

Proposition 2.10. *Let A and B be unital C^*-algebras and $(\alpha_*, \gamma)\colon KT_u(A) \to KT_u(B)$ be a morphism. Then there exists a group homomorphism $\beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B)$ such that*

$$\begin{array}{ccccccc} K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) & \xrightarrow{\mathrm{Th}_A} & \overline{K}_1^{\mathrm{alg}}(A) & \xrightarrow{\not\mathrm{a}_A} & K_1(A) \\ \big\downarrow{\scriptstyle \alpha_0} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \alpha_1} \\ K_0(B) & \xrightarrow{\rho_B} & \operatorname{Aff} T(B) & \xrightarrow{\mathrm{Th}_B} & \overline{K}_1^{\mathrm{alg}}(B) & \xrightarrow{\not\mathrm{a}_B} & K_1(B) \end{array} \tag{2.11}$$

[63] An abelian group G is divisible if for each $x \in G$ and $n \in \mathbb{N}$, there exists $y \in G$ with $nx = y$. This holds trivially for $\operatorname{Aff} T(A)$, and is inherited by quotients. For abelian groups, divisibility is equivalent to injectivity (see [285, Corollary 2.3.2], for example), and this provides a splitting $\overline{K}_1^{\mathrm{alg}}(A) \to \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A}$.

commutes. Moreover, if (α_*, γ) *is an isomorphism, then any such* β *is an isomorphism.*

Proof. As γ is continuous, since it is unital, positive, and linear, the maps (α_*, γ) induce the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A} \arrow[r, "\overline{\mathrm{Th}}_A"] \arrow[d, "\bar{\gamma}"] & \overline{K}_1^{\mathrm{alg}}(A) \arrow[r, "\not a_A"] \arrow[d, dashed, "\beta"] & K_1(A) \arrow[r] \arrow[d, "\alpha_1"] & 0 \\
0 \arrow[r] & \operatorname{Aff} T(B)/\overline{\operatorname{im}\rho_B} \arrow[r, "\overline{\mathrm{Th}}_B"] & \overline{K}_1^{\mathrm{alg}}(B) \arrow[r, "\not a_B"] & K_1(B) \arrow[r] & 0
\end{tikzcd}
\tag{2.12}
$$

with exact rows. Both rows of this diagram split because $\operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A}$ and $\operatorname{Aff} T(B)/\overline{\operatorname{im}\rho_B}$ are divisible abelian groups. Let

$$
p_A\colon \overline{K}_1^{\mathrm{alg}}(A) \to \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A} \quad \text{and} \quad i_B\colon K_1(B) \to \overline{K}_1^{\mathrm{alg}}(B) \tag{2.13}
$$

be splittings of $\overline{\mathrm{Th}}_A$ and $\not a_B$, respectively. Defining the dashed arrow in (2.12) by

$$
\beta \coloneqq i_B \circ \alpha_1 \circ \not a_A + \overline{\mathrm{Th}}_B \circ \bar{\gamma} \circ p_A \tag{2.14}
$$

ensures that (2.12), and hence (2.11), commutes. If (α_*, γ) is an isomorphism, then γ is an isomorphism and thus $\overline{\gamma}$ is an isomorphism. By the five lemma, any map β making (2.12) commute is necessarily an isomorphism. □

The rest of this subsection defines Th_A and indicates how to use the literature ([266] especially) to establish Properties 2.9 efficiently.

Recall that every element of $\operatorname{Aff} T(A)$ may be realized as $\hat{a}$ for some $a \in A_{\mathrm{sa}}$. (See Proposition 2.1.) We wish to define

$$
\mathrm{Th}_A(\hat{a}) \coloneqq [e^{2\pi i a}]_{\mathrm{alg}}, \qquad a \in A_{\mathrm{sa}}. \tag{2.15}
$$

To see this is well-defined, first note that

$$
e^{ia}e^{ib} = e^{i(a+b)} \text{ modulo } \overline{DU(A)} \tag{2.16}
$$

for $a, b \in A_{\mathrm{sa}}$ (using the Lie–Trotter formula as in [266, Equation (1.1)]), so that the map $a \mapsto [e^{2\pi i a}]_{\mathrm{alg}}$ is a continuous group homomorphism $A_{\mathrm{sa}} \to \overline{K}_1^{\mathrm{alg}}(A)$. It then suffices to show $[e^{2\pi i(v^*v - vv^*)}]_{\mathrm{alg}} = 0$ for every $v \in A$ (using Proposition 2.1). This follows by combining (2.16) with the polar decomposition trick found in the last few lines of the proof of [266, Lemma 3.1], which shows that for any $v \in M_n(A)$, $e^{2\pi i(v^*v - vv^*)} = 1_A$ modulo $\overline{DU_n(A)}$. Note that (2.16) also shows that the Thomsen map is a homomorphism. Since $\ker \not a_A$ is generated by exponentials, the definition (2.15) ensures that (2.7) is exact, so we have established (i) and (ii) of Properties 2.9.

As we can work equally well with elements $a \in M_n(A)_{sa}$ in (2.15),[64] we get that $\operatorname{im}\rho_A \subseteq \ker \mathrm{Th}_A$, as follows. Given projections $p, q \in M_n(A)$, we have $\widehat{p - q} = \rho_A([p]_0 - [q]_0)$, and so

$$
\mathrm{Th}_A(\rho_A([p]_0 - [q]_0)) = [e^{2\pi i(p-q)}]_{\mathrm{alg}} = [e^{2\pi i p}]_{\mathrm{alg}} + [e^{-2\pi i q}]_{\mathrm{alg}} = 0. \tag{2.17}
$$

Given $f \in \operatorname{Aff} T(A)$, one can control the norm of a choice of a satisfying $\hat{a} = f$ (see Proposition 2.1), and therefore the Thomsen map is continuous as a map into $U_1(A)/\overline{DU}_1(A)$, so $\overline{\operatorname{im}\rho_A} \subseteq \ker \mathrm{Th}_A$.

---

[64] We must also take care to follow our scaling conventions: $\hat{1}_{M_n(A)}$ is the constant function n on $T(A)$.

The reverse inclusion $\ker \mathrm{Th}_A \subseteq \overline{\operatorname{im}\rho_A}$ uses the de la Harpe–Skandalis determinant from [68]. We record its basic properties in a slightly unusual fashion, designed for use in Section 3.1.

Proposition 2.11. *Let A be a unital C^*-algebra. The de la Harpe–Skandalis determinant map gives a continuous group homomorphism*

$$\widetilde{\Delta}_A\colon U_\infty(C([0,1],A)) \longrightarrow \operatorname{Aff} T(A) \tag{2.18}$$

such that:

(i) $\widetilde{\Delta}_A(u)$ depends only on the homotopy class of u (in the space of unitary paths with endpoints fixed);
(ii) for a self-adjoint $h \in M_n(C([0,1],A))$,

$$\widetilde{\Delta}_A(e^{2\pi i h})(\tau) = \tau_n(h(1)-h(0)), \quad \tau \in T(A) \tag{2.19}$$

where τ_n is the canonical non-normalized extension of τ to $M_n(A)$;
(iii) identifying homotopy classes of $U_\infty(C_0((0,1),A)^\dagger)$ with $K_0(A)$ via the Bott map, the homomorphism $U_\infty(C_0((0,1),A)^\dagger) \to \operatorname{Aff} T(A)$ induced by (i) is precisely the pairing map ρ_A;
(iv) there is a continuous group homomorphism[65]

$$\det_A\colon U_\infty^{(0)}(A) \longrightarrow \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A} \tag{2.20}$$

given by $\det_A(u) := \widetilde{\Delta}_A(v) + \overline{\operatorname{im}\rho_A}$, where $v \in U_\infty(C([0,1],A))$ has $v(0) = 1_A$ and $v(1) = u$.

Proof. For a piecewise smooth path $u\colon [0,1] \to U_n(A)$, de la Harpe and Skandalis defined $\widetilde{\Delta}_A(u) \in \operatorname{Aff} T(A)$ by

$$\widetilde{\Delta}_A(u)(\tau) := \frac{1}{2\pi i}\int_0^1 \tau_n\big(\big(\tfrac{d}{dt}u(t)\big)u(t)^*\big)\,dt, \quad \tau \in T(A). \tag{2.21}$$

By [68, Lemme 1(c)], $\widetilde{\Delta}_A(u)(\tau)$ depends only on the homotopy class[66] (with endpoints fixed) of u. Then one can extend the definition of $\widetilde{\Delta}_A$ to all continuous paths $u\colon [0,1] \to U_n(A)$ by applying $\widetilde{\Delta}_A$ to a piecewise smooth path homotopic to u (again with endpoints fixed) so that (i) holds.[67] Then $\widetilde{\Delta}_A$ is well-defined on $U_\infty(C([0,1],A))$ and is a homomorphism by [68, Lemme 1(a)].

Condition (ii) is immediate when h is piecewise smooth, and so follows in general by replacing h with a piecewise smooth path homotopic to h with the same endpoints. Note that the Bott map associates a projection $p \in M_n(A)$ with the unitary $u \in U_n((C_0(0,1),A)^\dagger)$ given by $u(t) := e^{2\pi i t p}$ for $t \in [0,1]$, so (iii) follows from (ii). Continuity of $\widetilde{\Delta}_A$ on the collection of piecewise smooth paths in $U_n(A)$ is immediate, and then extends to $U_n(C([0,1],A))$. As this holds for all n, $\widetilde{\Delta}_A$ is continuous on $U_\infty(C([0,1],A))$.

[65]The codomain of $\det_A$ can be taken to be $\operatorname{Aff} T(A)/\operatorname{im}\rho_A$ instead (with the same proof), but this will not be needed.

[66]Note that the homotopies need not be through piecewise smooth paths.

[67]These always exist. In the case when $\|u(s)-u(0)\| < 2$ for all $s \in [0,1]$, use a continuous logarithm to write $u(s) = e^{2\pi i h(s)}u(0)$ for a continuous path h of self-adjoints. Then $k_t(s) = (1-t)h(s)+t((1-s)h(0)+sh(1))$ provides a homotopy between h and an affine path; exponentiating this gives a homotopy between u and a piecewise smooth path. The general case is obtained by applying this argument to finitely many subintervals.

Condition (iv) is essentially [68, Lemme 1(d)] and the subsequent definitions. Once one knows $\det_A$ is well-defined, it is a continuous homomorphism because $\widetilde{\Delta}_A$ is. Since $U_\infty^{(0)}(A) = \bigcup_n U_n^{(0)}(A)$ (see Footnote 59), given $u \in U_\infty^{(0)}(A)$, we can find $n \in \mathbb{N}$ and $v \in U_n(C[0,1],A)$ with $v(0) = 1_{M_n(A)}$ and $v(1) = u$. If $v_1, v_2 \in U_n(C([0,1],A))$ are two such paths, then $v_1 v_2^*$ can be viewed as an element of $U_n(C_0((0,1),A)^\dagger)$ and so, by (iii), $\widetilde{\Delta}_A(v_1 v_2^*) \in \operatorname{im}\rho_A$. Therefore $\widetilde{\Delta}_A(v_1) + \overline{\operatorname{im}\rho_A} = \widetilde{\Delta}_A(v_2) + \overline{\operatorname{im}\rho_A}$. ☐

Let us return to the proof that $\ker \mathrm{Th}_A \subseteq \overline{\operatorname{im}\rho_A}$. Since $\det_A$ is a continuous group homomorphism, $\overline{DU_\infty^{(0)}(A)} \subseteq \ker \det_A$, and hence $\det_A$ induces a continuous group homomorphism

$$\overline{\det}_A \colon \ker \not{a}_A = U_\infty^{(0)}(A)/\overline{DU_\infty^{(0)}(A)} \longrightarrow \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A}. \tag{2.22}$$

A computation shows $\overline{\det}_A(\mathrm{Th}_A(f)) = f + \overline{\operatorname{im}\rho_A}$ for all $f \in \operatorname{Aff} T(A)$ (using that $f = \hat{a}$ for some $a \in A_{\mathrm{sa}}$). This implies $\ker \mathrm{Th}_A \subseteq \overline{\operatorname{im}\rho_A}$ and completes the outline of Properties 2.9. This also proves

$$\overline{\det}_A = \overline{\mathrm{Th}}_A^{-1}. \tag{2.23}$$

While we will treat $\overline{K}_1^{\mathrm{alg}}(A)$ as a discrete abelian group throughout the rest of the paper, we end this subsection by clarifying why $\overline{K}_1^{\mathrm{alg}}(A)$ (unlike $U_\infty(A)$) is a topological group. The second part of the following proposition explains why it is not necessary to keep track of this topological data for the classification of full approximate embeddings, as all of the morphisms β we consider between $\overline{K}_1^{\mathrm{alg}}$-groups will satisfy the compatibility conditions (2.11).

Proposition 2.12. *Let A be a unital C^*-algebra. Then $\overline{K}_1^{\mathrm{alg}}(A)$ is a Hausdorff topological group in the topology induced by the inductive limit topology on $U_\infty(A)$. Moreover, given a unital C^*-algebra B and a KT_u-morphism $(\alpha_*, \gamma)\colon KT_u(A) \to KT_u(B)$, then any group homomorphism $\beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B)$ making (2.11) commute is automatically continuous.*

Proof. From (2.7), we have a group isomorphism

$$\overline{\mathrm{Th}}_A \colon \operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A} \longrightarrow \ker \not{a}_A, \tag{2.24}$$

that is continuous because Th_A is. It follows from (2.23) that its inverse is the continuous map $\overline{\det}_A$. Since addition is separately continuous on $\overline{K}_1^{\mathrm{alg}}(A)$, to show addition is jointly continuous, it suffices to show that if $(x_i), (y_i) \subseteq \overline{K}_1^{\mathrm{alg}}(A)$ are nets with $x_i \to 0$ and $y_i \to 0$, then $x_i + y_i \to 0$. Since $\not{a}_A$ is continuous and $K_1(A) = U_\infty(A)/U_\infty^{(0)}(A)$ is discrete, we have $x_i, y_i \in \ker \not{a}_A$ for all large i. Since $\overline{\det}_A$ is a homeomorphic group isomorphism and the addition on $\operatorname{Aff} T(A)/\overline{\operatorname{im}\rho_A}$ is jointly continuous, by (2.23) we have $x_i + y_i \to 0$. This shows $\overline{K}_1^{\mathrm{alg}}(A)$ is a topological group. As $\not{a}_A$ and $\overline{\det}_A$ are continuous, $\{0\}$ is closed in $\overline{K}_1^{\mathrm{alg}}(A)$, and hence $\overline{K}_1^{\mathrm{alg}}(A)$ is Hausdorff.

For the second statement, note that any map β making (2.11) commute also makes (2.12) commute, so that by applying the continuous map $\overline{\det}_A$, it follows that $\beta|_{\ker \not{a}_A}$ is continuous. Since $\ker \not{a}_A$ is open and $\overline{K}_1^{\mathrm{alg}}(A)$ is a topological group, it follows that β is continuous. ☐

Remark 2.13. As mentioned in Footnote 58, it may be more natural to equip $U_\infty(A)$ with the topological group inductive limit topology — the strongest topology such that $U_\infty(A)$ is a topological group and the natural maps $U_n(A) \to U_\infty(A)$ are continuous. This topology is awkward to describe explicitly but satisfies the universal property that if H is a topological group and $\phi_n\colon U_n(A) \to H$ are compatible continuous group homomorphisms, then there is a unique continuous group homomorphism $\phi\colon U_\infty(A) \to H$ extending the ϕ_n's. We briefly sketch a proof that using this topology instead in Definition 2.6 produces the same group $\overline{K}_1^{\mathrm{alg}}(A)$.

Let $U_\infty^g(A)$ denote $U_\infty(A)$ equipped with the topological group inductive limit topology. Define

$$\overline{K}_1^{\mathrm{alg},g}(A) = U_\infty^g(A)/\overline{DU_\infty^g(A)} \tag{2.25}$$

and note that $\overline{K}_1^{\mathrm{alg},g}(A)$ is a Hausdorff topological abelian group. The natural maps $U_n(A) \to \overline{K}_1^{\mathrm{alg},g}(A)$ are continuous and hence induce a continuous group homomorphism $U_\infty(A) \to \overline{K}_1^{\mathrm{alg},g}(A)$. As $\overline{K}_1^{\mathrm{alg},g}(A)$ is Hausdorff and abelian, this homomorphism annihilates $\overline{DU_\infty(A)}$ and hence induces a continuous homomorphism $\phi\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg},g}(A)$. A similar argument using that $\overline{K}_1^{\mathrm{alg}}(A)$ is a Hausdorff topological abelian group (by Proposition 2.12) shows that the natural maps $U_n(A) \to \overline{K}_1^{\mathrm{alg}}(A)$ induce a continuous group homomorphism $U_\infty^g(A) \to \overline{K}_1^{\mathrm{alg}}(A)$, and in turn a continuous group homomorphism $\psi\colon \overline{K}_1^{\mathrm{alg},g}(A) \to \overline{K}_1^{\mathrm{alg}}(A)$. Clearly ϕ and ψ are inverses of each other since they are both induced by the identity map on the set $U_\infty(A)$.

2.3. **Total K-theory.** Total K-theory comprises K-theory, K-theory with coefficients in $\mathbb{Z}/n\mathbb{Z}$ (written $\mathbb{Z}/n$ from now on) for all $n \geq 2$, and the natural connecting maps between these groups — the *Bockstein operations* — satisfying a number of 6-term exact sequences. To our knowledge, the first explicit use of K-theory with coefficients in the classification program was by Dadarlat and Loring as an obstruction to the classification of certain non-simple C^*-algebras using the Elliott invariant ([64]). Eilers subsequently used K-theory with coefficients (equipped with an order structure) to classify certain AD algebras [81].

For $n \geq 2$ and $i \in \{0,1\}$, the groups $K_i(A;\mathbb{Z}/n)$ and the Bockstein operations were introduced by Schochet in [255] and have subsequently been expressed in a number of equivalent forms. We will use two pictures of total K-theory in the paper. For the purpose of constructing our new pairing map connecting $K_0(A;\mathbb{Z}/n)$ to $\overline{K}_1^{\mathrm{alg}}(A)$ in Section 3.1, we use the definition[68]

$$K_i(A;\mathbb{Z}/n) \coloneqq K_{1-i}(A \otimes \mathbb{I}_n), \tag{2.26}$$

where $\mathbb{I}_n$ is the algebra

$$\mathbb{I}_n \coloneqq \{f \in C([0,1], M_n) : f(0) \in \mathbb{C}1_{M_n},\ f(1) = 0\}. \tag{2.27}$$

[68]This form of the definition is obtained in the joint work of Cuntz and Schochet found in [255, Section 6]; as noted there, one can define $K_i(\,\cdot\,;\mathbb{Z}/n) = K_i(\,\cdot \otimes C_n)$ using any separable nuclear C^*-algebra C_n in the UCT class with $K_*(C_n) \cong (\mathbb{Z}/n, 0)$. In our case, we use $C_n \coloneqq C_0((0,1)) \otimes \mathbb{I}_n$; as $K_*(\mathbb{I}_n) \cong (0, \mathbb{Z}/n)$, the suspension gives the dimension shift from K_i to K_{1-i}. Other possible choices for C_n include the Cuntz algebras $\mathcal{O}_{n+1}$ (as we shall briefly use in Lemma 3.16) or — as in Schochet's original definition — the mapping cone of a degree n map on $C_0(0,1)$.

Later in the paper, when we need to exploit the intimate links between total K-theory and KK-theory through Dadarlat and Loring's universal multicoefficient theorem, we will use the formulation in terms of $KK^i(\mathbb{I}_n, A)$ from [66, Section 1]. While it is well-known to experts that all pictures of total K-theory are naturally isomorphic, this is not easily extracted from the literature. We handle this point in Appendix A.

Using the definition in (2.26) one obtains natural Bockstein maps

$$\mu^{(n)}_{i,A}\colon K_i(A) \longrightarrow K_i(A;\mathbb{Z}/n) \text{ and } \nu^{(n)}_{i,A}\colon K_i(A;\mathbb{Z}/n) \longrightarrow K_{1-i}(A) \tag{2.28}$$

from the short exact sequence $0 \to C_0((0,1), M_n) \to \mathbb{I}_n \to \mathbb{C} \to 0$, which induces[69] a 6-term exact sequence

$$\begin{tikzcd} K_0(A) \arrow[r, "\mu^{(n)}_{0,A}"] & K_0(A;\mathbb{Z}/n) \arrow[r, "\nu^{(n)}_{0,A}"] & K_1(A) \arrow[d, "\times n"] \\ K_0(A) \arrow[u, "\times n"] & K_1(A;\mathbb{Z}/n) \arrow[l, "\nu^{(n)}_{1,A}"] & K_1(A) \arrow[l, "\mu^{(n)}_{1,A}"] \end{tikzcd} \tag{2.29}$$

that is natural in A. For $n, m \geq 2$, the remaining Bockstein operations

$$\begin{aligned} &\kappa^{(nm,n)}_{i,A}\colon K_i(A;\mathbb{Z}/n) \to K_i(A;\mathbb{Z}/nm) \quad \text{and} \\ &\kappa^{(n,nm)}_{i,A}\colon K_i(A;\mathbb{Z}/nm) \to K_i(A;\mathbb{Z}/n) \end{aligned} \tag{2.30}$$

are induced by the canonical inclusions $\mathbb{I}_n \xrightarrow{\cong} \mathbb{I}_n \otimes 1_{M_m} \hookrightarrow \mathbb{I}_{nm}$ and $\mathbb{I}_{nm} \hookrightarrow \mathbb{I}_n \otimes M_m$ respectively. These maps interact with the rows of (2.29) through the following commutative diagram with exact columns (which is easily checked directly from the definitions):

$$\begin{tikzcd} K_i(A) \arrow[r, equal] \arrow[d, "\times n"] & K_i(A) \arrow[r, "\times m"] \arrow[d, "\times nm"] & K_i(A) \arrow[d, "\times n"] \\ K_i(A) \arrow[r, "\times m"] \arrow[d, "\mu^{(n)}_{i,A}"] & K_i(A) \arrow[r, equal] \arrow[d, "\mu^{(nm)}_{i,A}"] & K_i(A) \arrow[d, "\mu^{(n)}_{i,A}"] \\ K_i(A;\mathbb{Z}/n) \arrow[r, "\kappa^{(nm,n)}_{i,A}"] \arrow[d, "\nu^{(n)}_{i,A}"] & K_i(A;\mathbb{Z}/nm) \arrow[r, "\kappa^{(n,nm)}_{i,A}"] \arrow[d, "\nu^{(nm)}_{i,A}"] & K_i(A;\mathbb{Z}/n) \arrow[d, "\nu^{(n)}_{i,A}"] \\ K_{1-i}(A) \arrow[r, equal] \arrow[d, "\times n"] & K_{1-i}(A) \arrow[r, "\times m"] \arrow[d, "\times nm"] & K_{1-i}(A) \arrow[d, "\times n"] \\ K_{1-i}(A) \arrow[r, "\times m"] & K_{1-i}(A) \arrow[r, equal] & K_{1-i}(A) \end{tikzcd} \tag{2.31}$$

[69] For this, tensor with A and apply the natural identification $K_i(C_0(0,1) \otimes M_n \otimes A) \cong K_{1-i}(A)$ from Bott periodicity and stability of K-theory. It is easily checked that the boundary maps are multiplication by n. These Bockstein maps are often denoted by ρ and β in the literature, but as we consistently use ρ and β for other maps, we have chosen to denote these Bockstein maps by μ and ν.

In addition, the maps κ_A fit into the exact sequence

$$\begin{tikzcd} K_0(A;\mathbb{Z}/m) \arrow[r, "\kappa_{0,A}^{(nm,m)}"] & K_0(A;\mathbb{Z}/nm) \arrow[r, "\kappa_{0,A}^{(n,nm)}"] & K_0(A;\mathbb{Z}/n) \arrow[d, "\mu_{1,A}^{(m)}\circ\nu_{0,A}^{(n)}"] \\ K_1(A;\mathbb{Z}/n) \arrow[u, "\mu_{0,A}^{(m)}\circ\nu_{1,A}^{(n)}"] & K_1(A;\mathbb{Z}/nm) \arrow[l, "\kappa_{1,A}^{(n,nm)}"] & K_1(A;\mathbb{Z}/m) \arrow[l, "\kappa_{1,A}^{(nm,m)}"] \end{tikzcd} \tag{2.32}$$

but since this does not play an explicit role in the paper,[70] we refer to [255, Proposition 2.6] for details on this.

Definition 2.14. The *total K-theory*, $\underline{K}(A)$, of a C^*-algebra A is the collection of abelian groups $K_i(A)$ and $K_i(A;\mathbb{Z}/n)$ for $i = 0,1$ and $n = 2,3,\dots$, together with all the Bockstein maps

$$\big\{\mu_{i,A}^{(n)}, \nu_{i,A}^{(n)}, \kappa_{i,A}^{(n,nm)}, \kappa_{i,A}^{(nm,n)} : n,m \geq 2, i \in \{0,1\}\big\}. \tag{2.33}$$

A Λ-*morphism* $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(B)$ consists of homomorphisms

$$\alpha_i\colon K_i(A) \to K_i(B) \quad \text{and} \quad \alpha_i^{(n)}\colon K_i(A;\mathbb{Z}/n) \to K_i(B;\mathbb{Z}/n) \tag{2.34}$$

for all $i \in \{0,1\}$ and $n \geq 2$ that intertwine all the Bockstein operations. The collection of these Λ-morphisms is written $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$.[71]

As noted in [255, Proposition 1.8], the sequence (2.29) collapses to a short exact sequence[72]

$$0 \longrightarrow K_i(A)\otimes\mathbb{Z}/n \xrightarrow{\overline{\mu}_{i,A}^{(n)}} K_i(A;\mathbb{Z}/n) \xrightarrow{\overline{\nu}_{i,A}^{(n)}} \mathrm{Tor}(K_{1-i}(A),\mathbb{Z}/n) \longrightarrow 0. \tag{2.35}$$

Explicitly, $\mathrm{Tor}(K_{1-i}(A),\mathbb{Z}/n)$ consists of those elements $x \in K_{1-i}(A)$ with $nx = 0$. The sequence (2.35) splits unnaturally ([255, Proposition 2.4]). Moreover, as is known to experts, these splittings may be chosen so that they preserve the Bockstein operations. This goes back to Bödigheimer ([19, 20]) in a purely abstract framework and is noted in the C^*-algebraic setting in [81, Lemma 2.2.8] and [83, Remark 4.6], resulting in Proposition 2.15 below. We explain how to extract this precise statement from the literature in Appendix A.3 (the proof follows that of Lemma A.5) since it is not entirely straightforward.[73]

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[70] The exactness of (2.32) is important in the proof of the universal multicoefficient theorem in [66].

[71] This notation comes from the point of view, set out in [66], that the Bockstein operations span a non-unital ring Λ, and that the objects of total K-theory are modules over this ring. We prefer to consider the collection of K-groups, the K-groups with coefficients, and the Bockstein maps, as it better fits with our overall approach to the classification invariant. From this perspective, the objects of the total K-theory category are a collection of abelian groups indexed by $G_{i,n}$ for $(i,n) \in \{0,1\} \times \{0,2,3,4,\dots\}$ where $G_{i,n}$ for $n \geq 2$ is n-torsion, together with Bockstein maps so that the sequences (2.29) and (2.32) are chain complexes and the diagrams (2.31) commute. With the Λ-morphisms as described above, this gives an abelian category.

[72] Alternatively, as $\mathbb{I}_n$ satisfies the UCT, (2.35) can be obtained from the Künneth exact sequence for $K_{1-i}(A\otimes\mathbb{I}_n)$ from [245].

[73] In order to be clear about exactly where the UCT assumption plays a role in our approach to the classification theorem, we highlight that Proposition 2.15 is a purely algebraic result that doesn't require the UCT. However, in the proof of the classification theorem, we will only apply Proposition 2.15 when A satisfies the UCT. In this case, as will be familiar to many readers, the result quickly follows from the UCT assumption. Briefly, any $\alpha_*\colon K_*(A) \to K_*(B)$ lifts to an element of $KK(A,B)$ by the UCT, and this induces a Λ-morphism $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(B)$, which restricts to α_*. The map $\underline{\alpha}$ is an isomorphism if α_* is (by applying the five lemma to (2.29)). For

Proposition 2.15. *Let A and B be C^*-algebras. Any morphism $\alpha_*\colon K_*(A) \to K_*(B)$ can be extended to a Λ-morphism $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(B)$, which is an isomorphism when α_* is.*

As with $\overline{K}_1^{\mathrm{alg}}(A)$, the additional information captured by $\underline{K}$ — compared with K_* — lies in its behavior on morphisms, and not in the constituent groups of $\underline{K}(A)$ (which are completely determined by $K_*(A)$).

We end this subsection with the following calculation using the Bockstein maps, which will be applied to the trace-kernel extension as part of the proof of Theorem 1.6. The relevant hypotheses will be verified in Section 6.3. If G is an abelian group, we write $\mathrm{Tor}(G)$ for the torsion subgroup of G consisting of those elements in G of finite order; i.e., $\mathrm{Tor}(G) = \bigcup_{n\geq 2} \mathrm{Tor}(G, \mathbb{Z}/n)$.

Lemma 2.16. *Let A be a C^*-algebra and let*

$$0 \to I \xrightarrow{j} E \xrightarrow{q} D \to 0 \tag{2.36}$$

be an extension of C^*-algebras such that $K_0(D)$ is uniquely divisible[74] and $K_1(D) = 0$. Then a Λ-morphism $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(I)$ belongs to $\ker \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(j)\big)$ if and only if the only non-zero component of $\underline{\alpha}$ is α_1 and this component vanishes on $\mathrm{Tor}(K_1(A))$. In fact, there is a natural[75] isomorphism

$$\ker \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(j)\big) \xrightarrow{\cong} \mathrm{Hom}\big(K_1(A)/\mathrm{Tor}\big(K_1(A)\big), \ker K_1(j)\big). \tag{2.37}$$

Proof. Since $K_0(D)$ is uniquely divisible, multiplication by n is an automorphism of $K_0(D)$ for all $n \geq 2$. This, together with $K_1(D) = 0$ and the exactness of (2.29), gives $K_i(D;\mathbb{Z}/n) = 0$ for all $i = 0,1$ and $n \geq 2$. Since $K_*(\,\cdot\,;\mathbb{Z}/n)$ takes short exact sequences to 6-term exact sequences (this follows from its definition as $K_{1-*}(\,\cdot \otimes \mathbb{I}_n)$), it follows that $\ker K_i(j;\mathbb{Z}/n) = 0$ for all $n \geq 2$ and $i \in 0,1$. Similarly, $\ker K_0(j) = 0$ since $K_1(D) = 0$.

Let $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(I)$ be in $\ker \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(j))$ with components α_i and $\alpha_i^{(n)}$ for $i = 0,1$ and $n \geq 2$. As these homomorphisms factor through $\ker K_i(j)$ and $\ker K_i(j;\mathbb{Z}/n)$ respectively, it follows that $\alpha_0 = 0$ and $\alpha_i^{(n)} = 0$, for $i = 0,1$. Because $\underline{\alpha}$ is a Λ-morphism, we have $\alpha_1 \circ \nu_{0,A}^{(n)} = \nu_{0,I}^{(n)} \circ \alpha_0^{(n)} = 0$. It follows that α_1 vanishes on the image of $\nu_{0,A}^{(n)}$ for every $n \geq 2$. By exactness of (2.29), the image of $\nu_{0,A}^{(n)}$ consists precisely of the n-torsion elements of $K_1(A)$, and thus α_1 vanishes on $\mathrm{Tor}(K_1(A))$. Thus the map sending $\underline{\alpha} \in \ker \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(j))$ to the map $K_1(A)/\mathrm{Tor}(K_1(A)) \to K_1(A)$ induced by α_1 is a well-defined injection.

For surjectivity in (2.37), suppose $\alpha_1'\colon K_1(A)/\mathrm{Tor}(K_1(A)) \to \ker K_1(j)$ is a homomorphism, and let α_1 be the composition

$$K_1(A) \longrightarrow K_1(A)/\mathrm{Tor}(K_1(A)) \xrightarrow{\alpha_1'} \ker K_1(j) \subseteq K_1(I). \tag{2.38}$$

We define $\underline{\alpha}$ by setting the other components, that is, $\alpha_0\colon K_0(A) \to K_0(I)$ and $\alpha_i^{(n)}\colon K_i(A;\mathbb{Z}/n) \to K_i(I;\mathbb{Z}/n)$, to be the zero maps. This will certainly lie in the

this reason we do not include [19, 20] in the list of references in Section 1.4.2, which includes only those that we rely on to prove Theorems A and B.

[74] That is, for each $x \in K_0(D)$ and $n \in \mathbb{Z}$ with $n \neq 0$, there exists a unique $y \in K_0(D)$ with $ny = x$.

[75] It is natural in both A and in extensions satisfying the specified hypotheses.

kernel of $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(j))$ once we have shown that these choices of components do intertwine the Bockstein operations, i.e., that $\underline{\alpha}$ is a Λ-morphism.

Since $\alpha_0 = 0$ and $\alpha_i^{(n)} = 0$, we only need to verify intertwinings involving α_1. That is, we must verify the commutativity of

$$\begin{tikzcd} K_1(A) \arrow[r, "\mu_{1,A}^{(n)}"] \arrow[d, "\alpha_1"] & K_1(A;\mathbb{Z}/n) \arrow[d, "\alpha_1^{(n)}=0"] \\ K_1(I) \arrow[r, "\mu_{1,I}^{(n)}"] & K_1(I;\mathbb{Z}/n) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K_0(A;\mathbb{Z}/n) \arrow[r, "\nu_{0,A}^{(n)}"] \arrow[d, "\alpha_0^{(n)}=0"'] & K_1(A) \arrow[d, "\alpha_1"] \\ K_0(I;\mathbb{Z}/n) \arrow[r, "\nu_{0,I}^{(n)}"] & K_1(I) \end{tikzcd} . \tag{2.39}$$

As noted above, the image of $\nu_{0,A}^{(n)}$ consists of n-torsion elements so is contained in $\mathrm{Tor}(K_1(A))$. Thus $\alpha_1 \circ \nu_{0,A}^{(n)} = 0$ and the right square in (2.39) commutes. For commutativity of the left square, we note that since α_1 takes values in $\ker K_1(j)$, naturality of the Bockstein maps $\mu_1^{(n)}$ gives

$$K_1(j;\mathbb{Z}/n) \circ \mu_{1,I}^{(n)} \circ \alpha_1 = \mu_{1,E}^{(n)} \circ K_1(j) \circ \alpha_1 = 0. \tag{2.40}$$

By injectivity of $K_1(j;\mathbb{Z}/n)$ (as observed at the beginning the proof), it follows that $\mu_{1,I}^{(n)} \circ \alpha_1 = 0$, so the left hand square of (2.39) also commutes. $\square$

3. The total invariant

This section assembles the total invariant $\underline{K}T_u(\,\cdot\,)$. We start out by constructing natural maps

$$\zeta_A^{(n)} \colon K_0(A;\mathbb{Z}/n) \to \overline{K}_1^{\mathrm{alg}}(A) \tag{3.1}$$

relating total and algebraic K-theory in Section 3.1. Such a system of maps was independently constructed by Gong, Lin, and Niu in [124] by other means; see Remark 3.4. Section 3.2 formalizes $\underline{K}T_u(\,\cdot\,)$ and proves an "extension of invariants" theorem (Theorem 3.9): (iso)morphisms at the level of $KT_u(\,\cdot\,)$ lift to $\underline{K}T_u(\,\cdot\,)$. This is crucial in deducing Theorem A and Corollary C from Theorem B. We end by observing that the inclusion of a C^*-algebra into its sequence algebra is injective on invariants (Lemma 3.16), for use in establishing Theorem B from Theorem 1.1.

3.1. Factorizing the Bockstein maps.

Let A be a unital C^*-algebra. There is a well-known bijection between projections $p \in A$ and self-adjoint unitaries $u \in A$ given by $p \mapsto 1 - 2p = e^{2\pi i p/2}$. Since self-adjoint unitaries are trivial in K_1, this identification is not interesting at the level of K-theory. But the map $[p]_0 \mapsto [1 - 2p]_{\mathrm{alg}} = [e^{2\pi i p/2}]_{\mathrm{alg}}$ does give a well-defined natural map $K_0(A) \to \overline{K}_1^{\mathrm{alg}}(A)$ that factorizes through $K_0(A) \otimes \mathbb{Z}/2$.[76] More generally, for any $n \geq 2$, we have a map $K_0(A) \to \overline{K}_1^{\mathrm{alg}}(A)$ given by $[p]_0 \mapsto [e^{2\pi i p/n}]_{\mathrm{alg}}$ that factors through $K_0(A) \otimes \mathbb{Z}/n$. Recall that the Bockstein maps give rise to the short exact sequence

$$0 \longrightarrow K_0(A) \otimes \mathbb{Z}/n \xrightarrow{\bar{\mu}_{0,A}^{(n)}} K_0(A;\mathbb{Z}/n) \xrightarrow{\bar{\nu}_{0,A}^{(n)}} \mathrm{Tor}(K_1(A), \mathbb{Z}/n) \longrightarrow 0 \tag{3.2}$$

that splits unnaturally (see (2.35)). This allows the maps above to be extended (a priori unnaturally) from $K_0(A) \otimes \mathbb{Z}/n$ to $K_0(A;\mathbb{Z}/n)$.

[76] The map is $[p]_0 \mapsto \mathrm{Th}_A(\rho_A([p]_0)/2)$, which shows that it is well-defined.

In the next two paragraphs and Proposition 3.1, we show that this can, in fact, be done naturally. Then, in Proposition 3.2, we show that the resulting map is compatible with the other structure maps.

Fix $n \geq 2$. We identify $(A \otimes \mathbb{I}_n)^\dagger$ with

$$\{f \in C([0,1], A \otimes M_n) : f(0) \in A \otimes 1_{M_n},\ f(1) \in \mathbb{C}1_{A \otimes M_n}\}. \tag{3.3}$$

Write $\mathrm{ev}_A^{(0,n)} \colon (A \otimes \mathbb{I}_n)^\dagger \to A$ for the map induced by evaluation at 0. (In particular, note that $\mathrm{ev}_A^{(0,n)}(1_{(A\otimes\mathbb{I}_n)^\dagger}) = 1_A$ and not $1_{A\otimes M_n}$.) We will abuse notation and also write $\mathrm{ev}_A^{(0,n)}$ for the induced map when taking matrix amplifications. Then, using the definition $K_0(A;\mathbb{Z}/n) = K_1(A \otimes \mathbb{I}_n)$, the Bockstein map $\nu_{0,A}^{(n)} \colon K_0(A;\mathbb{Z}/n) \to K_1(A)$ is the map $K_1(\mathrm{ev}_A^{(0,n)})$.

Let $u \in U_\infty((A \otimes \mathbb{I}_n)^\dagger)$; in spirit, we would like to define $\zeta_A^{(n)}([u]_1)$ to be $[\mathrm{ev}_A^{(0,n)}(u)]_{\mathrm{alg}}$. However, this would not be well-defined since the class $[\mathrm{ev}_A^{(0,n)}(u)]_{\mathrm{alg}}$ in $\overline{K}_1^{\mathrm{alg}}(A)$ is not invariant under homotopy, and thus we need a correcting term built using the de la Harpe–Skandalis determinant. Write $\mathrm{ev}_A^{(1,n)} \colon (A \otimes \mathbb{I}_n)^\dagger \to \mathbb{C}1_A \subseteq A$ for the map induced by evaluation at 1 (again with the normalization convention $\mathrm{ev}_A^{(1,n)}(1_{(A\otimes\mathbb{I}_n)^\dagger}) = 1_A$). Regarding u as element of $U_\infty(C([0,1],A))$, one obtains $\widetilde{\Delta}_A(u) \in \mathrm{Aff}\, T(A)$ as in Proposition 2.11. Now define $\widetilde{\zeta}_A^{(n)} \colon U_\infty((A\otimes\mathbb{I}_n)^\dagger) \to \overline{K}_1^{\mathrm{alg}}(A)$ by

$$\widetilde{\zeta}_A^{(n)}(u) \coloneqq [\mathrm{ev}_A^{(0,n)}(u)]_{\mathrm{alg}} - [\mathrm{ev}_A^{(1,n)}(u)]_{\mathrm{alg}} + \mathrm{Th}_A\big(\tfrac{1}{n}\widetilde{\Delta}_A(u)\big). \tag{3.4}$$

The factor of $\frac{1}{n}$ arises in (3.4) to account for the fact that when $u \in (A \otimes \mathbb{I}_n)^\dagger$, the computation in the last term is done viewing u as a path in $M_n(A)$, while the normalization conventions use elements of A for the first two terms.

Note that the last two terms in (3.4) lie in the kernel of $\not{a}_A$ (as $\mathrm{ev}_A^{(1,n)}(u) \in U_\infty(\mathbb{C})$, and Th_A takes values in $\ker \not{a}_A$). Accordingly,

$$\not{a}_A\big(\widetilde{\zeta}_A^{(n)}(u)\big) = [\mathrm{ev}_A^{(0,n)}(u)]_1 = \nu_{0,A}^{(n)}([u]_1). \tag{3.5}$$

Proposition 3.1. *Let A be a unital C^*-algebra and let $n \geq 2$. Then the map $\widetilde{\zeta}_A^{(n)}$ in (3.4) induces a group homomorphism $\zeta_A^{(n)} \colon K_0(A;\mathbb{Z}/n) \to \overline{K}_1^{\mathrm{alg}}(A)$ that is natural in A and factorizes the Bockstein map $\nu_{0,A}^{(n)}$ through $\overline{K}_1^{\mathrm{alg}}(A)$ as $\nu_{0,A}^{(n)} = \not{a}_A \circ \zeta_A^{(n)}$.*

Proof. As $\widetilde{\Delta}_A$ gives a homomorphism from $U_\infty(C([0,1],A))$ to $\mathrm{Aff}\, T(A)$ by Proposition 2.11, we have $\widetilde{\zeta}_A^{(n)}(uv) = \widetilde{\zeta}_A^{(n)}(u) + \widetilde{\zeta}_A^{(n)}(v)$ for all $u, v \in U_\infty((A \otimes \mathbb{I}_n)^\dagger)$. Therefore, to show $\widetilde{\zeta}_A^{(n)}$ induces a well-defined homomorphism $\zeta_A^{(n)}$ on $K_0(A;\mathbb{Z}/n)$ it suffices to check that $\widetilde{\zeta}_A^{(n)}(e^{2\pi i h}) = 0$ in $\overline{K}_1^{\mathrm{alg}}(A)$ for a self-adjoint $h \in M_k(A\otimes\mathbb{I}_n)^\dagger$ and $k \in \mathbb{N}$. For $\tau \in T(A)$, Proposition 2.11(ii) gives

$$\tfrac{1}{n}\widetilde{\Delta}_A(e^{2\pi i h})(\tau) = \tfrac{1}{n}\tau_{nk}\big(h(1) - h(0)\big) = \tau_k\big(\,\mathrm{ev}_A^{(1,n)}(h) - \mathrm{ev}_A^{(0,n)}(h)\big) \tag{3.6}$$

where τ_{nk} and τ_k are the non-normalized extensions of τ to $M_{kn}(A)$ and $M_k(A)$, and the second equality is a consequence of our normalization conventions. Thus, $\mathrm{ev}_A^{(1,n)}(h) - \mathrm{ev}_A^{(0,n)}(h)$ induces the affine function $\frac{1}{n}\widetilde{\Delta}_A(e^{2\pi i h})$, and so the definition of the Thomsen map in (2.15) (working at the matrix amplification level as discussed in the text above (2.17)) gives

$$
\begin{aligned}
\mathrm{Th}_A\big(\tfrac{1}{n}\widetilde{\Delta}_A(e^{2\pi i h})\big) &= [e^{2\pi i(\mathrm{ev}_A^{(1,n)}(h)-\mathrm{ev}_A^{(0,n)}(h))}]_{\mathrm{alg}} \\
&\overset{(2.16)}{=} [\mathrm{ev}_A^{(1,n)}(e^{2\pi i h})]_{\mathrm{alg}} - [\mathrm{ev}_A^{(0,n)}(e^{2\pi i h})]_{\mathrm{alg}}.
\end{aligned} \tag{3.7}
$$

Hence $\tilde{\zeta}_A^{(n)}(e^{2\pi i h}) = 0$ and so $\zeta_A^{(n)}$ is well-defined. Now $\nu_{0,A}^{(n)} = \not{a}_A \circ \zeta_A^{(n)}$ follows from (3.5).

Naturality of $\zeta^{(n)}$ follows as all components of the construction of $\zeta^{(n)}$ in (3.4) are natural. $\square$

The next proposition shows that the maps $\zeta_A^{(n)}$ satisfy three further compatibility relations with the other structure maps.

Proposition 3.2. *If A is a unital C^*-algebra and $m, n \geq 2$, then the diagrams*

$$
\begin{tikzcd}
K_0(A) \arrow[r, "\mu_{0,A}^{(n)}"] \arrow[d, "\frac{1}{n}\rho_A"] & K_0(A;\mathbb{Z}/n) \arrow[r, "\nu_{0,A}^{(n)}"] \arrow[d, "\zeta_A^{(n)}"] & K_1(A) \arrow[d, equal] \\
\mathrm{Aff}\, T(A) \arrow[r, "\mathrm{Th}_A"] & \overline{K}_1^{\mathrm{alg}}(A) \arrow[r, "\not{a}_A"] & K_1(A)
\end{tikzcd} \tag{3.8}
$$

and

$$
\begin{tikzcd}
K_0(A;\mathbb{Z}/nm) \arrow[r, "\kappa_{0,A}^{(n,nm)}"] \arrow[dr, "m\zeta_A^{(nm)}"'] & K_0(A;\mathbb{Z}/n) \arrow[r, "\kappa_{0,A}^{(nm,n)}"] \arrow[d, "\zeta_A^{(n)}"] & K_0(A;\mathbb{Z}/nm) \arrow[dl, "\zeta_A^{(nm)}"] \\
& \overline{K}_1^{\mathrm{alg}}(A) &
\end{tikzcd} \tag{3.9}
$$

commute.

Proof. Proposition 3.1 already shows that the right square in (3.8) commutes.

We first show the commutativity of the left square of (3.8). Recall that after identifying $K_0(A) \cong K_1(C_0((0,1),A))$ by Bott periodicity, $\mu_{0,A}^{(n)}$ is given by the inclusion of $C((0,1), A \otimes M_n)$ into $A \otimes \mathbb{I}_n$. Therefore, given a unitary $u \in U_\infty(C((0,1),A)^\dagger)$, and regarding it as a path $[0,1] \to U_\infty(A)$ with $u(0) = u(1) \in U_\infty(\mathbb{C})$, $\mu_{0,A}^{(n)}([u]_1)$ is the class of u in $K_1(A \otimes \mathbb{I}_n) = K_0(A;\mathbb{Z}/n)$. By Proposition 2.11(iii), $\widetilde{\Delta}_A(u) = \rho_A([u]_1)$ (where we again identify $K_0(A) \cong K_1(C_0((0,1),A))$). Since $\mathrm{ev}_A^{(0,n)}(u) = \mathrm{ev}_A^{(1,n)}(u)$, we have

$$
\zeta_A^{(n)}(\mu_{0,A}^{(n)}([u]_1)) = \widetilde{\zeta}_A^{(n)}(u) = \mathrm{Th}_A\left(\tfrac{1}{n}\rho_A([u]_1)\right). \tag{3.10}
$$

Next we deal with the right triangle of (3.9). Recall that $\kappa_A^{(nm,n)}$ is induced by the canonical inclusion $\iota\colon \mathbb{I}_n \xrightarrow{\cong} \mathbb{I}_n \otimes 1_{M_m} \hookrightarrow \mathbb{I}_{nm}$. For $u \in U_\infty((A \otimes \mathbb{I}_n)^\dagger)$ and $i = 0, 1$, we have $\mathrm{ev}_A^{(i,nm)}(\iota(u)) = \mathrm{ev}_A^{i,n}(u)$. Since $\widetilde{\Delta}_A(u \otimes 1_{M_m}) = m\widetilde{\Delta}_A(u)$, it follows that $(\zeta_A^{(nm)} \circ \kappa_A^{(nm,n)})([u]_1) = \zeta_A^{(n)}([u]_1)$.

Finally, we handle the left-hand triangle of (3.9). Since $\kappa_A^{(n,nm)}$ is induced by the inclusion $\mathbb{I}_{nm} \subseteq \mathbb{I}_n \otimes M_m$, for $u \in U_k((A \otimes \mathbb{I}_{nm})^\dagger)$, our normalization conventions give $\mathrm{ev}^{(i,n)}_{M_m(A)}(u) = \mathrm{ev}^{(i,nm)}_A(u) \otimes 1_{M_m}$ for $i = 0, 1$. Then

$$\begin{aligned}
&\zeta^{(n)}_A\left(\kappa^{(n,nm)}_A([u]_1)\right) = \\
&\quad= \left[\mathrm{ev}^{(0,n)}_{M_m(A)}(u)\right]_{\mathrm{alg}} - \left[\mathrm{ev}^{(1,n)}_{M_m(A)}(u)\right]_{\mathrm{alg}} + \mathrm{Th}_A\left(\tfrac{1}{n}\widetilde{\Delta}_A(u)\right) \\
&\quad= m\Big(\left[\mathrm{ev}^{(0,nm)}_A(u)\right]_{\mathrm{alg}} - \left[\mathrm{ev}^{(1,nm)}_A(u)\right]_{\mathrm{alg}} + \mathrm{Th}_A\left(\tfrac{1}{nm}\widetilde{\Delta}_A(u)\right)\Big) \\
&\quad= m\zeta^{(nm)}_A([u]_1),
\end{aligned} \tag{3.11}$$

as required. □

Remark 3.3. As $\zeta^{(n)}_A \circ \mu^{(n)}_{0,A} = \mathrm{Th}_A \circ \left(\frac{1}{n}\rho_A\right) : K_0(A) \to \overline{K}^{\mathrm{alg}}_1(A)$ by Proposition 3.2, it follows that if $p \in M_n(A)$ is a projection, then

$$\zeta^{(n)}_A\left(\mu^{(n)}_{0,A}([p]_0)\right) = \mathrm{Th}_A(\tfrac{1}{n}\hat{p}) = [e^{2\pi i p/n}]_{\mathrm{alg}}. \tag{3.12}$$

This shows that $\zeta^{(n)}_A \circ \overline{\mu}^{(n)}_{0,A}$ is the map $K_0(A) \otimes \mathbb{Z}/n \to \overline{K}^{\mathrm{alg}}_1(A)$ described in the opening paragraph of this subsection. When $K_1(A)$ has no n-torsion elements, $\bar{\mu}^{(n)}_{0,A}$ is an isomorphism, so $\zeta^{(n)}_A$ is completely determined by (3.12). In particular, if $A \coloneqq \mathbb{C}$ (or $A \coloneqq \mathcal{Z}$), then identifying $K_0(A; \mathbb{Z}/n) \cong \mathbb{Z}/n$ and $\overline{K}^{\mathrm{alg}}_1(A) \cong \mathbb{T}$, the map $\zeta^{(n)}_A$ is the embedding of n-th roots of unity in the circle.

Remark 3.4. As noted in the introduction, while this paper was being written, Gong, Lin, and Niu independently discovered the need for compatibility with natural maps $K_0(A; \mathbb{Z}/n) \to \overline{K}^{\mathrm{alg}}_1(A)$ in [124] — see Remark 6.6 therein.[77] (By a complete coincidence, the same symbol ζ was adopted in [124] and by us.)

Gong, Lin, and Niu construct their maps abstractly through a splitting of the Thomsen extension for $A \otimes \mathbb{I}_n$. It turns out that the two maps are identical (taking care of our different conventions regarding the orientation of the interval in $\mathbb{I}_n$). It will be more convenient to us to demonstrate this in the second paper in this series on non-unital C^*-algebras, where we will set out the Thomsen extension for non-unital algebras (such as $A \otimes \mathbb{I}_n$).

3.2. **The total invariant.** With all the ingredients in place, we now formalize the invariant needed to classify embeddings.

Definition 3.5. Let A be a unital C^*-algebra. The *total invariant* of A, denoted $\underline{K}T_u(A)$, is[78]

$$\underline{K}T_u(A) \coloneqq (\underline{K}(A), \mathrm{Aff}\, T(A), \overline{K}^{\mathrm{alg}}_1(A), [1_A]_0, \rho_A, \mathrm{Th}_A, \not{a}_A, (\zeta^{(n)}_A)_{n\geq 2}) \tag{3.13}$$

with the convention that $\mathrm{Aff}\, T(A) = 0$ if $T(A) = \emptyset$. A morphism $(\underline{\alpha}, \beta, \gamma) : \underline{K}T_u(A) \to \underline{K}T_u(B)$ consists of

- $\underline{\alpha} \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ with $\alpha_0([1_A]_0) = [1_B]_0$,
- a positive linear map $\gamma\colon\ \mathrm{Aff}\, T(A) \to \mathrm{Aff}\, T(B)$,[79] and

[77] The notation in [124] is in terms of unitary groups modulo the closure of the commutators, and there is often a finite stable rank condition which allows them to work with $U_n(A)$ instead of $U_\infty(A)$.

[78] Recall our convention that the Bockstein maps $\mu^{(n)}_{i,A}, \nu^{(n)}_{i,A}, \kappa^{(m,n)}_{i,A}$ form part of $\underline{K}(A)$.

[79] Note that the definition implies that the map γ is unital since

$$\gamma(1) = \gamma(\rho_A([1_A]_0)) \overset{(3.14)}{=} \rho_B(\alpha_0([1_A]_0)) = \rho_B([1_B]_0) = 1.$$

- a group homomorphism $\beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B)$,

such that

$$\begin{tikzcd} K_0(A) \arrow[r, "\rho_A"] \arrow[d, "\alpha_0"] & \operatorname{Aff} T(A) \arrow[r, "\mathrm{Th}_A"] \arrow[d, "\gamma"] & \overline{K}_1^{\mathrm{alg}}(A) \arrow[r, "\not\partial_A"] \arrow[d, "\beta"] & K_1(A) \arrow[d, "\alpha_1"] \\ K_0(B) \arrow[r, "\rho_B"] & \operatorname{Aff} T(B) \arrow[r, "\mathrm{Th}_B"] & \overline{K}_1^{\mathrm{alg}}(B) \arrow[r, "\not\partial_B"] & K_1(B) \end{tikzcd} \tag{3.14}$$

and

$$\begin{tikzcd} K_0(A;\mathbb{Z}/n) \arrow[r, "\zeta_A^{(n)}"] \arrow[d, "\alpha_0^{(n)}"] & \overline{K}_1^{\mathrm{alg}}(A) \arrow[d, "\beta"] \\ K_0(B;\mathbb{Z}/n) \arrow[r, "\zeta_B^{(n)}"] & \overline{K}_1^{\mathrm{alg}}(B) \end{tikzcd} \tag{3.15}$$

commute.

As with KT_u-morphisms, we say that $(\underline{\alpha},\beta,\gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B)$ is *faithful* if the induced map $\gamma^*\colon T(B) \to T(A)$ (given by Kadison duality) satisfies that $\gamma^*(\tau)$ is faithful for all $\tau \in T(B)$.

It is important for us to note that $\underline{K}T_u$ is a functor; the point is not the exact specification of the target category,[80] but rather that unital *-homomorphisms do induce $\underline{K}T_u$-morphisms by naturality of the $\zeta^{(n)}$ (together with functoriality of KT_u). This is vital in being able to prove the existence portion of the classification of full approximate embeddings in the presence of torsion in K_1. Moreover, $\underline{K}T_u(\,\cdot\,)$ is invariant under approximate unitary equivalence as each component is.

Proposition 3.6. *Given a unital *-homomorphism $\phi\colon A \to B$,*

$$\underline{K}T_u(\phi) \coloneqq (\underline{K}(\phi), \overline{K}_1^{\mathrm{alg}}(\phi), \operatorname{Aff} T(\phi))\colon \underline{K}T_u(A) \to \underline{K}T_u(B) \tag{3.16}$$

is a $\underline{K}T_u$-morphism. In this way, the assignment of $\underline{K}T_u(A)$ to a unital C^*-algebra A provides a functor from the category of unital C^*-algebras and unital *-homomorphisms. Furthermore, if $\phi,\psi\colon A \to B$ are approximately unitarily equivalent, then $\underline{K}T_u(\phi) = \underline{K}T_u(\psi)$.

Remark 3.7. Let B be a unital simple purely infinite C^*-algebra. Then $T(B) = \emptyset$ and exactness of (2.9) shows that $\not\partial_B$ is an isomorphism. Thus, on these algebras, the invariants $\underline{K}T_u(B)$ and $(\underline{K}(B), [1_B]_0)$ are equivalent (in the sense of Footnote 54). In particular, given any unital C^*-algebra A, there is a one-to-one correspondence between $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(B)$ with $\alpha_0([1_A]_0) = [1_B]_0$ and morphisms $(\underline{\alpha},\beta,\gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B)$, given by defining $\beta \coloneqq \not\partial_B^{-1} \circ \alpha_1 \circ \not\partial_A$ and $\gamma \coloneqq 0$. This allows the classification of nuclear embeddings of unital separable exact C^*-algebras with the UCT into unital Kirchberg algebras by total K-theory (see [238, 221, 159, 110] — in this level of generality, the result is recorded as [110, Theorem 8.12]) to be viewed as a classification using $\underline{K}T_u$.

---

[80] The target category consists of 8-tuples of the appropriate objects as in (3.13) such that all the diagrams in Proposition 3.2 commute, $\rho(1) = 1$, and the top row of (2.11) is exact at the third entry ($\overline{K}_1^{\mathrm{alg}}$) and almost exact at the second, in the sense that the image of ρ is dense in the kernel of Th.

Remark 3.8. For each integer $n \geq 2$, there are examples of triples $(\underline{\alpha}, \beta, \gamma)$ that satisfy (3.14) but not (3.15). Set $A \coloneqq \mathbb{I}_n^\dagger$ and let $B \coloneqq \mathcal{Z}$. Let $\psi\colon \mathbb{I}_n^\dagger \to \mathcal{Z}$ be an embedding inducing the Lebesgue trace on $\mathbb{I}_n^\dagger$ (such an embedding exists, for example, by [254, Theorem 4.11]). Define $\underline{\alpha} \coloneqq \underline{K}(\psi)$ and $\gamma \coloneqq \operatorname{Aff} T(\psi)$. As

$$K_1(\mathbb{I}_n^\dagger) \cong \mathbb{Z}/n \quad \text{and} \quad \operatorname{Aff} T(\mathcal{Z})/\overline{\operatorname{im}\rho_{\mathcal{Z}}} \cong \mathbb{R}/\mathbb{Z}, \tag{3.17}$$

there is an embedding $r\colon K_1(\mathbb{I}_n^\dagger) \to \operatorname{Aff} T(\mathcal{Z})/\overline{\operatorname{im}\rho_{\mathcal{Z}}}$. Define

$$\beta \coloneqq \overline{K}_1^{\mathrm{alg}}(\psi) + \overline{\mathrm{Th}}_{\mathcal{Z}} \circ r \circ \not a_{\mathbb{I}_n^\dagger} \colon \overline{K}_1^{\mathrm{alg}}(\mathbb{I}_n^\dagger) \to \overline{K}_1^{\mathrm{alg}}(\mathcal{Z}). \tag{3.18}$$

Then β satisfies (3.14). Also, by (3.8) and the fact that $\underline{K}T_u(\psi)$ is an $\underline{K}T_u$-morphism (Proposition 3.6), we have

$$\beta \circ \zeta^{(n)}_{\mathbb{I}_n^\dagger} = \zeta^{(n)}_{\mathcal{Z}} \circ \alpha_0^{(n)} + \overline{\mathrm{Th}}_{\mathcal{Z}} \circ r \circ \nu^{(n)}_{0,\mathbb{I}_n^\dagger}. \tag{3.19}$$

Since $K_1(\mathbb{I}_n^\dagger) \cong \mathbb{Z}/n$, exactness of (2.29) implies that $\nu^{(n)}_{0,\mathbb{I}_n^\dagger}$ is surjective. As $\overline{\mathrm{Th}}_{\mathcal{Z}}$ is injective and $r \neq 0$, it follows that $\overline{\mathrm{Th}}_{\mathcal{Z}} \circ r \circ \nu^{(n)}_{0,\mathbb{I}_n^\dagger} \neq 0$. Hence $\beta \circ \zeta^{(n)}_{\mathbb{I}_n^\dagger} \neq \zeta^{(n)}_{\mathcal{Z}} \circ \alpha_0^{(n)}$; i.e., (3.15) does not commute.

For unital C^*-algebras A and B, a morphism $(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B)$ certainly restricts to a morphism $(\alpha_*, \gamma)\colon KT_u(A) \to KT_u(B)$. To obtain classification results for C^*-algebras using KT_u from those for $\underline{K}T_u$, we need to be able to go in the other direction and extend (in a non-canonical fashion) KT_u-(iso)morphisms to $\underline{K}T_u$-(iso)morphisms. Results of this form (often using the UCT to extend morphisms from K-theory to total K-theory rather than via Bödigheimer's result, Proposition 2.15), have been used in the classification program previously in cases where $\underline{K}$ or $\overline{K}_1^{\mathrm{alg}}$ is needed to obtain uniqueness of morphisms — see the proof of [125, Theorem 21.9], for example. The new ingredient we need is to construct an extension that intertwines the maps $\zeta^{(n)}$ of the previous subsection.

Theorem 3.9 (Extension of (iso)morphisms from KT_u to $\underline{K}T_u$)**.** *Let A and B be unital C^*-algebras. Then any morphism $(\alpha_*, \gamma)\colon KT_u(A) \to KT_u(B)$ can be extended to a morphism $(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B)$. In addition, if (α_*, γ) is an isomorphism, then any such extension $(\underline{\alpha}, \beta, \gamma)$ is an isomorphism.*

Proof. By Proposition 2.15, we may extend α_* to $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(B)$, and by Proposition 2.10, there exists $\beta'\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B)$ such that

$$\beta' \circ \mathrm{Th}_A = \mathrm{Th}_B \circ \gamma \quad \text{and} \quad \not a_B \circ \beta' = \alpha_1 \circ \not a_A. \tag{3.20}$$

Therefore, β' makes the first diagram (3.14) commute from the definition of a $\underline{K}T_u$-morphism, but it need not make (3.15) commute. The rest of the proof consists of identifying an appropriate correcting term to address this.

For $n \geq 2$, consider the maps

$$\beta' \circ \zeta_A^{(n)} - \zeta_B^{(n)} \circ \alpha_0^{(n)} \colon K_0(A; \mathbb{Z}/n) \to \overline{K}_1^{\mathrm{alg}}(B). \tag{3.21}$$

The plan is to prove that these maps factor through maps $s_n\colon \operatorname{Tor}(K_1(A), \mathbb{Z}/n) \to \ker \not a_B$ as compositions $s_n \circ \overline{\nu}^{(n)}_{0,A}$, and moreover that the resulting s_n are compatible, and so define a homomorphism $s\colon \operatorname{Tor}(K_1(A)) \to \ker \not a_B$. This will provide the needed correcting term.

Fix $n \geq 2$ for the next two paragraphs. Since $\underline{\alpha}$ is a Λ-homomorphism and (α_*, γ) is a morphism on KT_u, we get

$$\begin{aligned} \zeta_B^{(n)} \circ \alpha_0^{(n)} \circ \mu_{0,A}^{(n)} &\overset{(\Lambda)}{=} \zeta_B^{(n)} \circ \mu_{0,B}^{(n)} \circ \alpha_0 \\ &\overset{(3.8)}{=} \mathrm{Th}_B \circ \tfrac{1}{n}\rho_B \circ \alpha_0 \\ &\overset{(KT_u)}{=} \mathrm{Th}_B \circ \tfrac{1}{n}\gamma \circ \rho_A. \end{aligned} \tag{3.22}$$

Since γ is $\mathbb{R}$-linear, it follows that $\frac{1}{n}\gamma = \gamma \circ \frac{1}{n}$. Hence, continuing the computations, we get

$$\begin{aligned} \mathrm{Th}_B \circ \tfrac{1}{n}\gamma \circ \rho_A &= \mathrm{Th}_B \circ \gamma \circ \tfrac{1}{n}\rho_A \\ &\overset{(3.20)}{=} \beta' \circ \mathrm{Th}_A \circ \tfrac{1}{n}\rho_A \\ &\overset{(3.8)}{=} \beta' \circ \zeta_A^{(n)} \circ \mu_{0,A}^{(n)}. \end{aligned} \tag{3.23}$$

Hence the map $\beta' \circ \zeta_A^{(n)} - \zeta_B^{(n)} \circ \alpha_0^{(n)}$ vanishes on the image of $\mu_{0,A}^{(n)}$. By exactness of (2.29), this image is $\ker \nu_{0,A}^{(n)}$, and so the map factors through $\mathrm{Tor}(K_1(A), \mathbb{Z}/n)$ via the map $\overline{\nu}_{0,A}^{(n)} \colon K_0(A; \mathbb{Z}/n) \twoheadrightarrow \mathrm{Tor}(K_1(A), \mathbb{Z}/n)$ from (2.35).

Similarly, we find that

$$\begin{aligned} \not{a}_B \circ \zeta_B^{(n)} \circ \alpha_0^{(n)} &\overset{(3.8)}{=} \nu_{0,B}^{(n)} \circ \alpha_0^{(n)} \\ &\overset{(\Lambda)}{=} \alpha_1 \circ \nu_{0,A}^{(n)} \\ &\overset{(3.8)}{=} \alpha_1 \circ \not{a}_A \circ \zeta_A^{(n)} \\ &\overset{(3.20)}{=} \not{a}_B \circ \beta' \circ \zeta_A^{(n)} \end{aligned} \tag{3.24}$$

and so the range of $\beta' \circ \zeta_A^{(n)} - \zeta_B^{(n)} \circ \alpha_0^{(n)}$ is contained in $\ker \not{a}_B \subseteq \overline{K}_1^{\mathrm{alg}}(B)$. It follows that there is a unique homomorphism

$$s_n \colon \mathrm{Tor}(K_1(A), \mathbb{Z}/n) \longrightarrow \ker \not{a}_B \subseteq \overline{K}_1^{\mathrm{alg}}(B) \tag{3.25}$$

such that

$$s_n \circ \overline{\nu}_{0,A}^{(n)} = \beta' \circ \zeta_A^{(n)} - \zeta_B^{(n)} \circ \alpha_0^{(n)}. \tag{3.26}$$

Allowing n to vary, we identify the torsion subgroup $\mathrm{Tor}(K_1(A))$ of $K_1(A)$ with the union $\bigcup_{n \geq 2} \mathrm{Tor}(K_1(A), \mathbb{Z}/n)$. Let $s_{nm,n}$ denote the restriction of s_{nm} to $\mathrm{Tor}(K_1(A), \mathbb{Z}/n)$. We will show that $s_{nm,n} = s_n$ for all $n, m \geq 2$, so that these maps induce a homomorphism $s \colon \mathrm{Tor}(K_1(A)) \to \ker \not{a}_B$. By uniqueness of s_n in (3.26), it suffices to show that

$$s_{nm,n} \circ \overline{\nu}_{0,A}^{(n)} = \beta' \circ \zeta_A^{(n)} - \zeta_B^{(n)} \circ \alpha_0^{(n)}. \tag{3.27}$$

As noted in (2.31), $\nu_{0,A}^{(nm)} \circ \kappa_{0,A}^{(nm,n)} = \nu_{0,A}^{(n)} \colon K_0(A; \mathbb{Z}/n) \to K_1(A)$, so it follows that $\overline{\nu}_{0,A}^{(n)}$ is the corestriction of $\overline{\nu}_{0,A}^{(nm)} \circ \kappa_A^{(nm,n)}$ to $\mathrm{Tor}(K_1(A), \mathbb{Z}/n)$. Hence

$$\begin{aligned} s_{nm,n} \circ \overline{\nu}_{0,A}^{(n)} &= s_{nm} \circ \overline{\nu}_{0,A}^{(nm)} \circ \kappa_A^{(nm,n)} \\ &\overset{(3.26)}{=} (\beta' \circ \zeta_A^{(nm)} - \zeta_B^{(nm)} \circ \alpha_0^{(nm)}) \circ \kappa_A^{(nm,n)}. \end{aligned} \tag{3.28}$$

By commutativity of (3.9), we have $\beta' \circ \zeta_A^{(nm)} \circ \kappa_A^{(nm,n)} = \beta' \circ \zeta_A^{(n)}$. Also

$$\zeta_B^{(nm)} \circ \alpha_0^{(nm)} \circ \kappa_{0,A}^{(nm,n)} \overset{(\Lambda)}{=} \zeta_B^{(nm)} \circ \kappa_{0,B}^{(nm,n)} \circ \alpha_0^{(n)} \overset{(3.9)}{=} \zeta_B^{(n)} \circ \alpha_0^{(n)}. \tag{3.29}$$

Combining these last two equations with (3.28), we obtain (3.27), as desired. In conclusion, the maps s_n for $n \geq 2$ induce a homomorphism

$$s\colon \mathrm{Tor}(K_1(A)) \longrightarrow \ker \not\partial_B. \tag{3.30}$$

Since $\mathrm{Aff}\, T(B)/\overline{\mathrm{im}\, \rho_B}$ is divisible, and hence injective (see [285, Corollary 2.3.2]), so is $\ker \not\partial_B$, because the two groups are isomorphic via $\overline{\mathrm{Th}}_B$. Accordingly, s can be extended to a homomorphism

$$\hat{s}\colon K_1(A) \longrightarrow \ker \not\partial_B \subseteq \overline{K}_1^{\mathrm{alg}}(B). \tag{3.31}$$

We now use this to correct the definition of β' by setting

$$\beta := \beta' - \hat{s} \circ \not\partial_A \colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B). \tag{3.32}$$

It remains to prove that $(\underline{\alpha}, \beta, \gamma)$ is a morphism of $\underline{K}T_u$. We already know that (α_*, γ) is a morphism of KT_u, and that $\underline{\alpha}$ is a Λ-morphism, so we only need to show that the last two squares of (3.14) commute and that (3.15) commutes. Since $\not\partial_A \circ \overline{\mathrm{Th}}_A = 0$, we have

$$\beta \circ \mathrm{Th}_A \overset{(3.32)}{=} \beta' \circ \mathrm{Th}_A \overset{(3.20)}{=} \mathrm{Th}_A \circ \gamma. \tag{3.33}$$

Moreover,

$$\not\partial_B \circ \beta \overset{(3.31)}{=} \not\partial_B \circ \beta' \overset{(3.20)}{=} \alpha_1 \circ \not\partial_A. \tag{3.34}$$

Therefore, the last two squares of (3.14) commute.

For commutativity of (3.15), fix $n \geq 2$. Recall that $\hat{s}$ is an extension of s_n and that $\overline{\nu}_{0,A}^{(n)}$ is the corestriction of $\nu_{0,A}^{(n)}$ to $\mathrm{Tor}(K_1(A), \mathbb{Z}/n)$, which is the domain of s_n. It follows that $\hat{s} \circ \nu_{0,A}^{(n)} = s_n \circ \overline{\nu}_{0,A}^{(n)}$. Hence

$$\hat{s} \circ \not\partial_A \circ \zeta_A^{(n)} \overset{(3.8)}{=} s_n \circ \overline{\nu}_{0,A}^{(n)} \overset{(3.26)}{=} \beta' \circ \zeta_A^{(n)} - \zeta_B^{(n)} \circ \alpha_0^{(n)}, \tag{3.35}$$

and so

$$\begin{aligned} \beta \circ \zeta_A^{(n)} &\overset{(3.32)}{=} (\beta' - \hat{s} \circ \not\partial_A) \circ \zeta_A^{(n)} \\ &\overset{(3.35)}{=} \beta' \circ \zeta_A^{(n)} - (\beta' \circ \zeta_A^{(n)} - \zeta_B^{(n)} \circ \alpha_0^{(n)}) = \zeta_B^{(n)} \circ \alpha_0^{(n)}, \end{aligned} \tag{3.36}$$

establishing commutativity of (3.15). Therefore $(\underline{\alpha}, \beta, \gamma)$ defines a well-defined morphism $\underline{K}T_u(A) \to \underline{K}T_u(B)$.

Finally, Propositions 2.10 and 2.15 ensure that $\underline{\alpha}$ and β are isomorphisms if α_* and γ are, which implies that $(\underline{\alpha}, \beta, \gamma)$ is an isomorphism whenever (α_*, γ) is. $\square$

Remark 3.10. Note that the proof of Theorem 3.9 shows that given a KT_u-morphism $(\alpha_*, \gamma) : KT_u(A) \to KT_u(B)$ and any $\underline{\alpha} : \underline{K}(A) \to \underline{K}(B)$ extending α_*, there is an $\underline{K}T_u$-morphism of the form $(\underline{\alpha}, \beta, \gamma)$.

The following lemma is used in the computation of $KL(A, J_B)$ in Theorem 1.6 to show that every $\underline{K}T_u$-morphism in part (ii) comes from an appropriate KL-class. Its proof illustrates the role played by the maps $\zeta^{(n)}$.

Lemma 3.11. *Let A and B be unital C^*-algebras. If*

$$(\underline{\alpha}, \beta, \gamma), (\underline{\alpha}', \beta', \gamma') \colon \underline{K}T_u(A) \longrightarrow \underline{K}T_u(B) \tag{3.37}$$

are morphisms such that $\underline{\alpha} = \underline{\alpha}'$ and $\gamma = \gamma'$, then there is a unique homomorphism

$$r\colon K_1(A)/\mathrm{Tor}(K_1(A)) \longrightarrow \ker \not\partial_B \tag{3.38}$$

such that $\beta - \beta' \colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B)$ *is the composition*

$$\overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{\not a_A} K_1(A) \twoheadrightarrow K_1(A)/\mathrm{Tor}(K_1(A)) \xrightarrow{r} \ker \not a_B \rightarrowtail \overline{K}_1^{\mathrm{alg}}(B). \tag{3.39}$$

Proof. We have $(\beta - \beta') \circ \mathrm{Th}_A = \mathrm{Th}_B \circ (\gamma - \gamma') = 0$. It follows that $\beta - \beta'$ vanishes on $\operatorname{im} \mathrm{Th}_A = \ker \not a_A$ and thus factors uniquely as $\beta - \beta' = \hat{r} \circ \not a_A$, for some map $\hat{r} \colon K_1(A) \to \overline{K}_1^{\mathrm{alg}}(B)$. It suffices to show that $\hat{r}$ vanishes on $\mathrm{Tor}(K_1(A))$ and takes values in $\ker \not a_B$. Since for any $n \geq 2$ we have

$$\begin{aligned} \hat{r} \circ \nu_{0,A}^{(n)} &\overset{(3.8)}{=} \hat{r} \circ \not a_A \circ \zeta_A^{(n)} = (\beta - \beta') \circ \zeta_A^{(n)} \\ &\overset{(3.15)}{=} \zeta_B^{(n)} \circ (\alpha_0^{(n)} - \alpha_0'^{(n)}) = 0, \end{aligned} \tag{3.40}$$

it follows that $\hat{r}$ vanishes on the image of $\nu_{0,A}^{(n)}$. By exactness of (2.29), this image is $\mathrm{Tor}(K_1(A), \mathbb{Z}/n)$, and as such, $\hat{r}$ vanishes on $\mathrm{Tor}(K_1(A))$. Notice that

$$\not a_B \circ \hat{r} \circ \not a_A = \not a_B \circ (\beta - \beta') \overset{(3.20)}{=} (\alpha_1 - \alpha_1') \circ \not a_A = 0. \tag{3.41}$$

Since $\not a_A$ is surjective, it follows that $\hat{r}$ takes values in $\ker \not a_B$. □

Although we will not use it in this paper, we note that compatibility with the maps $\zeta^{(n)}$ is automatic when $K_1(A)$ is torsion-free. Gong, Lin, and Niu made a similar observation, by means of a different approach in their framework (see [124, Corollary 5.13 with condition (4)]).

Proposition 3.12. *Suppose that* A *and* B *are unital* C^**-algebras with* $K_1(A)$ *torsion-free. Then the condition of commutativity of* (3.15) *is automatic in the definition of a* $\underline{K}T_u$*-morphism.*

Proof. Let $(\underline{\alpha}, \beta, \gamma)$ be as in the definition of a $\underline{K}T_u$-morphism, except we only assume that (3.14) commutes. We will show that commutativity of (3.15) follows. Fix $n \geq 2$. We have

$$\beta \circ \zeta_A^{(n)} \circ \mu_{0,A}^{(n)} \overset{(3.8)}{=} \beta \circ \mathrm{Th}_A \circ \tfrac{1}{n} \rho_A \overset{(3.14)}{=} \mathrm{Th}_B \circ \gamma \circ \tfrac{1}{n} \rho_A. \tag{3.42}$$

Since γ is $\mathbb{R}$-linear we have $\gamma \circ \frac{1}{n} = \frac{1}{n}\gamma$. Therefore, continuing the computation above, we get

$$\begin{aligned} \mathrm{Th}_B \circ \tfrac{1}{n} \gamma \circ \rho_A &\overset{(3.14)}{=} \mathrm{Th}_B \circ \tfrac{1}{n} \rho_B \circ \alpha_0 \\ &\overset{(3.8)}{=} \zeta_B^{(n)} \circ \mu_{0,B}^{(n)} \circ \alpha_0 \overset{(\Lambda)}{=} \zeta_B^{(n)} \circ \alpha_0^{(n)} \circ \mu_{0,A}^{(n)}. \end{aligned} \tag{3.43}$$

Multiplication by n is an injective endomorphism on $K_1(A)$ because $K_1(A)$ is torsion-free, and so $\mu_{0,A}^{(n)} \colon K_0(A) \to K_0(A; \mathbb{Z}/n)$ is surjective by exactness of (2.29). Therefore, $\beta \circ \zeta_A^{(n)} = \zeta_B^{(n)} \circ \alpha_0^{(n)}$ as desired. □

3.3. **The sequence algebra.** In this short section we define the sequence algebra B_∞ of a C^*-algebra, record Kirchberg's ϵ-test for later use, and observe that $B \hookrightarrow B_\infty$ is injective on $\underline{K}T_u$.

Definition 3.13. Let B be a C^*-algebra. The *sequence algebra* of B, denoted B_∞, is the quotient $\ell^\infty(B)/c_0(B)$ of bounded sequences modulo those tending to zero. There is a canonical embedding

$$\iota_B \colon B \to B_\infty \tag{3.44}$$

induced by embedding B into $\ell^\infty(B)$ as constant sequences.

As is standard in the literature, we will typically use representative sequences in $\ell^\infty(B)$ to denote elements of B_∞. One important feature of the sequence algebra is that one can often use reindexing arguments to convert approximate properties into exact ones. Our preference is to formulate this using a version of Kirchberg's ϵ-test from [163] suited to sequence algebras. The form we quote below can be proven by an easy modification of the proof of [163, Lemma A.1] or [167, Lemma 3.1].

Lemma 3.14 (Kirchberg's ϵ-test)**.** *Let $(X_n)_{n=1}^\infty$ be a sequence of non-empty sets and write $X := \prod_{n=1}^\infty X_n$. Given functions $f_n^{(k)}\colon X_n \to [0,\infty]$ for $k,n \in \mathbb{N}$, define functions $f^{(k)}\colon X \to [0,\infty]$ by $f^{(k)}((x_n)_{n=1}^\infty) := \limsup_n f_n^{(k)}(x_n)$. Suppose that for all $\epsilon > 0$ and $k_0 \in \mathbb{N}$, there exists $x \in X$ with $f^{(k)}(x) < \epsilon$ for $k = 1,\ldots,k_0$. Then there exists $x \in X$ with $f^{(k)}(x) = 0$ for all $k \in \mathbb{N}$.*

The following application of the ϵ-test is completely standard. (See the proof of [21, Lemma 1.17], for example.)

Lemma 3.15. *Let A and B be C^*-algebras with A separable, and let $\phi,\psi\colon A \to B_\infty$ be *-homomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if they are unitarily equivalent.*

We end by checking that the inclusion $\iota_B : B \hookrightarrow B_\infty$ is injective on the total invariant.

Lemma 3.16. *If B is a unital C^*-algebra, then*

$$\underline{K}T_u(\iota_B)\colon \underline{K}T_u(B) \longrightarrow \underline{K}T_u(B_\infty) \tag{3.45}$$

is a monomorphism.

Proof. It suffices to prove that each of $\underline{K}(\iota_B)$, $\overline{K}_1^{\mathrm{alg}}(\iota_B)$, and $\mathrm{Aff}\,T(\iota_B)$ is injective.

For $\mathrm{Aff}\,T(\iota_B)$, when $T(B) = \emptyset$, the result is vacuous, so assume $T(B) \neq \emptyset$. Let $\tau \in T(B)$, let ω be a free ultrafilter on $\mathbb{N}$ and define $\tau_\omega\colon B_\infty \to \mathbb{C}$ by $\tau_\omega((b_n)_{n=1}^\infty) := \lim_{n\to\omega} \tau(b_n)$. Then $\tau_\omega \in T(B_\infty)$ and $\tau_\omega \circ \iota_B = \tau$, proving that $T(\iota_B)$ is surjective, and hence $\mathrm{Aff}\,T(\iota_B)$ is injective by duality.

For $\overline{K}_1^{\mathrm{alg}}(\iota_B)$, take a unitary $u \in M_r(B)$ with $[\iota_B(u)]_{\mathrm{alg}} = 0$ in $\overline{K}_1^{\mathrm{alg}}(B_\infty)$. Given $\epsilon > 0$, there are positive integers k and s and unitaries $v_1,\ldots,v_k,w_1,\ldots,w_k$ in $M_{r+s}(B_\infty)$ such that

$$\|\iota_B(u) \oplus 1_{M_s(B_\infty)} - v_1w_1v_1^*w_1^*\cdots v_kw_kv_k^*w_k^*\| < \epsilon. \tag{3.46}$$

For $i \in \{1,\ldots,k\}$, let $(v_{i,n})_{n=1}^\infty$ and $(w_{i,n})_{n=1}^\infty$ be sequences of unitaries in $M_{r+s}(B)$ lifting v_i and w_i, respectively. Then, for sufficiently large n, we have

$$\|u \oplus 1_{M_s(B)} - v_{1,n}w_{1,n}v_{1,n}^*w_{1,n}^*\cdots v_{k,n}w_{k,n}v_{k,n}^*w_{k,n}^*\| < \epsilon. \tag{3.47}$$

Thus, $[u]_{\mathrm{alg}} = 0$ in $\overline{K}_1^{\mathrm{alg}}(B)$, proving that $\overline{K}_1^{\mathrm{alg}}(\iota_B)$ is injective.

For injectivity of $\underline{K}(\iota_B)$, we start by establishing injectivity of $K_1(\iota_B)$. Suppose that r and s are positive integers, that $u \in U_r(B)$, and that $\iota_B(u) \oplus 1_{M_s(B_\infty)}$ is homotopic to the identity in $U_{r+s}(B_\infty)$. Then there is a finite sequence $v_1,\ldots,v_k$ in $U_{r+s}(B_\infty)$ with $v_1 = u \oplus 1_{M_s(B_\infty)}$ and $v_k = 1_{M_{r+s}(B_\infty)}$ such that $\|v_{j+1} - v_j\| < 2$ for each $j = 1,\ldots,k-1$. Lifting $v_2,\ldots,v_{k-1}$ to sequences $(v_{2,n})_{n=1}^\infty,\ldots,(v_{k-1,n})_{n=1}^\infty$ in $U_{r+s}(B)$, and setting $v_{k,n} := 1_{M_{r+s}(B)}$, $v_{1,n} := u \oplus 1_{M_s(B)}$, we can find some $n \in \mathbb{N}$, such that $\|v_{j+1,n} - v_{j,n}\| < 2$ for all $j = 1,\ldots,k-1$, so that $u \oplus 1_{M_s(B)}$ is homotopic to the identity in $U_{r+s}(B)$, and hence $[u]_1 = 0$ in $K_1(B)$. Hence $K_1(\iota_B)$ is injective. A very similar argument shows that $K_0(\iota_B)$ is injective.

As we noted in Footnote 68, we could equally well have defined total K-theory in terms of $K_i(A\otimes\mathcal{O}_{n+1})$: the functors $K_i(\,\cdot\,;\mathbb{Z}/n)$ and $K_i(\,\cdot\otimes\mathcal{O}_{n+1})$ are naturally isomorphic by [255, Theorem 6.4].[81]

Since $\mathcal{O}_{n+1}$ is nuclear, the canonical maps from B_∞ and $\mathcal{O}_{n+1}$ to $(B\otimes\mathcal{O}_{n+1})_\infty$ with commuting ranges combine to yield a *-homomorphism $\theta\colon B_\infty\otimes\mathcal{O}_{n+1}\to(B\otimes\mathcal{O}_{n+1})_\infty$. Then we have the factorization $\iota_{B\otimes\mathcal{O}_{n+1}}=\theta\circ(\iota_B\otimes\mathrm{id}_{\mathcal{O}_{n+1}})$. Since $K_i(\iota_{B\otimes\mathcal{O}_{n+1}})$ is injective from the previous paragraph,[82] so too is $K_i(\iota_B;\mathbb{Z}/n)$. $\square$

4. $\mathcal{Z}$-stability and Cuntz semigroup techniques

The key regularity hypothesis in the classification of embeddings theorem (Theorem B) is $\mathcal{Z}$-stability of the codomain B, i.e., $B\cong B\otimes\mathcal{Z}$, where $\mathcal{Z}$ is the Jiang–Su algebra from [144]. We give a brief review of the Jiang–Su algebra in Section 4.1 and then turn to $\mathcal{Z}$-stability, setting out in Sections 4.1–4.4 the various consequences that we use elsewhere in the paper. Section 4.2 is devoted to a modern proof of the unpublished result of Jiang ([143]) that $\mathcal{Z}$-stable C^*-algebras are K_1-injective, for use in Section 5.

The road from $\mathcal{Z}$-stability to absorption passes through the Cuntz semigroup and the corona factorization property. We provide the relevant background in Section 4.3. Finally, Section 4.4 collates the various structural properties of sequence algebras of $\mathcal{Z}$-stable C^*-algebras that we need from the literature.

4.1. **The Jiang–Su algebra.** The Jiang–Su algebra $\mathcal{Z}$ was originally constructed in [144] as an inductive limit of prime dimension drop algebras[83] with unital connecting maps chosen carefully to ensure simplicity and uniqueness of trace. Since every prime dimension drop algebra has the same K-theory as $\mathbb{C}$, it follows that $KT_u(\mathcal{Z})\cong KT_u(\mathbb{C})$. Building on the analysis of inductive limits developed in the 1990s, Jiang and Su classified simple inductive limits of dimension drop algebras, concluding that $\mathcal{Z}$ is independent of choices made in the construction. See also [260, Chapter 15] for an expository account of $\mathcal{Z}$. The following fundamental fact about $\mathcal{Z}$ is one of the major black boxes in this paper.

Theorem 4.1 (Jiang–Su, [144, Theorems 7.6 and 8.8])**.** *The Jiang–Su algebra $\mathcal{Z}$ is strongly self-absorbing,*[84] *i.e., there is an isomorphism $\mathcal{Z}\to\mathcal{Z}\otimes\mathcal{Z}$ that is approximately unitarily equivalent to* $\mathrm{id}_{\mathcal{Z}}\otimes 1_{\mathcal{Z}}\colon\mathcal{Z}\to\mathcal{Z}\otimes\mathcal{Z}$.

To establish Theorem 4.1, Jiang and Su use the explicit inductive limit structure of $\mathcal{Z}$, and their classification of limits of dimension drop algebras, to obtain a *-homomorphism from $\mathcal{Z}\otimes\mathcal{Z}$ to $\mathcal{Z}$. Via the intertwining argument, they then show that this is approximately unitarily equivalent to an isomorphism.

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[81] Strictly speaking, this result should be applied twice, as in [255], $K_i(A;\mathbb{Z}/n)$ is defined to be $K_i(A\otimes C_n)$, where C_n is an abelian C^*-algebra with $K_*(C_n)\cong(\mathbb{Z}/n,0)$.

[82] Since $\mathcal{O}_{n+1}$ is unital (in contrast to $\mathbb{I}_n$), the previous paragraph applies.

[83] A dimension drop algebra is a C^*-algebra of the form

$$\{f\in C([0,1],M_p\otimes M_q): f(0)\in M_p\otimes\mathbb{C}1_{M_q},\ f(1)\in\mathbb{C}1_{M_p}\otimes M_q\},$$

for $p,q\in\mathbb{N}$. It is called a *prime* dimension drop algebra when p,q are coprime.

[84] The formalism of strongly self-absorbing algebras was subsequently introduced by Toms and Winter in [274]. Instead of proving the statement of Theorem 4.1, Jiang and Su prove that $\mathcal{Z}\cong\mathcal{Z}\otimes\mathcal{Z}$ and all unital endomorphisms of $\mathcal{Z}$ are approximately inner, which easily implies strong self-absorption (and is in fact equivalent to it, by [274, Corollary 1.12]).

We record the following facts about tensoring with $\mathcal{Z}$. The KK-equivalence fact is observed by Jiang and Su in [144, Lemma 2.11]. In the separable case, the isomorphism of invariants follows from this and uniqueness of trace on $\mathcal{Z}$; the non-separable case follows.

Proposition 4.2. *For any unital C^*-algebra D, the first-factor embedding induces both an isomorphism $KT_u(D) \cong KT_u(D \otimes \mathcal{Z})$, and (when D is separable), a KK-equivalence.*

More recently, two essentially self-contained approaches to Theorem 4.1 have been given. Ghasemi ([117], building on [73, 196]) tackles strong self-absorption via the notion of Fraïssé limits from model theory. Schemaitat ([254]), following a direction outlined in Winter's 2011 CBMS lectures, takes a more traditional classification approach, starting with classification results for UHF algebras to obtain a specific set of existence and uniqueness theorems. To do this, Schemaitat uses Rørdam and Winter's picture of the Jiang–Su algebra: an inductive limit of generalized dimension drop algebras with strongly self-absorbing fibers, developed in the mid-2000s ([244]). It is this picture of $\mathcal{Z}$ that is used in Winter's localization technique ([292], see Section 1.2.5). Since we will use this description of $\mathcal{Z}$ in Section 4.2, we recall it here.

Let $p, q \geq 2$ be coprime, and consider the UHF algebras M_{p^∞} and M_{q^∞}. Define the generalized dimension drop algebra

$$\mathcal{Z}_{p^\infty,q^\infty} \subseteq C([0,1], M_{p^\infty} \otimes M_{q^\infty}) \tag{4.1}$$

to be the subalgebra of functions $f \in C([0,1], M_{p^\infty} \otimes M_{q^\infty})$ satisfying

$$f(0) \in M_{p^\infty} \otimes \mathbb{C}1_{M_{q^\infty}} \quad \text{and} \quad f(1) \in \mathbb{C}1_{M_{p^\infty}} \otimes M_{q^\infty}. \tag{4.2}$$

By [254, Section 4], there is a unital *-endomorphism θ of $\mathcal{Z}_{p^\infty,q^\infty}$ such that

$$\tau \circ \theta = \tau_{\mathrm{Leb}}, \quad \tau \in T(\mathcal{Z}_{p^\infty,q^\infty}), \tag{4.3}$$

where τ_{Leb} is given by

$$\tau_{\mathrm{Leb}}(f) \coloneqq \int_0^1 \tau_{M_{p^\infty} \otimes M_{q^\infty}}(f(t))\,dt, \quad f \in \mathcal{Z}_{p^\infty,q^\infty}. \tag{4.4}$$

Then [244, Theorem 3.4(ii)] shows that $\mathcal{Z}$ arises as the stationary inductive limit:[85]

$$\mathcal{Z} \cong \varinjlim \big(\mathcal{Z}_{p^\infty,q^\infty} \xrightarrow{\theta} \mathcal{Z}_{p^\infty,q^\infty} \xrightarrow{\theta} \mathcal{Z}_{p^\infty,q^\infty} \xrightarrow{\theta} \cdots\big). \tag{4.5}$$

Remark 4.3. Schemaitat's work ([254]) shows directly that there is a unique inductive limit of algebras $\mathcal{Z}_{p^\infty,q^\infty}$ where p, q are coprime with trace-collapsing maps as in (4.3). With the benefit of hindsight, it is perhaps more natural to define $\mathcal{Z}$ to be such an inductive limit and, as in [254, Theorem 5.15], access Winter's work [290] to show that this is the minimal strongly self-absorbing C^*-algebra. Most of the other properties of $\mathcal{Z}$-stable C^*-algebras used in the classification theorem can be readily obtained from this viewpoint.

Now we turn to $\mathcal{Z}$-stability. First, the definition.

Definition 4.4. A C^*-algebra B is *$\mathcal{Z}$-stable* (or *$\mathcal{Z}$-absorbing*) if $B \cong B \otimes \mathcal{Z}$.

[85] In fact, in [244, Theorem 3.4(ii)], θ is only required to be trace-collapsing (not necessarily inducing the Lebesgue trace). Rørdam and Winter also proved the existence of these less specific trace-collapsing maps, produced as $\mathcal{Z}_{p^\infty,q^\infty} \to \mathcal{Z} \to \mathcal{Z}_{p^\infty,q^\infty}$.

One immediate consequence of (strong) self-absorption is that $\mathcal{Z}$ is itself $\mathcal{Z}$-stable. Moreover, the strong form of the isomorphism $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$ immediately strengthens the isomorphism in Definition 4.4 as follows (see [274, Theorem 2.2]).

Proposition 4.5. *Let B be a unital C^*-algebra. The following are equivalent:*

(i) B is $\mathcal{Z}$-stable.

(ii) There exists an isomorphism $B \to B \otimes \mathcal{Z}$ that is approximately unitarily equivalent to the first-factor embedding $\mathrm{id}_B \otimes 1_{\mathcal{Z}} \colon B \to B \otimes \mathcal{Z}$.

It is using this (or similar formulations, such as [242, Lemma 4.4]) that strong self-absorption of $\mathcal{Z}$ is generally used to obtain structural properties of $\mathcal{Z}$-stable C^*-algebras, such as [242, Theorem 4.5]. This form of the isomorphism is also a key ingredient in the proof of the $\mathcal{Z}$-stable KK-uniqueness theorem in Section 5.4.

The first use of $\mathcal{Z}$-stability in the unital classification theorem is through the dichotomy it provides between purely infinite and stably finite simple C^*-algebras, as shown in [122, Theorem 3]. This is a special case of a more general dichotomy result of Kirchberg for simple C^*-algebras that can be written non-trivially as a tensor product ([17, Corollary 3.9(i)]; see also [240, Theorem 4.1.10(ii)]).

Theorem 4.6. *Let B be a simple exact $\mathcal{Z}$-stable C^*-algebra. Then B is either purely infinite or stably finite.*

We will later need to use matrix amplifications of codomain C^*-algebras, and for this, we record the following.

Remark 4.7. If B is a unital simple separable nuclear (resp. exact) and $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$, then $M_n(B)$ also enjoys these properties, for any $n \in \mathbb{N}$.

4.2. **K_1-injectivity.** In this section we give a short proof of an unpublished result of Jiang that we will use in the $\mathcal{Z}$-stable KK-uniqueness theorem in Section 5.4.[86] Recall that a unital C^*-algebra D is *K_1-injective* if whenever $u \in U_1(D)$ vanishes in K_1, then $u \in U_1^{(0)}(D)$. That is, no matrix amplification is needed to construct a homotopy from u to 1_D.

Theorem 4.8 (Jiang, [143])**.** *If D is a unital C^*-algebra, then $D \otimes \mathcal{Z}$ is K_1-injective.*

Before proving the theorem, it is helpful to isolate the following fact regarding the non-stable K-theory of C^*-algebras that absorb a UHF algebra of infinite type.

Lemma 4.9. *Let D be a unital C^*-algebra and $n \in \mathbb{N}$ with $n \geq 2$. Then $D \otimes M_{n^\infty}$ is K_1-injective and $K_0(D \otimes M_{n^\infty})$ is generated by*

$$\{[p]_0 : p \text{ is a projection in } D \otimes M_{n^\infty}\}. \tag{4.6}$$

Proof. Write $B \coloneqq D \otimes M_{n^\infty}$ and suppose $u \in U_1(B)$ satisfies $[u]_1 = 0$. Then there is an $m \geq 1$ such that $u \oplus 1_B^{\oplus(m-1)}$ is homotopic to $1_B^{\oplus m}$ in $U_m(B)$. Enlarging m if necessary, we may assume $m = n^k$ for some $k \geq 1$. We have

$$u^{\oplus m} = (u \oplus 1_B^{\oplus(m-1)})(1_B \oplus u^{\oplus(m-1)}) \sim_h 1_B \oplus u^{\oplus(m-1)}, \tag{4.7}$$

and by repeating this, it follows that $u^{\oplus m}$ is homotopic to the identity in $U_m(B)$. Put another way, this says that $u \otimes 1_{M_{n^k}}$ is homotopic to the identity in $U_1(B \otimes M_{n^k})$. Therefore, $u \otimes 1_{M_{n^\infty}}$ is homotopic to the identity in the unitary group of $B \otimes M_{n^\infty}$.

[86]When we apply Theorem 4.8 in the proof of Theorem 5.15, the algebra D will be properly infinite, so we could alternatively apply Rohde's later (and also unpublished) result on the K_1-injectivity of properly infinite $\mathcal{Z}$-stable C^*-algebras from her Ph.D. thesis ([234]).

Fix an isomorphism

$$\theta\colon B \to B \otimes M_{n^\infty} \tag{4.8}$$

that is approximately unitarily equivalent to the embedding $x \mapsto x \otimes 1_{M_{n^\infty}}$.[87] Then take a unitary $v \in B \otimes M_{n^\infty}$ with

$$\|\theta(u) - v(u \otimes 1_{M_{n^\infty}})v^*\| < 1 \tag{4.9}$$

so that $\theta(u)$ is homotopic to $v(u \otimes 1_{M_{n^\infty}})v^*$ in $U_1(B \otimes M_{n^\infty})$, which, in turn, is homotopic to $1_{B\otimes M_{n^\infty}}$. Applying θ^{-1} shows that u is homotopic to 1_B in $U_1(B)$.

The statement regarding K_0 follows in a similar fashion. If $p \in B \otimes M_m$ is a projection, we may again assume $m = n^k$ for some k. This time, take an isomorphism

$$\theta\colon B \to B \otimes M_{n^k} \tag{4.10}$$

that is approximately unitarily equivalent to the embedding $x \mapsto x \otimes 1_{M_{n^k}}$.[88] Set $q \coloneqq \theta^{-1}(p) \in B$. As $p = \theta(q)$ is approximately unitarily equivalent to $q \otimes 1_{M_{n^k}}$ in $B \otimes M_{n^k}$, $[p]_0 = n^k[q]_0$, and so $[p]_0$ is in the group generated by K_0-classes of projections in B. $\square$

Proof of Theorem 4.8. $D \otimes \mathcal{Z}$ is a stationary inductive limit of the algebra $D \otimes \mathcal{Z}_{2^\infty,3^\infty}$; see (4.1). It is therefore enough to show $D \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is K_1-injective. To this end, choose $u \in U_1(D \otimes \mathcal{Z}_{2^\infty,3^\infty})$ with $[u]_1 = 0$ in $K_1(D \otimes \mathcal{Z}_{2^\infty,3^\infty})$.

Consider the exact sequence[89]

$$0 \to D \otimes SM_{6^\infty} \to D \otimes \mathcal{Z}_{2^\infty,3^\infty} \xrightarrow{\sigma} D \otimes (M_{2^\infty} \oplus M_{3^\infty}) \to 0, \tag{4.11}$$

where SM_{6^∞} is the suspension $C_0((0,1), M_{6^\infty}) \subseteq \mathcal{Z}_{2^\infty,3^\infty}$. Note that $[\sigma(u)]_1 = 0$ in $K_1(D \otimes (M_{2^\infty} \oplus M_{3^\infty}))$. Lemma 4.9 applies to both $D \otimes M_{2^\infty}$ and $D \otimes M_{3^\infty}$. Because $D \otimes (M_{2^\infty} \oplus M_{3^\infty}) \cong D \otimes M_{2^\infty} \oplus D \otimes M_{3^\infty}$, it follows that $\sigma(u)$ is in the path component of the identity in the unitary group of this algebra. Hence, there is a unitary v in the path component of the identity in $U_1(D \otimes \mathcal{Z}_{2^\infty,3^\infty})$ with $\sigma(v) = \sigma(u)$. After replacing u with uv^*, we may assume $\sigma(u) = 1_{D\otimes(M_{2^\infty}\oplus M_{3^\infty})}$ so that u is a unitary in $(D \otimes SM_{6^\infty})^\dagger$.

Since $[u]_1 = 0$ in $K_1(D \otimes \mathcal{Z}_{2^\infty,3^\infty})$, the 6-term exact sequence gives an element $x \in K_0(D \otimes (M_{2^\infty} \oplus M_{3^\infty}))$ such that $\exp(x) = [u]_1$ in $K_1(D \otimes SM_{6^\infty})$. As above, another application of Lemma 4.9 shows there are projections $p_1, \dots, p_n, q_1, \dots, q_n$ in $D \otimes (M_{2^\infty} \oplus M_{3^\infty})$ with $x = \sum_j([p_j]_0 - [q_j]_0)$. Then, in $K_1(D \otimes SM_{6^\infty})$, we have

$$[u]_1 = [e^{2\pi i h_1} \cdots e^{2\pi i h_n} e^{-2\pi i k_1} \cdots e^{-2\pi i k_n}]_1 \tag{4.12}$$

where the h_j, k_j are positive contractions in $D \otimes \mathcal{Z}_{2^\infty,3^\infty}$ lifting p_j, q_j, respectively. Replacing u with

$$ue^{-2\pi i h_1} \cdots e^{-2\pi i h_n} e^{2\pi i k_1} \cdots e^{2\pi i k_n}, \tag{4.13}$$

if necessary, we may assume that $u \in (D \otimes SM_{6^\infty})^\dagger$ with $[u]_1 = 0$ in $K_1(D \otimes SM_{6^\infty})$ and with scalar part equal to 1.

With these reductions in place, we will now show u is in the path component of the identity in the unitary group of $(D \otimes SM_{6^\infty})^\dagger$. We identify $D \otimes SM_{6^\infty}$ with the

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[87] This follows immediately from strong self-absorption of M_{n^∞} which, in contrast to strong self-absorption of $\mathcal{Z}$, is elementary.

[88] For this, use $B \cong B \otimes M_{n^\infty} \cong B \otimes M_{n^k} \otimes M_{n^\infty} \cong B \otimes M_{n^k}$, where the first and last isomorphisms are approximately unitarily equivalent to the first-factor embedding.

[89] It is exact as each of SM_{6^∞}, $\mathcal{Z}_{2^\infty,3^\infty}$, and $M_{2^\infty} \oplus M_{3^\infty}$ is nuclear.

subalgebra of $C(\mathbb{T}, D \otimes M_{6^\infty})$ consisting of functions vanishing at 1. Then u defines an element in $C(\mathbb{T}, D\otimes M_{6^\infty})$ with trivial K_1-class. By a third, and final, application of Lemma 4.9, u is in the path component of the identity of $U_1(C(\mathbb{T}, D\otimes M_{6^\infty}))$. Let $(v_t)_{t\in[0,1]}$ be a path of unitaries in $C(\mathbb{T}, D \otimes M_{6^\infty})$ with $v_0 = 1_{D\otimes M_{6^\infty}}$ and $v_1 = u$. Define $u_t = v_t(1)^* v_t \in C(\mathbb{T}, D \otimes M_{6^\infty})$. Then $u_t(1) = 1_{D\otimes M_{6^\infty}}$ for all $t \in [0, 1]$, so $(u_t)_{t\in[0,1]}$ is a path of unitaries in $(D \otimes SM_{6^\infty})^\dagger$. Moreover, $u_0 = 1_{(D\otimes SM_{6^\infty})^\dagger}$ and $u_1 = u$, as required. □

4.3. **The Cuntz semigroup and the corona factorization property.** Considerable information about a C^*-algebra is carried in an algebraic object — the Cuntz semigroup — constructed from its positive elements. The ideas for this semigroup originated in [51] and were subsequently developed in [12, 235, 220, 166, 242, 29, 49] and other works; see the survey [3].

Given a C^*-algebra D and positive elements $a, b \in D \otimes \mathcal{K}$, write $a \precsim b$ if there exists a sequence $(x_n)_{n=1}^\infty$ in $D \otimes \mathcal{K}$ with $x_n^* b x_n \to a$. Say a and b are *Cuntz equivalent*, written $a \sim b$, if $a \precsim b$ and $b \precsim a$. The *Cuntz semigroup*, $\mathrm{Cu}(D)$, is defined by $(D \otimes \mathcal{K})_+/{\sim}$. This is an abelian semigroup (with addition given by identifying $\mathcal{K} \cong M_2(\mathcal{K})$ and then adding representatives diagonally) that inherits an order $\leq$ from $\precsim$. When consulting the literature, it is important to distinguish between the complete[90] Cuntz semigroup $\mathrm{Cu}(D)$ introduced in [49] and the earlier incomplete version $W(D)$, which is defined in a similar fashion as above using $\bigcup_{n=1}^\infty M_n(D)_+$ in place of $(D \otimes K)_+$. Note that $\mathrm{Cu}(D) \cong W(D \otimes \mathcal{K})$. In order to accurately cite the results we need, we use both constructions.

Given $\tau \in T(D)$ and $a \in M_n(D)_+$ define

$$d_\tau(a) := \lim_{r\to\infty} \tau_n(a^{1/r}), \tag{4.14}$$

where τ_n is the non-normalized extension of τ to $M_n(D)$. This provides a *state* on $W(D)$,[91] i.e., an order-preserving semigroup homomorphism $W(D) \to \mathbb{R}$ (which also maps the class of the unit to 1 when D is unital). A particularly useful concept, strict comparison, allows Cuntz comparison to be recovered from tracial data. This idea goes back to [8, Section 6], which gives an account of comparison conditions for simple C^*-algebras in the spirit of Murray and von Neumann's comparison theory for von Neumann algebra factors. There are several variations of "strict comparison" in the literature. We use the following form, which appears in [21, Definition 1.5].

Definition 4.10. Let D be a C^*-algebra. Say that D has *strict comparison with respect to bounded traces* when for all $n \in \mathbb{N}$ and $a, b \in M_n(D)_+$, we have

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in T(D) \implies a \precsim b. \tag{4.15}$$

We recall in Lemma 4.12 below a well-known application of strict comparison, namely, to catalyze checking fullness of *-homomorphisms. Let us first define fullness, which is the simplicity hypothesis in the classification of approximate embeddings.

Definition 4.11. Let B, D be C^*-algebras. A *-homomorphism $\phi\colon B \to D$ is *full* if $\phi(b)$ generates D as an ideal, for every non-zero $b \in B$.

[90] Here, complete means that every increasing sequence has a supremum.

[91] It also gives a state on $\mathrm{Cu}(D)$ when one works with the extension of τ to a lower semicontinuous trace on $D \otimes \mathcal{K}$. See [103, Proposition 4.2], which shows (using work going back to [51, 12]) that all states on $\mathrm{Cu}(D)$ are of this form for a suitable extended quasitrace on D.

The first statement of the following lemma is trivial, and the second is relatively standard; see [269, Lemma 2.2], for example.

Lemma 4.12. *Let $\theta\colon B \to D$ be a *-homomorphism between C^*-algebras. If θ is full, then $\tau \circ \theta$ is faithful for every $\tau \in T(D)$. The converse holds if D is unital and has strict comparison with respect to bounded traces.*

While the order structure of the Cuntz semigroup of a general C^*-algebra — even a simple one — can be quite badly behaved (see [281, 270]), Rørdam showed the situation is much better for a $\mathcal{Z}$-stable C^*-algebra. In the second part of the following proposition, Rørdam uses Haagerup's theorem ([131]) that quasitraces are traces for exact C^*-algebras. This is the place where this fundamental result appears in our proof of the unital classification theorem.

Proposition 4.13 (Rørdam, [242])**.** *Let D be a $\mathcal{Z}$-stable C^*-algebra. Then*

(i) $\mathrm{Cu}(D)$ *is almost unperforated: whenever $x, y \in \mathrm{Cu}(D)$ satisfy $nx \leq my$ for some $n > m$ in $\mathbb{N}$, we have $x \leq y$.*

(ii) If, in addition, D is unital, simple, and exact with $T(D) \neq \emptyset$, then D has strict comparison of positive elements with respect to bounded traces.[92]

Proof. Theorem 4.5 of [242] shows that the incomplete Cuntz semigroup $W(D)$ is almost unperforated when D is $\mathcal{Z}$-stable. Part (i) follows by applying this to $D \otimes \mathcal{K}$, since $\mathrm{Cu}(D) \cong W(D \otimes \mathcal{K})$. Part (ii) follows by applying [242, Corollary 4.6] to each $M_n(D)$. □

Now we turn to the corona factorization property of Kucerovsky and Ng from [172], which we use to verify absorption in Section 5 and beyond. A σ-unital C^*-algebra D has the *corona factorization property* if every (norm-)full projection in the multiplier algebra of $D \otimes \mathcal{K}$ is properly infinite. This is a relatively mild regularity hypothesis, and is weaker than $\mathcal{Z}$-stability, although a tight reference for this seems difficult to come by. The route through the literature that we give below uses Ortega, Perera, and Rørdam's characterization of the corona factorization property in terms of the complete Cuntz semigroup ([212, Theorem 5.11]).

Proposition 4.14. *Let D be a σ-unital $\mathcal{Z}$-stable C^*-algebra. Then D has the corona factorization property.*

Proof. By Proposition 4.13(i), $\mathrm{Cu}(D)$ is almost unperforated. By [212, Remark 2.4], this is equivalent to 0-comparison for $\mathrm{Cu}(D)$ in the sense of [212, Definition 2.8]. In particular, $\mathrm{Cu}(D)$ has the ω-comparison property of [212, Definition 2.11] (see the sentence immediately following that definition). Then we can combine [212, Proposition 2.17] and [212, Theorem 5.11] to conclude that D has the corona factorization property. □

4.4. **Sequence algebras of $\mathcal{Z}$-stable C^*-algebras.** We end this section by recording some structural properties that a sequence algebra B_∞ enjoys when the underlying C^*-algebra B is $\mathcal{Z}$-stable. Such a sequence algebra cannot be $\mathcal{Z}$-stable,[93] and

[92] Working with an appropriate definition of strict comparison (one using extended quasitraces instead of bounded traces), this holds more generally for all $\mathcal{Z}$-stable C^*-algebras.

[93] This is due to Ghasemi ([116]). The fact that sequence algebras satisfy Pedersen's SAW* condition used by Ghasemi is a standard application of Kirchberg ϵ-test (cf. [163, Corollary 1.7]).

so we work with the following version of $\mathcal{Z}$-stability in terms of separable subalgebras, in the spirit of Blackadar's notion of separable inheritability ([11, Section II.8.5]).

Definition 4.15 (cf. [253, Definition 1.4])**.** A C^*-algebra D is *separably $\mathcal{Z}$-stable* if, for every separable C^*-subalgebra D_0 of D, there exists a separable $\mathcal{Z}$-stable C^*-subalgebra $D_1 \subseteq D$ containing D_0.

When B is separable and unital, an intertwining argument which seems to be originally due to Elliott (see [166, Theorem 8.2], [237, Page 34], and [274, Section 2]) shows that $\mathcal{Z}$-stability of B can be characterized in terms of an embedding of $\mathcal{Z}$ into the central sequence algebra $B_\infty \cap B'$. This is in the spirit of McDuff's characterization of separably acting II_1 factors absorbing the hyperfinite II_1 factor tensorially. It gives rise to a local characterization of $\mathcal{Z}$-stability for unital separable C^*-algebras, that extends to characterize separable $\mathcal{Z}$-stability; this is why separable $\mathcal{Z}$-stability is well-suited to working with non-separable C^*-algebras (see [253, Lemma 1.11]). As a consequence, one has the following.

Proposition 4.16 ([253, Proposition 1.12])**.** *If B is a unital separable $\mathcal{Z}$-stable C^*-algebra, then B_∞ is separably $\mathcal{Z}$-stable.*

Now we turn to traces on sequence algebras, with the aim of using Kirchberg's ϵ-test to observe that the image of $K_0(B_\infty)$ is closed in $\operatorname{Aff} T(B_\infty)$ when B satisfies the codomain assumptions in the classification of embeddings theorem. Once this is done, Thomsen's work described in Section 2.2 (in particular, Properties 2.9) will give an exact sequence

$$K_0(B_\infty) \xrightarrow{\rho_{B_\infty}} \operatorname{Aff} T(B_\infty) \xrightarrow{\mathrm{Th}_{B_\infty}} \overline{K}_1^{\mathrm{alg}}(B_\infty) \xrightarrow{a_{B_\infty}} K_1(B_\infty) \to 0 \tag{4.16}$$

which we use in the computation of $KL(A, J_B)$ in Section 8. First, we recall the notion of limit traces; these are the traces on B_∞ that are computationally tractable.

Definition 4.17. The set of *limit traces* on B_∞, written $T_\infty(B)$, consists of all $\tau \in T(B_\infty)$ of the form $\tau((b_n)_{n=1}^\infty) = \lim_{n\to\omega} \tau_n(b_n)$, where $(\tau_n)_{n=1}^\infty$ is a sequence in $T(B)$ and ω is a free ultrafilter on $\mathbb{N}$.

Proposition 4.18 (Ozawa, cf. [207, Theorem 1.2])**.** *If B is an exact $\mathcal{Z}$-stable C^*-algebra, then the convex hull of $T_\infty(B)$ is weak*-dense in $T(B_\infty)$.*

Proof. Ozawa's [213, Theorem 8] shows this for ultraproducts of $\mathcal{Z}$-stable separable exact C^*-algebras, and the case of sequence algebras is essentially identical. One follows the proof of [213, Theorem 8] verbatim with $A \coloneqq B_\infty$ and $\Sigma \coloneqq \operatorname{co} T_\infty(B)$ and changing "Let $I \in \mathcal{U}$ (or $I = \mathbb{N}$ in case $A = \prod A_n$)" to "Let I be cofinite". $\square$

The following proposition collects additional properties of the sequence algebra that will be useful later.

Proposition 4.19. *Let B be a unital simple separable exact $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Then*

(i) B_∞ has stable rank one, and
(ii) B_∞ has strict comparison of positive elements with respect to bounded traces.

Proof. (i) follows directly from [195, Lemma 19.2.2], since B has stable rank one by [242, Theorem 6.7].[94]

For (ii), note that B has strict comparison of positive elements by bounded traces by Rørdam's result recalled in Proposition 4.13(ii). Using this, the version of (ii) for the ultrapower B_ω has been observed in [21, Lemma 1.23]. The result for sequence algebras is obtained by following this short proof verbatim, with the following changes: replace B_ω by B_∞, $T_\omega(B_\omega)$ by $T_\infty(B)$, and $I \in \omega$ in [21, (1.38)] by "I is cofinite." □

Now we can show exactness of (4.16).

Proposition 4.20. *Let B be a unital simple separable exact $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Then $\rho_{B_\infty}(K_0(B_\infty))$ is closed in* $\mathrm{Aff}\, T(B_\infty)$*, and so* (4.16) *is exact.*

Proof. Let $g \in \overline{\rho_{B_\infty}(K_0(B_\infty))}$. We will show that $g \in \rho_{B_\infty}(K_0(B_\infty))$. Without loss of generality, we may assume that $0 < g < 1$.[95] As a consequence of Proposition 2.1, there exists a self-adjoint $c = (c_n)_{n=1}^\infty \in B_\infty$ such that $\tau(c) = g(\tau)$ for all $\tau \in T(B_\infty)$.

Let X be the unit ball of B and define $f_n\colon X \to [0,\infty)$ by

$$f_n(b) := \|b - b^*\| + \|b^2 - b\| + \sup_{\tau \in T(B)} |\tau(b - c_n)|, \tag{4.17}$$

and then define $f\colon \prod_{n=1}^\infty X \to [0,\infty)$ by

$$f((b_n)_{n=1}^\infty) := \limsup_{n\to\infty} f_n(b_n). \tag{4.18}$$

For a projection $p = (p_n)_{n=1}^\infty \in B_\infty$, one computes

$$f\left((p_n)_{n=1}^\infty\right) = \sup_{\tau \in T_\infty(B)} |\tau(p) - g(\tau)| \overset{\text{Prop. 4.18}}{=} \sup_{\tau \in T(B_\infty)} |\tau(p) - g(\tau)|. \tag{4.19}$$

By hypothesis, for any $\epsilon > 0$, there exists $x \in K_0(B_\infty)$ such that $\|\rho_{B_\infty}(x) - g\|_\infty < \epsilon$. For ϵ sufficiently small (so that $\rho_{B_\infty}(x)$ is bounded away from 0 and 1), Proposition 4.19(ii) implies that $x = [p]_0$ for some projection $p \in B_\infty$,[96] and thus $f(p) < \epsilon$.

By Kirchberg's ϵ-test (Lemma 3.14), there exists $b \in \prod_{n=1}^\infty X$ such that $f(b) = 0$. Hence b represents a projection in B_∞ such that $\tau(b) = g(\tau)$ for all $\tau \in T(B_\infty)$. Therefore, (4.16) is exact at $\mathrm{Aff}T(B)$, and hence exact by Properties 2.9. □

5. Absorption and KK-existence and uniqueness theorems

The main goal of this section is to prove the $\mathcal{Z}$-stable KK- and KL-uniqueness theorems (collected together as Theorem 5.15) in Section 5.4. This proves Theorem 1.4 from the introduction. We start by setting out the Cuntz–Thomsen picture of KK-theory in Section 5.1, leaving certain proofs for the non-separable setting to Appendix B. We follow this by recalling salient facts about absorption in Section 5.2 and record the KK-existence theorem in Section 5.3. Although it is not needed for

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[94] Note that exactness is not needed.

[95] Indeed, we may add $n = \rho_{B_\infty}([1_{M_n(B)}]_0)$ for sufficiently large n, and replace B with $M_m(B)$ for sufficiently large m, using Remark 4.7.

[96] First, $x = [r]_0 - [s]_0$ for some projections r, s in $M_m(B_\infty)$. Since $\tau(x) > 0$ for all $\tau \in T(B_\infty)$, strict comparison implies $s \precsim r$, and hence by [235, Proposition 2.1], there is v such that $v^*v = s$ and $vv^* \leq r$. Then $x = [p]_0$ where $p = r - vv^*$. Now, as $\tau(x) < 1$ for all $\tau \in T(B_\infty)$, we have $p \precsim 1_{B_\infty}$, and hence p is equivalent to a projection in B_∞, as claimed.

this paper, we end the section by outlining in Section 5.5 the minor modifications needed to obtain a $\mathcal{Z}$-stable KK_{nuc}-uniqueness theorem as this will be crucial to our future work.

5.1. **The Cuntz–Thomsen picture of Kasparov's KK-theory.** We start by collecting those aspects of KK- and KL-theory that we need. As was the case in Kasparov's original definition of KK-theory ([152]) using Fredholm modules, $KK(A,I)$ is typically defined for separable A and σ-unital I.[97] However, we need to allow the non-σ-unital trace-kernel ideal J_B in the second variable. As suggested by Skandalis in [257, Section 3] (see also [258, Definition 5.5]), this can be achieved by taking inductive limits over separable C^*-subalgebras. We set out how to do this here. All definitions and results in this subsection are standard when the second variable is σ-unital; we provide proofs of how to extend these to the general case in Appendix B.

Of the many equivalent ways of defining KK-theory (see [10, Chapter 17], or [142]), the Cuntz–Thomsen picture is particularly well-suited to classification problems (as demonstrated, for example, by Dadarlat and Eilers in [62, 63]). Originally, this approach was developed by means of reductions to Kasparov's Fredholm module construction.[98] Thomsen later gave a self-contained treatment in [264].

Definition 5.1. Let A and I be C^*-algebras with A separable. An (A,I)-*Cuntz pair* is a pair (ϕ,ψ) of *-homomorphisms $A\to E$ for some C^*-algebra E containing I as an ideal, such that $\phi(a)-\psi(a)\in I$ for all $a\in A$. We will often write $(\phi,\psi)\colon A\rightrightarrows E\rhd I$ for such a Cuntz pair. We call (ϕ,ψ) *unital* when all of A, E, ϕ, and ψ are unital.

Definition 5.2. Let A and I be C^*-algebras with A separable and I σ-unital. Write $\mathcal{M}(I\otimes\mathcal{K})$ for the multiplier algebra of the stabilization $I\otimes\mathcal{K}$. Two Cuntz pairs

$$(\phi_i,\psi_i)\colon A\rightrightarrows\mathcal{M}(I\otimes\mathcal{K})\rhd I\otimes\mathcal{K},\quad i=0,1, \tag{5.1}$$

are said to be *homotopic* if there is a Cuntz pair[99]

$$(\phi,\psi)\colon A\rightrightarrows C_\sigma\big([0,1],\mathcal{M}(I\otimes\mathcal{K})\big)\rhd C([0,1],I\otimes\mathcal{K}) \tag{5.2}$$

such that evaluation at i induces the Cuntz pairs (ϕ_i,ψ_i) for $i=0,1$. The Cuntz–Thomsen picture of KK-theory is then

$$KK(A,I)\coloneqq\{(\phi,\psi)\colon A\rightrightarrows\mathcal{M}(I\otimes\mathcal{K})\rhd I\otimes\mathcal{K}\}/\text{homotopy}. \tag{5.3}$$

We write $[\phi,\psi]_{KK(A,I)}$ for the class in $KK(A,I)$ of $(\phi,\psi)\colon A\rightrightarrows\mathcal{M}(I\otimes\mathcal{K})\rhd I\otimes\mathcal{K}$. Any Cuntz pair $(\phi,\psi)\colon A\rightrightarrows E\rhd I$ induces a class in $KK(A,I)$,[100] which we

[97] The σ-unitality hypothesis is to obtain the Kasparov product through Kasparov's stabilization and technical theorems ([152, 151]). Some approaches to KK-theory, such as Cuntz's quasihomomorphism picture from [54], and in particular, Cuntz's approach from [56] which we use in Appendix B, proceed without a σ-unitality restriction on I. That said, the equivalence of the various forms of $KK(A,I)$ is typically only recorded for separable A and σ-unital I.

[98] This is implicit in [54, Section 5] and set out in [10, 17.6.2]. See also [62, Section 3.1].

[99] $C_\sigma\big([0,1],\mathcal{M}(I\otimes\mathcal{K})\big)$ is the algebra of strictly continuous functions $[0,1]\to\mathcal{M}(I\otimes\mathcal{K})$ (these are automatically bounded by the principle of uniform boundedness). By [1, Corollary 3.4], one has $C_\sigma\big([0,1],\mathcal{M}(I\otimes\mathcal{K})\big)\cong\mathcal{M}\big(C([0,1],I\otimes\mathcal{K})\big)$.

[100] The ideal $I\otimes\mathcal{K}\lhd E\otimes\mathcal{K}$ induces a canonical map $\mu\colon E\otimes\mathcal{K}\to\mathcal{M}(I\otimes\mathcal{K})$. Writing e for a rank one projection in $\mathcal{K}$, $[\phi,\psi]_{KK(A,I)}$ is defined to be the class of $(\mu\circ(\phi\otimes e),\mu\circ(\psi\otimes e))\colon A\rightrightarrows\mathcal{M}(I\otimes\mathcal{K})\rhd I\otimes\mathcal{K}$.

also denote $[\phi, \psi]_{KK(A,I)}$. An important special case is when $\phi\colon A \to I$ is a *-homomorphism, so that $(\phi, 0)$ is an (A, I)-Cuntz pair. We write $[\phi]_{KK(A,I)}$ for the induced class in $KK(A, I)$.

Addition in $KK(A, I)$ is defined by means of an orthogonal direct sum. There is a copy of $\mathcal{B}(\mathcal{H})$ in $\mathcal{M}(I \otimes \mathcal{K})$, so one can find $s_1, s_2 \in \mathcal{M}(I \otimes \mathcal{K})$ satisfying $s_i^* s_i = s_1 s_1^* + s_2 s_2^* = 1_{\mathcal{M}(I\otimes\mathcal{K})}$. Given $\phi_1, \phi_2\colon A \to \mathcal{M}(I \otimes \mathcal{K})$, write[101]

$$(\phi_1 \oplus \phi_2)(a) \coloneqq s_1\phi_1(a)s_1^* + s_2\phi_2(a)s_2^*, \quad a \in A. \tag{5.4}$$

Then, the sum in $KK(A, I)$ is well-defined by

$$[\phi_1, \psi_1]_{KK(A,I)} + [\phi_2, \psi_2]_{KK(A,I)} \coloneqq [\phi_1 \oplus \phi_2, \psi_1 \oplus \psi_2]_{KK(A,I)}.\text{[102]} \tag{5.5}$$

This turns $KK(A, I)$ into an abelian group. The zero element is $[\phi, \phi]_{KK(A,I)}$, where $\phi\colon A \to \mathcal{M}(I \otimes \mathcal{K})$ is any *-homomorphism ([142, Lemma 4.1.5]).

A key feature is that $KK(\,\cdot\,,\,\cdot\,)$ is a bifunctor. Contravariance in the first variable is straightforward: a *-homomorphism $\theta\colon A_1 \to A_2$ induces a homomorphism $KK(\theta, I)\colon KK(A_2, I) \to KK(A_1, I)$ by precomposing Cuntz pairs (and homotopies thereof) by θ. A direct definition of $KK(A, \theta)$ (for $\theta\colon I_1 \to I_2$) is possible, though subtle, in the Cuntz–Thomsen picture (see [264] or [142, Section 4.1]), though is straightforward in the pictures introduced by Cuntz ([54, 56]). All of our computations involving functoriality in the second variable will use the statement in Proposition 5.4(iv) below.

We now (re)define $KK(A, I)$ to cover the case that I is not σ-unital.

Definition 5.3. Let A and I be C^*-algebras with A separable. Then

$$KK(A, I) \coloneqq \varinjlim_{I_0 \text{ sep.}} KK(A, I_0), \tag{5.6}$$

where I_0 ranges over all separable C^*-subalgebras of I, ordered by inclusion, and with connecting maps $KK(A, \iota_{I_0 \subseteq I_1})$ for $I_0 \subseteq I_1$.

Given a Cuntz pair $(\phi, \psi)\colon A \rightrightarrows E \rhd I$, we define $[\phi, \psi]_{KK(A,I)}$ to be the image of $[\phi|^{E_0}, \psi|^{E_0}]_{KK(A,I_0)}$ in the right-hand side of (5.6), where $E_0 \subseteq E$ and $I_0 \subseteq I$ are separable C^*-subalgebras such that $I_0 \lhd E_0$, $\phi(A) \cup \psi(A) \subseteq E_0$, and $(\phi - \psi)(A) \subseteq I_0$. (In Proposition B.4, we show that this is well-defined.)

Definition 5.3 allows the functoriality in the second variable to be extended to non-σ-unital I, giving a bifunctor. We note that when I is σ-unital but not separable, a priori Definitions 5.2 and 5.3 give two competing definitions of $KK(A, I)$. In Proposition B.6, we show that these two definitions naturally agree.

The following facts are standard when I is separable. The proof that they extend to general I is deferred to Proposition B.9.

Proposition 5.4. *Let A and I be C^*-algebras with A separable, let E be a C^*-algebra containing I as an ideal, and let $(\phi, \psi)\colon A \rightrightarrows E \rhd I$ be an (A, I)-Cuntz pair.*

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[101] As a map, $\phi_1 \oplus \phi_2$ depends on the choice of s_1 and s_2. However, this sum is well-defined up to unitary equivalence (which implies homotopy equivalence, since the unitary group of $\mathcal{M}(I\otimes\mathcal{K})$ is path connected in the strict topology by [10, Proposition 12.2.2], for example). Note that $\oplus$ is associative up to unitary equivalence. Moreover, the Cuntz isometries s_1, s_2 implement an isomorphism $I \otimes \mathcal{K} \cong M_2(I \otimes \mathcal{K})$ (and hence $\mathcal{M}(I \otimes \mathcal{K}) \cong M_2(\mathcal{M}(I \otimes \mathcal{K}))$), and $\oplus$ is defined by pulling back the usual direct sum in $M_2(\mathcal{M}(I \otimes \mathcal{K}))$ to $\mathcal{M}(I \otimes \mathcal{K})$.

[102] In this definition, the same choice of Cuntz isometries s_1 and s_2 should be used to define both direct sums.

(i) *Let* $\iota_I^{(2)}\colon I \to M_2(I)$ *be the top-left corner inclusion. Then the induced map* $KK(A,\iota_I^{(2)})\colon KK(A,I) \to KK(A,M_2(I))$ *is an isomorphism taking* $[\phi,\psi]_{KK(A,I)}$ *to* $[\iota_E^{(2)}\circ\phi,\iota_E^{(2)}\circ\psi]_{KK(A,M_2(I))}$.

(ii) *Given a unitary* $u \in I^\dagger$,

$$[\phi,\psi]_{KK(A,I)} = [\mathrm{Ad}\,u\circ\phi,\psi]_{KK(A,I)}. \tag{5.7}$$

(iii) $[\phi,\psi]_{KK(A,I)} + [\psi,\phi]_{KK(A,I)} = 0$.

(iv) *If* $J \lhd F$ *and* $\theta\colon I \to J$ *is a* **-homomorphism that extends to a* **-homomorphism* $\bar\theta\colon E \to F$, *then* $KK(A,\theta)([\phi,\psi]_{KK(A,I)}) = [\bar\theta\circ\phi,\bar\theta\circ\psi]_{KK(A,J)}$. *In particular,*[103] $KK(A,\iota_{I\subseteq E})([\phi,\psi]_{KK(A,I)}) = [\phi]_{KK(A,E)} - [\psi]_{KK(A,E)}$ *and if* $\phi\colon A \to I$ *and* $\theta\colon I \to J$ *are* **-homomorphisms, then*

$$KK(A,\theta)([\phi]_{KK(A,I)}) = [\theta\circ\phi]_{KK(A,J)}. \tag{5.8}$$

(v) *Let* A *be nuclear, and suppose*

$$\mathsf{e}\colon 0 \longrightarrow I \xrightarrow{j_{\mathsf{e}}} E \xrightarrow{q_{\mathsf{e}}} D \longrightarrow 0 \tag{5.9}$$

is an extension of C^**-algebras. Then the sequence*

$$KK(A,I) \xrightarrow{KK(A,j_{\mathsf{e}})} KK(A,E) \xrightarrow{KK(A,q_{\mathsf{e}})} KK(A,D) \tag{5.10}$$

is exact. The same holds when A *is* KK*-equivalent to a nuclear* C^**-algebra.*

There is a well-known action of KK on K-theory,

$$\Gamma^{(A,I)}\colon KK(A,I) \to \mathrm{Hom}(K_*(A),K_*(I)) \tag{5.11}$$

such that if $\phi\colon A \to I$ is a *-homomorphism, then

$$\Gamma_i^{(A,I)}([\phi]_{KK(A,I)}) = K_i(\phi). \tag{5.12}$$

This map is defined using the Kasparov product in the case where I is σ-unital, and it is extended to the non-separable case by taking the limit; this is laid out (and (5.12) is justified) in (B.32).

Next, we recall the KL-groups, first conceived to help classify *-homomorphisms up to approximate unitary equivalence. Rørdam's original definition ([238, Section 5]) required a UCT assumption. This was relaxed by Dadarlat in [61], who defined $KL(A,I)$ to be the quotient of $KK(A,I)$ by the closure of $\{0\}$ in various equivalent topologies on $KK(A,I)$. We use a direct description of this closure, also from [61], which we give without any separability assumptions on I.

Definition 5.5 (cf. [61, Theorem 3.5, Theorem 4.1, and Section 5])**.** Let A and I be C^*-algebras with A separable. Write $\overline{\mathbb{N}} \coloneqq \mathbb{N}\cup\{\infty\}$ for the one-point compactification of $\mathbb{N}$ and ev_n for evaluation at a point $n \in \overline{\mathbb{N}}$. Define

$$Z_{KK(A,I)} \coloneqq \left\{ KK(A,\mathrm{ev}_\infty)(\kappa)\colon \begin{array}{l} \kappa \in KK(A,C(\overline{\mathbb{N}},I)) \text{ and} \\ KK(A,\mathrm{ev}_n)(\kappa) = 0\,\forall n\in\mathbb{N} \end{array} \right\} \tag{5.13}$$

and

$$KL(A,I) \coloneqq KK(A,I)/Z_{KK(A,I)}. \tag{5.14}$$

[103]For the first in particular, take $J = F \coloneqq E$, $\theta \coloneqq \iota_{I\subseteq E}$, and $\bar\theta \coloneqq \mathrm{id}_E$. For the second one, use $E \coloneqq I$, $F \coloneqq J$, and $\bar\theta \coloneqq \theta$.

With this definition, the bifunctor structure descends from $KK(\,\cdot\,,\cdot\,)$ to $KL(\,\cdot\,,\,\cdot\,)$, just as it does when I is separable. We can also express KL as an inductive limit over separable subalgebras, analogous to how we defined KK in Definition 5.3; we prove the following in Appendix B, immediately following Lemma B.8.

Proposition 5.6. *Let A and I be C^*-algebras with A separable. Then*

$$KL(A,I) \cong \varinjlim_{I_0 \text{ sep.}} KL(A,I_0) \tag{5.15}$$

where I_0 ranges over all separable C^*-subalgebras of I, ordered by inclusion, and the isomorphism is induced by the inclusions $I_0 \to I$.

Bootstrapping standard facts for KK (from Proposition 5.4), we arrive at the following KL version (see Proposition B.9).

Proposition 5.7. *Let A and I be C^*-algebras with A separable, let E be a C^*-algebra containing I as an ideal, and let $(\phi,\psi)\colon A \rightrightarrows E \rhd I$ be an (A,I)-Cuntz pair. Then:*

(i) Let $\iota_I^{(2)}\colon I \to M_2(I)$ be the top-left corner inclusion. Then the induced map $KL(A,\iota_I^{(2)})\colon KL(A,I) \to KL(A,M_2(I))$ is an isomorphism taking $[\phi,\psi]_{KL(A,I)}$ to $[\iota_E^{(2)}\circ\phi,\iota_E^{(2)}\circ\psi]_{KL(A,M_2(I))}$.

(ii) Given a unitary in $I^\dagger$,

$$[\phi,\psi]_{KL(A,I)} = [\operatorname{Ad} u\circ\phi,\psi]_{KL(A,I)}. \tag{5.16}$$

(iii) $[\phi,\psi]_{KL(A,I)} + [\psi,\phi]_{KL(A,I)} = 0$.

(iv) If $J \lhd F$ and $\theta\colon I \to J$ is a *-homomorphism that extends to a *-homomorphism $\bar\theta\colon E\to F$, then $KL(A,\theta)([\phi,\psi]_{KL(A,I)}) = [\bar\theta\circ\phi,\bar\theta\circ\psi]_{KL(A,J)}$. In particular, $KL(A,\iota_{I\subseteq E})([\phi,\psi]_{KL(A,I)}) = [\phi]_{KL(A,E)} - [\psi]_{KL(A,E)}$, and if $\phi\colon A\to I$ and $\theta\colon I\to J$ are *-homomorphisms, then

$$KL(A,\theta)([\phi]_{KL(A,I)}) = [\theta\circ\phi]_{KL(A,J)}. \tag{5.17}$$

5.2. **Absorption.** Now we turn to absorption. Loosely speaking, an absorbing *-homomorphism is one for which the conclusion of Voiculescu's non-commutative Weyl–von Neumann Theorem holds. The relevance of absorption can be seen in the early developments of the theory: in [223], for example, Pimsner, Popa, and Voiculescu prove an absorption theorem for extensions of $C(X)\otimes\mathcal{K}$ for X finite dimensional, in part to generalize Brown–Douglas–Fillmore theory; Kasparov's generalization of Voiculescu's theorem in [151] provides (nuclearly) absorbing *-homomorphisms.

Whereas we set out KK-theory without restriction on the second variable, absorption results are another matter. For these, it is vital that one has a countable approximate unit.

Throughout the rest of the section, I will be σ-unital and stable. We write $\mathcal{Q}(I)$ for the corona algebra $\mathcal{M}(I)/I$, and $q_I\colon \mathcal{M}(I)\to\mathcal{Q}(I)$ for the quotient map. We can form the direct sum of two maps $\phi_1,\phi_2\colon A\to\mathcal{M}(I)$ as in (5.4). Likewise, given maps $\psi_1,\psi_2\colon A\to\mathcal{Q}(I)$, we can form the direct sum $\psi_1\oplus\psi_2\colon A\to\mathcal{Q}(I)$ (using isometries coming from $\mathcal{M}(I)$). Both of these are well defined up to unitary equivalence (in the latter case by unitaries from $\mathcal{M}(I)$). Therefore, the notions of absorption in the next definition do not depend on the isometries chosen.[104]

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[104]Part (ii) of Definition 5.8 goes back to Kasparov (phrased in the language of extensions) as [152, p. 560, Definition 2]. To our knowledge, absorbing *-homomorphisms into general multiplier

Definition 5.8. Let A and I C^*-algebras with A separable and I σ-unital and stable.

(i) A *-homomorphism $\phi\colon A \to \mathcal{M}(I)$ is *absorbing* if for every *-homomorphism $\psi\colon A \to \mathcal{M}(I)$, there is a sequence of unitaries $(u_n)_{n=1}^\infty \subseteq \mathcal{M}(I)$ such that, for each $a \in A$, we have

$$\big(n \mapsto u_n(\phi(a) \oplus \psi(a))u_n^* - \phi(a)\big) \in C_0(\mathbb{N}, I).^{105} \tag{5.18}$$

(ii) A *-homomorphism $\theta\colon A \to \mathcal{Q}(I)$ is *absorbing* if for every *-homomorphism $\psi\colon A \to \mathcal{M}(I)$ there is a unitary $u \in \mathcal{M}(I)$ such that $\mathrm{Ad}(q_I(u)) \circ (\theta \oplus (q_I \circ \psi)) = \theta$.

The next three results are not new, but nor are they entirely straightforward to extract from the literature. The proof below follows the computations in [62] and [63], which in turn uses ideas from [151]. The "asymptotic absorption" condition in (iii) below originates in [62], and Corollary 5.11 below is implicit there.

Proposition 5.9. *Let A and I be C^*-algebras with A separable and I σ-unital and stable, and let $\phi\colon A \to \mathcal{M}(I)$ be a *-homomorphism. The following are equivalent:*

(i) ϕ is absorbing;

(ii) for all *-homomorphisms $\psi\colon A \to \mathcal{M}(I)$, there is a unitary $u \in \mathcal{M}(I)$ such that

$$u(\phi(a) \oplus \psi(a))u^* - \phi(a) \in I, \quad a \in A; \tag{5.19}$$

(iii) for all *-homomorphisms $\psi\colon A \to \mathcal{M}(I)$, there is a norm-continuous path $(u_t)_{t\geq 0} \subseteq \mathcal{M}(I)$ of unitaries such that

$$\big(t \mapsto u_t(\phi(a) \oplus \psi(a))u_t^* - \phi(a)\big) \in C_0([0,\infty), I), \quad a \in A. \tag{5.20}$$

Proof. The implications (iii)$\Rightarrow$(i)$\Rightarrow$(ii) are clear. Now assume that (ii) holds and fix a *-homomorphism $\psi\colon A \to \mathcal{M}(I)$. Because I is stable, we may find a sequence $(v_n)_{n=1}^\infty \subseteq \mathcal{M}(I)$ of isometries such that $\sum_{n=1}^\infty v_n v_n^* = 1_{\mathcal{M}(I)}$ (with convergence in the strict topology),[106] so that, in particular,

$$v_n^* x v_n \to 0, \quad x \in I. \tag{5.21}$$

Define *-homomorphisms $\psi_\infty, (\psi_\infty)_\infty\colon A \to \mathcal{M}(I)$ by

$$\psi_\infty(a) \coloneqq \sum_{n=1}^\infty v_n \psi(a) v_n^*, \quad a \in A, \tag{5.22}$$

and

$$(\psi_\infty)_\infty(a) \coloneqq \sum_{n=1}^\infty v_n \psi_\infty(a) v_n^* = \sum_{m,n=1}^\infty v_n v_m \psi(a) v_m^* v_n^*, \quad a \in A. \tag{5.23}$$

Note that, for each $n \in \mathbb{N}$ and $a \in A$,

$$v_n^*(\psi_\infty)_\infty(a) v_n = \psi_\infty(a). \tag{5.24}$$

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algebras (as in part (i) of the definition) started to appear explicitly around the turn of the millennium, and are formalized in [267].

[105] This is merely a concise way to encode the two conditions that $u_n(\phi(a) \oplus \psi(a))u_n^*$ is equal to $\phi(a)$ modulo I for each n, and that $\|u_n(\phi(a) \oplus \psi(a))u_n^* - \phi(a)\| \to 0$ as $n \to \infty$. We will use it throughout this section.

[106] Use that $1_{\mathcal{M}(I)} \otimes \mathcal{B}(\mathcal{H}) \subseteq \mathcal{M}(I \otimes \mathcal{K})$.

Now fix isometries $s_1, s_2 \in \mathcal{M}(I)$ with $s_1 s_1^* + s_2 s_2^* = 1_{\mathcal{M}(I)}$ so that

$$\big(\phi \oplus (\psi_\infty)_\infty\big)(a) = s_1 \phi(a) s_1^* + s_2 (\psi_\infty)_\infty(a) s_2^*, \quad a \in A. \tag{5.25}$$

Using (ii), let $u \in \mathcal{M}(I)$ be a unitary with

$$u\big(\phi(a) \oplus (\psi_\infty)_\infty(a)\big)u^* - \phi(a) \in I, \quad a \in A, \tag{5.26}$$

and note that this implies

$$s_2^* u^* \phi(a) u s_2 - (\psi_\infty)_\infty(a) \in I, \quad a \in A. \tag{5.27}$$

Then $w_n \coloneqq u s_2 v_n$ is an isometry for each $n \geq 1$ and, for $n \neq m$, we have $w_n^* w_m = v_n^* v_m = 0$. Furthermore, for each $a \in A$,

$$n \mapsto w_n^* \phi(a) w_n - \psi_\infty(a) \overset{(5.24)}{=} v_n^*\big(s_2^* u^* \phi(a) u s_2^* - (\psi_\infty)_\infty(a)\big)v_n \tag{5.28}$$

lies in $C_0(\mathbb{N}, I)$ by (5.27) and (5.21). Then, for all $a \in A$, we have

$$\begin{aligned}
&\Big(n \mapsto \big(w_n \psi_\infty(a) - \phi(a) w_n\big)^* \big(w_n \psi_\infty(a) - \phi(a) w_n\big)\Big) \\
&\quad = \Big(n \mapsto \psi_\infty(a^*)\big(\psi_\infty(a) - w_n^* \phi(a) w_n\big) \\
&\qquad + \big(\psi_\infty(a^*) - w_n^* \phi(a^*) w_n\big)\psi_\infty(a) \\
&\qquad + \big(w_n^* \phi(a^* a) w_n - \psi_\infty(a^* a)\big)\Big) \in C_0(\mathbb{N}, I).
\end{aligned} \tag{5.29}$$

Therefore,

$$\big(n \mapsto w_n \psi_\infty(a) - \phi(a) w_n\big) \in C_0(\mathbb{N}, I) \tag{5.30}$$

by the C^*-identity. Condition (iii) now follows from [62, Lemma 2.3] (which uses [63, Lemma 2.16]). $\square$

The two notions of absorption in Definition 5.8 are closely related.

Corollary 5.10. *Suppose A and I are C^*-algebras with A separable and I σ-unital and stable. Then a *-homomorphism $\phi\colon A \to \mathcal{M}(I)$ is absorbing if and only if $q_I \circ \phi\colon A \to \mathcal{Q}(I)$ is absorbing.*

Proof. The forward direction is clear. The backward direction follows from (ii)$\Rightarrow$(i) of Proposition 5.9. $\square$

The asymptotic absorption in Proposition 5.9(iii) leads to the following asymptotic uniqueness result for absorbing *-homomorphisms, which we use in the $\mathcal{Z}$-stable KK- and KL-uniqueness theorems.

Corollary 5.11. *Suppose A and I are C^*-algebras with A separable and I σ-unital and stable. If $\phi, \psi\colon A \to \mathcal{M}(I)$ are absorbing representations, then there is a norm-continuous path $(u_t)_{t \geq 0} \subseteq \mathcal{M}(I)$ of unitaries such that*

$$\big(t \mapsto u_t \phi(a) u_t^* - \psi(a)\big) \in C_0([0, \infty), I). \tag{5.31}$$

Proof. By Proposition 5.9(iii) applied to both ϕ and ψ, there are norm-continuous paths $(v_t)_{t \geq 0}$ and $(w_t)_{t \geq 0}$ of unitaries in $\mathcal{M}(I)$ such that for $a \in A$,

$$\begin{aligned}
&\big(t \mapsto v_t(\phi(a) \oplus \psi(a))v_t^* - \phi(a)\big) \in C_0([0, \infty), I), \quad \text{and} \\
&\big(t \mapsto w_t(\phi(a) \oplus \psi(a))w_t^* - \psi(a)\big) \in C_0([0, \infty), I).
\end{aligned} \tag{5.32}$$

The result follows with $u_t \coloneqq w_t v_t^*$. $\square$

Next we turn to tools for verifying absorption of maps $A \to \mathcal{Q}(I)$ when A is separable and I is σ-unital and stable. These date back to Voiculescu's theorem ([283]) for the case $I = \mathcal{K}$, which was then generalized by Kasparov to produce absorbing maps when at least one of A or I is nuclear ([151, Theorem 6]).[107] Kirchberg first characterized absorption for *-homomorphisms under certain pure infiniteness criteria ([161, Theorem 6]; see also [240, Theorem 8.3.1]). In the presence of nuclearity, Elliott and Kucerovsky used Kirchberg's work (see [99, Lemma 11]) to give a beautiful characterization ([99, Theorem 6]) of unitally absorbing *-homomorphisms (defined by requiring all maps in Definition 5.8 to be unital) by a condition they call *pure largeness.* We will explore pure largeness in more detail in our subsequent work, but, for the maps appearing in this paper, Kucerovsky and Ng's corona factorization property ([172]) allows a clean characterization of absorption in terms of unitizably full maps. We recall the definition below.

Definition 5.12. A *-homomorphism $\phi\colon A \to D$ between C^*-algebras with D unital is said to be *unitizably full* if the unitized map $\phi^\dagger\colon A^\dagger \to D$ is full.[108]

With this in place, the characterization we use is as follows. The short proof is deceiving — the real work is outsourced to references.[109] It is fundamental to our abstract approach to C^*-algebra classification in just the same way that a corresponding version for C^*-algebras that absorb the universal UHF algebra was vital in [252, 253].

Theorem 5.13. *If A is a separable nuclear C^*-algebra and I is a σ-unital stable $\mathcal{Z}$-stable C^*-algebra, then a *-homomorphism $A \to \mathcal{M}(I)$ or $A \to \mathcal{Q}(I)$ is absorbing if and only if it is unitizably full.*

Proof. By Proposition 4.14, I has the corona factorization property. The statement for a *-homomorphism $\phi\colon A \to \mathcal{Q}(I)$ is [111, Theorem 2.6] (which is phrased in the language of extensions) after noting that, since A is nuclear, absorption and nuclear absorption of ϕ are equivalent. The multiplier version follows from Corollary 5.10 and the fact that (by stability of I) an element $x \in \mathcal{M}(I)$ is full if and only if its image in $\mathcal{Q}(I)$ is full.[110] □

5.3. ***KK*-existence.** The KK-existence theorem we use comes in two forms, both of which are essentially folklore to experts. The first shows that all KK-classes can be realized by Cuntz pairs with a fixed absorbing *-homomorphism in the second variable, while the second uses a result of Dadarlat to rephrase a standard fact from extension theory in terms of KK-theory.

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[107]The same result when I is separable but without the nuclearity assumption was later obtained by Thomsen ([267]).

[108]When A is unital, $A^\dagger$ is obtained by adding an additional unit: $A^\dagger \coloneqq A \oplus \mathbb{C}$. Note that no unital map can be unitizably full.

[109]Indeed, the heavy lifting is performed in [99, Theorem 6], which characterizes unitally nuclearly absorbing maps. The non-unital version we need is then obtained by unitizing, but this is not handled correctly in [99, Corollary 16]. The correct result, as noted in [111] is that a map $\phi\colon A \to \mathcal{Q}(I)$ is nuclearly absorbing if and only if its forced unitization $\phi^\dagger$ is purely large. When I has the corona factorization property, $\phi^\dagger$ is purely large if and only if $\phi^\dagger$ is full, i.e., if and only if ϕ is unitizably full ([172, Theorem 1.4]). A very different reformulation is used to establish the corona factorization property from regularity conditions such as $\mathcal{Z}$-stability.

[110]This is standard: if the image of x in $\mathcal{Q}(I)$ is full, there are $y_1, \dots, y_n, z_1, \dots, z_n \in \mathcal{M}(I)$ such that $c \coloneqq 1_{\mathcal{M}(I)} - \sum_{i=1}^n y_i x z_i \in I$. As I is stable we may pick an isometry $v \in \mathcal{M}(I)$ such that $\|v^* c v\| < 1$. Hence $\|v^* c v\| = \|1_{\mathcal{M}(I)} - \sum_{i=1}^n v^* y_i x z_i v\| < 1$ which implies that x is full.

Theorem 5.14 (KK-existence)**.** *Suppose A is a separable C^*-algebra and I is a σ-unital stable C^*-algebra.*

(i) If $\psi\colon A \to \mathcal{M}(I)$ is an absorbing *-homomorphism and $\kappa \in KK(A,I)$, then there is an absorbing *-homomorphism $\phi\colon A \to \mathcal{M}(I)$ such that (ϕ,ψ) is an (A,I)-Cuntz pair and $[\phi,\psi]_{KK(A,I)} = \kappa$.

(ii) Suppose A is nuclear. If $\theta\colon A \to \mathcal{Q}(I)$ is an absorbing *-homomorphism, then there is a (necessarily) absorbing *-homomorphism $\phi\colon A \to \mathcal{M}(I)$ lifting θ if and only if $[\theta]_{KK(A,\mathcal{Q}(I))} = 0$.

Part (i) is standard — a version for weakly nuclear, nuclearly absorbing *-homomorphisms is given in [253, Proposition 2.6], for example, and the same proof works here. The result has its origins in Higson's Paschke duality result ([135]) for K-homology, showing that if $\kappa \in KK(A,\mathcal{K})$ and $\psi\colon A \to \mathcal{M}(\mathcal{K})$ is absorbing, then there is a unitary $u \in \mathcal{M}(\mathcal{K})$ commuting with $\psi(A)$ modulo $\mathcal{K}$ such that $\kappa = [\psi,\psi,u]_{KK(A,\mathcal{K})}$ in the Fredholm picture of KK-theory. The result for general I was obtained by Thomsen in [267] in the proof of his Paschke duality result.

Part (ii) is a consequence of Dadarlat's isomorphism between $\mathrm{Ext}(A,I)$ and $KK(A,\mathcal{Q}(I))$ for A separable and nuclear and I stable and σ-unital from [59]. In the proof below, we identify extensions with their Busby invariants. Recall that, under the hypotheses of Theorem 5.14, an extension of A by I with Busby invariant $\theta\colon A \to \mathcal{Q}(I)$ splits if and only if θ lifts to $\phi\colon A \to \mathcal{M}(I)$. Moreover, the class of θ in $\mathrm{Ext}(A,I)$ vanishes precisely when $\theta \oplus \psi$ splits for some split extension ψ. See [10, Chapter 15] for details.

Proof of Theorem 5.14(ii). If θ lifts to a *-homomorphism $\phi\colon A \to \mathcal{M}(I)$, then $[\theta]_{KK(A,\mathcal{Q}(I))} = 0$ since $KK(A,\mathcal{M}(I)) = 0$ by [59, Proposition 4.1]. For the converse, if $[\theta]_{KK(A,\mathcal{Q}(I))}$ vanishes, then the extension with Busby invariant θ has the trivial class in $\mathrm{Ext}(A,I)$, by applying the isomorphism between $KK(A,\mathcal{Q}(I))$ and $\mathrm{Ext}(A,I)$ from [59, Proposition 4.2]. Thus there is a split extension, say with Busby invariant $q_I \circ \psi\colon A \to \mathcal{Q}(I)$ for some *-homomorphism $\psi\colon A \to \mathcal{M}(I)$, such that $\theta \oplus (q_I \circ \psi)$ splits, and so has a lift to $\mathcal{M}(I)$. But since θ is absorbing, there is a unitary $u \in \mathcal{M}(I)$ with $q_I(u)(\theta \oplus (q_I \circ \psi))q_I(u)^* = \theta$. Thus θ has a lift $\phi\colon A \to \mathcal{M}(I)$, which is necessarily absorbing by Proposition 5.10, since θ is absorbing. $\square$

5.4. KK-uniqueness.

Now we turn to our $\mathcal{Z}$-stable KK- and KL-uniqueness theorems. The first part of the following result is (i)$\Rightarrow$(ii) of Theorem 1.4 from the introduction. The other implication will be proved at the end of this subsection.

Theorem 5.15 ($\mathcal{Z}$-stable KK- and KL-uniqueness)**.** *Let A and I be C^*-algebras with A separable and I σ-unital and stable. Let $(\phi,\psi)\colon A \rightrightarrows \mathcal{M}(I) \rhd I$ be a Cuntz pair with ϕ and ψ absorbing.*

(i) If $[\phi,\psi]_{KK(A,I)} = 0$, then there is a norm-continuous path $(u_t)_{t\geq 0}$ of unitaries in $(I \otimes \mathcal{Z})^\dagger$ such that

$$\|u_t(\phi(a) \otimes 1_{\mathcal{Z}})u_t^* - \psi(a) \otimes 1_{\mathcal{Z}}\| \to 0, \quad a \in A. \tag{5.33}$$

(ii) If $[\phi,\psi]_{KL(A,I)} = 0$, then there is a sequence $(u_n)_{n=1}^\infty$ of unitaries in $(I \otimes \mathcal{Z})^\dagger$ such that

$$\|u_n(\phi(a) \otimes 1_{\mathcal{Z}})u_n^* - \psi(a) \otimes 1_{\mathcal{Z}}\| \to 0, \quad a \in A. \tag{5.34}$$

Our proof of (i) follows the same strategy as Dadarlat and Eilers' original KK-uniqueness theorem for $I = \mathcal{K}$ and no $\mathcal{Z}$-stabilization ([62, Theorem 3.12]), making heavy use of Paschke duality ([215], and generalizations [267, Theorem 3.2], [62, Lemma 3.5]). More precisely, given a Cuntz pair $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(I) \triangleright I$ of absorbing representations, take a unitary path $(u_t)_{t\geq 0}$ as in Corollary 5.11. Then $q_I(u_0)$ commutes with $q_I(\phi(A))$. Paschke duality ensures that $[\phi, \psi]_{KK(A,I)} = 0$ if and only if the class of $q_I(u_0)$ is trivial in $K_1(\mathcal{Q}(I) \cap q_I(\phi(A))')$. If $q_I(u_0)$ is in the path component of the identity in $\mathcal{Q}(I) \cap q_I(\phi(A))'$, then ϕ and ψ are properly asymptotically unitarily equivalent;[111] this is extracted from the proof of [62, Theorem 3.12] as Lemma 5.16(i) below.

Accordingly, this strategy gives KK-uniqueness whenever $\mathcal{Q}(I) \cap q_I(\phi(A))'$ is K_1-injective.[112] It is an open problem whether all such relative commutants are K_1-injective (see Question 5.17 and the discussion thereafter).

Dadarlat and Eilers' stable KK-uniqueness theorem (recalled in the remarks after Theorem 1.4) now follows from the fact that $\mathcal{Q}(I) \cap q_I(\phi(A))'$ is properly infinite,[113] so that $[q_I(u_0)]_1 = 0$ implies that $q_I(u_0) \oplus 1_{\mathcal{Q}(I)}$ is in the path component of the identity in $U_2(\mathcal{Q}(I) \cap q_I(\phi(A))')$ (see [243, Exercise 8.11(ii)]). The requirement for the direct summand $1_{\mathcal{Q}(I)}$ in this argument leads to the summand of η in condition (iii) of the stable uniqueness theorem (Theorem 1.4).

For our $\mathcal{Z}$-stable KK-uniqueness theorem, we bypass the problem of whether $\mathcal{Q}(I) \cap q_I(\phi(A))'$ is K_1-injective, by working with its $\mathcal{Z}$-stabilization — which is K_1-injective by Jiang's result (Theorem 4.8). This allows us to conclude that $u_0 \otimes 1_{\mathcal{Z}}$ is in the path component of the identity in the $\mathcal{Z}$-stabilization of $\mathcal{Q}(I) \cap q_I(\phi(A))'$. So in spirit, our $\mathcal{Z}$-stable KK-uniqueness theorem is obtained by tensoring on a copy of $1_{\mathcal{Z}}$, whereas the Dadarlat–Eilers theorem adjoins a direct summand of the identity.

The KL-uniqueness theorem (part (ii) of Theorem 5.15) follows from the KK-uniqueness theorem (part (i)). The strategy, which goes back to Lin ([177, 181]) and Dadarlat ([61]), is to relate vanishing in KL with approximate vanishing in KK. This is exemplified in the proof of [61, Theorem 5.1], where Dadarlat uses a reduction to KK-classification to reprove that KL detects approximate uniqueness of morphisms between Kirchberg algebras; our argument for (ii) is in the same spirit.

We now give an abstract version of Dadarlat and Eilers' argument.[114]

Lemma 5.16. *Let A be a unital separable C^*-algebra, and let*

$$0 \longrightarrow I \longrightarrow E \xrightarrow{\ q\ } D \longrightarrow 0 \tag{5.35}$$

[111] ϕ and ψ are properly asymptotically unitarily equivalent when there is a continuous path $(u_t)_{t\geq 0}$ of unitaries in $I^\dagger$ with $u_t\phi(a)u_t^* \to \psi(a)$ for all $a \in A$.

[112] K_1-injectivity allows one to know that $q_I(u_0)$ is in the path component of the identity whenever $[q_I(u_0)]_1 = 0$.

[113] Indeed, since ϕ is absorbing, ϕ is unitarily equivalent to $\phi \oplus \phi$ modulo I by Corollary 5.11, and if u is a unitary implementing this equivalence and s_1 and s_2 are the Cuntz isometries defining the direct sum, then us_1 and us_2 are isometries in $\mathcal{Q}(I) \cap q_I(\phi(A))'$ with orthogonal range projections.

[114] Dadarlat and Eilers' strategy has its origins in the large body of work on automorphisms and derivations on operator algebras undertaken in the 1960s ([148, 246, 149, 211]; see also [219, Sections 8.6 and 8.7]).

be an extension of C^*-algebras with E unital. Suppose $\phi, \psi\colon A \to E$ are unital *-homomorphisms such that $q \circ \phi = q \circ \psi$ and that this composition is injective. Suppose further that there is a continuous path $(v_t)_{t\geq 0}$ of unitaries in E such that

$$\big(t \mapsto v_t\phi(a)v_t^* - \psi(a)\big) \in C_0([0,\infty), I), \quad a \in A. \tag{5.36}$$

(i) If the path $(v_t)_{t\geq 0}$ satisfying (5.36) can be chosen such that $q(v_0)$ lies in the path component of the identity in the unitary group of $D \cap q(\phi(A))'$, then it can be chosen with $v_t \in I^\dagger \subseteq E$ for all $t \geq 0$.

(ii) If the path $(v_t)_{t\geq 0}$ satisfying (5.36) can be chosen with $[q(v_0)]_1 = 0$ in $K_1(D \cap q(\phi(A))')$, then there is a path $(u_t)_{t\geq 0}$ of unitaries in $(I \otimes \mathcal{Z})^\dagger$ such that for all $a \in A$,

$$\big(t \mapsto u_t(\phi(a) \otimes 1_{\mathcal{Z}})u_t^* - \psi(a) \otimes 1_{\mathcal{Z}}\big) \in C_0([0,\infty), I \otimes \mathcal{Z}). \tag{5.37}$$

Proof. (i): A path $(v_t)_{t\geq 0}$ of unitaries satisfying (5.36) such that $q(v_0)$ is in the path component of the identity in $D \cap q(\phi(A))'$ can be adjusted so that $v_0 \in I^\dagger$.[115]

As

$$\begin{aligned} v_0^*v_t\phi(a)v_t^*v_0 - \phi(a) = v_0^*(v_t\phi(a)v_t^* - \psi(a))v_0 \\ + (v_0^*\psi(a)v_0 - \phi(a)) \in I, \quad a \in A, \end{aligned} \tag{5.38}$$

$\phi(A) + I$ is a C^*-subalgebra of $\mathcal{M}(I)$ which is invariant under $\mathrm{Ad}(v_0^*v_t)$ for all $t \geq 0$. Then, since A is separable, we can find a separable C^*-subalgebra $C \subseteq \phi(A) + I$ containing $\phi(A)$ such that C is invariant under $\mathrm{Ad}(v_0^*v_t)$ for all $t \geq 0$.[116] Then $(\mathrm{Ad}(v_0^*v_t))_{t\geq 0}$ is a norm-continuous path of automorphisms of C starting at id_C. By [62, Proposition 2.15], there is a continuous path $(w_t')_{t\geq 0}$ in $C \subseteq \phi(A) + I$ of unitaries such that

$$\|\mathrm{Ad}(w_t')(c) - \mathrm{Ad}(v_0^*v_t)(c)\| \to 0, \quad c \in C. \tag{5.39}$$

Set $w_t \coloneqq v_0w_t'$. Then w_t is a unitary in $\phi(A) + I$ for all $t \geq 0$ and

$$\|w_t\phi(a)w_t^* - \psi(a)\| \to 0, \quad a \in A. \tag{5.40}$$

Write $w_t = \phi(a_t) + x_t$, where $a_t \in A$ and $x_t \in I$, so that $q(\phi(a_t)) = q(w_t)$ is a unitary. Since $q \circ \phi$ is injective, a_t (and hence x_t) are uniquely determined by w_t, and $(a_t)_{\geq 0}$ is a continuous path of unitaries in A. Also, since $q \circ \phi = q \circ \psi$, by (5.40) we have

$$\|q(\phi(a_t))q(\phi(a))q(\phi(a_t))^* - q(\phi(a))\| \to 0, \quad a \in A, \tag{5.41}$$

so another application of injectivity of $q \circ \phi$ gives

$$\|a_taa_t^* - a\| \to 0, \quad a \in A. \tag{5.42}$$

Now, define

$$u_t \coloneqq w_t\phi(a_t)^* = (1_{I^\dagger} + x_t\phi(a_t)^*) \in I^\dagger. \tag{5.43}$$

[115]Indeed, the hypothesis gives self-adjoint elements $\bar{h}_1, \dots, \bar{h}_n \in D \cap q(\phi(A))'$ such that $q(v_0) = e^{i\bar{h}_1} \cdots e^{i\bar{h}_n}$. Lifting these to self-adjoints $h_1, \dots, h_n \in E$ with $q(h_i) = \bar{h}_i$ for all $i = 1, \dots, n$, define $v_t \coloneqq v_0e^{ith_n} \cdots e^{ith_1}$ for $-1 \leq t < 0$. Then $(v_t)_{t\geq -1}$ is a continuous path of unitaries in E with $v_{-1} \in I^\dagger$ and

$$\big(t \mapsto v_t\phi(a)v_t^* - \psi(a)\big) \in C_0([-1,\infty), I).$$

Hence after shifting the index, we may assume the given path $(v_t)_{t\geq 0}$ satisfies $v_0 \in I^\dagger$.

[116]Let $(t_n)_{n=1}^\infty$ be an enumeration of $[0,\infty) \cap \mathbb{Q}$. Starting with $C_0 = \phi(A)$, choose separable C^*-subalgebras $C_0 \subseteq C_1 \subseteq \dots$ of $\phi(A) + I$ such that for each $n \in \mathbb{N}$, C_n is invariant under $\mathrm{Ad}(v_0^*v_{t_m})$ for $m = 1, \dots, n$. Then $C = \overline{\bigcup_{n=1}^\infty C_n}$ provides the required separable invariant subalgebra.

Then $(u_t)_{t\geq 0}$ is a continuous path of unitaries in $I^\dagger$, and combining (5.40) and (5.42), we have, for all $a \in A$,

$$\begin{aligned}\|u_t\phi(a)u_t^* - \psi(a)\| &\leq \|w_t(\phi(a_t^*)\phi(a)\phi(a_t))w_t^* - w_t\phi(a)w_t^*\| \\ &\quad + \|w_t\phi(a)w_t^* - \psi(a)\| \to 0,\end{aligned} \tag{5.44}$$

as required.

(ii): Let $(v_t)_{t\geq 0}$ be a path of unitaries as in (ii). By nuclearity of $\mathcal{Z}$, we have a short exact sequence

$$0 \longrightarrow I \otimes \mathcal{Z} \longrightarrow E \otimes \mathcal{Z} \xrightarrow{q\otimes \mathrm{id}_{\mathcal{Z}}} D \otimes \mathcal{Z} \longrightarrow 0. \tag{5.45}$$

We will complete the proof by applying (i) to this extension, the *-homomorphisms $\phi \otimes 1_{\mathcal{Z}}$ and $\psi \otimes 1_{\mathcal{Z}}$, and the path $(v_t \otimes 1_{\mathcal{Z}})_{t\geq 0}$.

One has a containment of relative commutants

$$\theta\colon (D \cap q(\phi(A))') \otimes \mathcal{Z} \hookrightarrow (D \otimes \mathcal{Z}) \cap (q \otimes \mathrm{id}_{\mathcal{Z}})\big((\phi \otimes 1_{\mathcal{Z}})(A)\big)'.\text{[117]} \tag{5.46}$$

Since $q(v_0)$ has trivial class in $K_1(D \cap q(\phi(A))')$, its image $q(v_0) \otimes 1_{\mathcal{Z}}$ is also trivial in $K_1((D \cap q(\phi(A))') \otimes \mathcal{Z})$. Jiang's K_1-injectivity of $\mathcal{Z}$-stable C^*-algebras (Theorem 4.8) implies that $q(v_0) \otimes 1_{\mathcal{Z}}$ is in the path component of the identity in $(D \cap q(\phi(A))') \otimes \mathcal{Z}$. Applying θ, it follows that $(q \otimes \mathrm{id}_{\mathcal{Z}})(v_0 \otimes 1_{\mathcal{Z}})$ is in the path component of the identity in $(D\otimes\mathcal{Z})\cap(q\otimes\mathrm{id}_{\mathcal{Z}})(\phi\otimes 1_{\mathcal{Z}})(A)'$. The result now follows from (i). □

We now prove the $\mathcal{Z}$-stable KK- and KL-uniqueness theorems.

Proof of Theorem 5.15. (i): In this proof, we use $\oplus$ to denote the usual direct sum (using a matrix amplification), rather than the definition in (5.4); see Footnote 101. Since ϕ and ψ are absorbing, Corollary 5.11 provides a norm-continuous path $(u_t)_{t\geq 0} \subseteq \mathcal{M}(I)$ of unitaries such that

$$\big(t \mapsto u_t\phi(a)u_t^* - \psi(a)\big) \in C_0([0,\infty), I). \tag{5.47}$$

By Lemma 5.16(ii), it suffices to show $[q_I(u_0)]_1 = 0$ in $K_1(\mathcal{Q}(I) \cap q_I(\phi^\dagger(A^\dagger))')$.

Now, $(\phi, \mathrm{Ad}(u_0) \circ \phi)$ is a Cuntz pair, and it is homotopic (via $(\phi, \mathrm{Ad}(u_t) \circ \phi)$) to (ϕ, ψ) (this is [62, Lemma 3.1]), so that $[\phi, \mathrm{Ad}(u_0) \circ \phi]_{KK(A,I)} = [\phi, \psi]_{KK(A,I)} = 0$. By considering the split exact sequence

$$0 \longrightarrow KK(\mathbb{C}, I) \longleftrightarrow KK(A^\dagger, I) \longrightarrow KK(A, I) \longrightarrow 0, \tag{5.48}$$

it follows that $[\phi^\dagger, \mathrm{Ad}(u_0) \circ \phi^\dagger]_{KK(A^\dagger,I)} = 0$.[118] Using the hypothesis that I is stable, [62, Lemma 3.5][119] gives a unital *-homomorphism $\theta\colon A^\dagger \to \mathcal{M}(I)$ and

[117] Using the completely positive approximation property for $\mathcal{Z}$, one can check that θ is surjective, though we do not need this. In general, the relative commutant of a tensor product of two inclusions of C^*-algebras need not be the tensor product of the relative commutants, even when one inclusion is of the form $\mathbb{C}1_C \subseteq C$; see [4].

[118] This follows, as the maps $KK(A^\dagger, I) \to KK(A, I)$ and $KK(A^\dagger, I) \to KK(\mathbb{C}, I)$ arise from the inclusions of A and $\mathbb{C}$ into $A^\dagger$, and so are given by restricting Cuntz pairs on $A^\dagger$ to A and $\mathbb{C}$ respectively (by definition). Thus $[\phi^\dagger, \mathrm{Ad}(u_0)\circ\phi^\dagger]_{KK(A^\dagger,I)}$ has image $[\phi, \mathrm{Ad}(u_0)\circ\phi]_{KK(A,I)} = 0$ in $KK(A, I)$ and image $[1_{\mathcal{M}(I)}, 1_{\mathcal{M}(I)}]_{KK(\mathbb{C},I)} = 0$ in $KK(\mathbb{C}, I)$. Hence $[\phi^\dagger, \mathrm{Ad}(u_0) \circ \phi^\dagger]_{KK(A^\dagger,I)} = 0$.

[119] [62, Lemma 3.5] is stated in terms of the Fredholm picture of KK-theory. We briefly explain how to translate following the discussion in [62, Section 3.1]. A Cuntz pair (ψ_1, ψ_2) gives the Fredholm triple $(\psi_1, \psi_2, 1_{\mathcal{M}(I)})$, and a cycle (ψ_1, ψ_2, u) with u a unitary in $\mathcal{M}(I)$ corresponds to

a norm-continuous path $(v_t)_{0\leq t\leq 1} \subseteq M_2(\mathcal{M}(I))$ with $v_0 = 1_{\mathcal{M}(I)} \oplus 1_{\mathcal{M}(I)}$ and $v_1 = u_0 \oplus 1_{\mathcal{M}(I)}$ such that each $q_{M_2(I)}(v_t)$ is a unitary in $M_2(\mathcal{Q}(I))$ commuting with $q_{M_2(I)}((\phi^\dagger \oplus \theta)(A))$. Thus $q_I(u_0) \oplus 1_{\mathcal{Q}(I)}$ is homotopic to $1_{\mathcal{Q}(I)} \oplus 1_{\mathcal{Q}(I)}$ in $M_2(\mathcal{Q}(I)) \cap q_{M_2(I)}((\phi^\dagger \oplus \theta)(A))'$. Taking the direct sum of this homotopy with $1_{\mathcal{Q}(I)}$, gives a homotopy between $q_I(u_0) \oplus 1_{\mathcal{Q}(I)} \oplus 1_{\mathcal{Q}(I)}$ and $1_{\mathcal{Q}(I)} \oplus 1_{\mathcal{Q}(I)} \oplus 1_{\mathcal{Q}(I)}$ in $M_3(\mathcal{Q}(I)) \cap q_{M_3(I)}((\phi^\dagger \oplus \theta \oplus \phi^\dagger)(A))'$.

Since ϕ is absorbing and I is stable, $\theta|_A \oplus \phi$ and ϕ are unitarily equivalent modulo I,[120] and hence so too are $\theta \oplus \phi^\dagger$ and $\phi^\dagger$. It follows that $q_I(u_0) \oplus 1_{\mathcal{Q}(I)}$ is homotopic to $1_{\mathcal{Q}(I)} \oplus 1_{\mathcal{Q}(I)}$ in the unitary group of

$$M_2(\mathcal{Q}(I)) \cap ((q_I \circ \phi^\dagger) \oplus (q_I \circ \phi^\dagger))(A^\dagger)' = M_2(\mathcal{Q}(I) \cap q_I(\phi^\dagger(A^\dagger))'). \tag{5.49}$$

Accordingly, $[q_I(u_0)]_1 = 0$ in $K_1(\mathcal{Q}(I) \cap q_I(\phi^\dagger(A^\dagger))')$ as required.

(ii): Suppose $[\phi, \psi]_{KL(A,I)} = 0$. Then there exists $\kappa \in KK(A, C(\overline{\mathbb{N}}, I))$ such that $KK(A, \mathrm{ev}_n)(\kappa) = 0$ for $n \in \mathbb{N}$ and $KK(A, \mathrm{ev}_\infty)(\kappa) = [\phi, \psi]_{KK(A,I)}$ (see Definition 5.5, and recall that $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$). By [1, Corollary 3.4],

$$\mathcal{M}(C(\overline{\mathbb{N}}, I)) \cong C_\sigma(\overline{\mathbb{N}}, \mathcal{M}(I)), \tag{5.50}$$

where the right hand side is the C^*-algebra of (norm-bounded) strictly continuous functions from $\overline{\mathbb{N}}$ to $\mathcal{M}(I)$ (see Footnote 99). Let $\Psi\colon A \to C_\sigma(\overline{\mathbb{N}}, \mathcal{M}(I))$ be given by $\Psi(a)(n) \coloneqq \psi(a)$ for all $n \in \overline{\mathbb{N}}$. It is absorbing by [61, Proposition 3.2].[121]

The KK-existence theorem (Theorem 5.14), gives an absorbing *-homomorphism $\Phi\colon A \to \mathcal{M}(C(\overline{\mathbb{N}}, I))$ with $[\Phi, \Psi]_{KK(A, C(\overline{\mathbb{N}}, I))} = \kappa$. Viewing Φ as taking values in $C_\sigma(\overline{\mathbb{N}}, \mathcal{M}(I))$, let $\phi_n \coloneqq \mathrm{ev}_n \circ \Phi\colon A \to \mathcal{M}(I)$. Then (ϕ_n, ψ) is a Cuntz pair for all $n \in \overline{\mathbb{N}}$. Since ψ is absorbing, Corollary 5.10 (applied twice) implies that ϕ_n is also absorbing for all $n \in \overline{\mathbb{N}}$.

Fix a finite subset $\mathcal{F}$ of A and $\epsilon > 0$. As A is separable, it suffices to find a unitary $u \in (I \otimes \mathcal{Z})^\dagger$ such that $\|u(\phi(a) \otimes 1_{\mathcal{Z}})u^* - \psi(a) \otimes 1_{\mathcal{Z}}\| < \epsilon$ for all $a \in \mathcal{F}$. A priori, for each $a \in A$, $\phi_n(a) \to \phi_\infty(a)$ strictly in $\mathcal{M}(I)$. However, since $\Phi(a) - \Psi(a) \in C(\overline{\mathbb{N}}, I)$, we have

$$\phi_n(a) - \phi_\infty(a) = (\phi_n(a) - \psi(a)) + (\psi(a) - \phi_\infty(a)) \in I, \quad a \in A, \tag{5.51}$$

so the convergence $\phi_n(a) \to \phi_\infty(a)$ actually happens in norm. Then we can find $n \in \mathbb{N}$ such that $\|\phi_n(a) - \phi_\infty(a)\| < \epsilon/3$ for $a \in \mathcal{F}$. Since $[\phi_n, \psi]_{KK(A,I)} = KK(A, \mathrm{ev}_n)(\kappa) = 0$ (by Proposition 5.4(iv)), we can use part (i) to find a unitary $v \in (I \otimes \mathcal{Z})^\dagger$ with

$$\|v(\phi_n(a) \otimes 1_{\mathcal{Z}})v^* - \psi(a) \otimes 1_{\mathcal{Z}}\| < \epsilon/3, \quad a \in \mathcal{F}. \tag{5.52}$$

Also

$$[\phi_\infty, \psi]_{KK(A,I)} = KK(A, \mathrm{ev}_\infty)(\kappa) = [\phi, \psi]_{KK(A,I)}. \tag{5.53}$$

---

the Cuntz pair $(\mathrm{Ad}(u) \circ \psi_1, \psi_2)$. Therefore, the Cuntz pair $(\phi^\dagger, \mathrm{Ad}(u_0) \circ \phi^\dagger)$ corresponds to the cycle $(\phi^\dagger, \phi^\dagger, u_0^*)$ and, since $[\phi^\dagger, \mathrm{Ad}(u_0) \circ \phi^\dagger]_{KK(A^\dagger, I)} = 0 = [\phi^\dagger, \phi^\dagger]_{KK(A^\dagger, I)}$, we can apply [62, Lemma 3.5] to the cycles $(\phi^\dagger, \phi^\dagger, u_0)$ and $(\phi^\dagger, \phi^\dagger, 1_{\mathcal{M}(I)})$. This gives the specified θ and $(v_t)_{0\leq t\leq 1}$.

[120] To be precise, there is a unitary 2×1 matrix u over $\mathcal{M}(A)$ such that $u^*(\theta(a) \oplus \phi(a))u - \phi(a) \in I$ for all $a \in A$ (and consequently, $\theta(a) \oplus \phi(a) - u\phi(a)u^* \in M_2(I)$ for all $a \in A$). This follows from Proposition 5.9, since if s_1, s_2 are Cuntz isometries then $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ is a 2×1 unitary which translates $\oplus$ used here to $\oplus$ as defined in (5.4) and used in Proposition 5.9.

[121] Note that although the statement of [61, Proposition 3.2] requires that I is separable, the proof only uses that I is σ-unital.

Accordingly, $[\phi_\infty, \phi]_{KK(A,I)} = 0$ (by Proposition 5.4(iii)) and so another application of part (i) gives a unitary $w \in (I \otimes \mathcal{Z})^\dagger$ with

$$\|w(\phi(a) \otimes 1_{\mathcal{Z}})w^* - \phi_\infty(a) \otimes 1_{\mathcal{Z}}\| < \epsilon/3, \quad a \in \mathcal{F}. \tag{5.54}$$

Set $u \coloneqq vw \in (I \otimes \mathcal{Z})^\dagger$. Then

$$\|u(\phi(a) \otimes 1_{\mathcal{Z}})u^* - \psi(a) \otimes 1_{\mathcal{Z}}\| < \epsilon, \quad a \in \mathcal{F}. \tag{5.55}$$

□

We do not know if either the tensor factor $1_{\mathcal{Z}}$ in Theorem 1.4(ii) or the direct summand η in Theorem 1.4(iii) are necessary. This leads to the *KK-uniqueness problem.*

Question 5.17 (The KK-uniqueness problem)**.** Let A and I be C^*-algebras with A separable and I σ-unital and stable, and let $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(I) \rhd I$ be a Cuntz pair with ϕ and ψ absorbing and $[\phi, \psi]_{KK(A,I)} = 0$. Must there be a norm-continuous path $(u_t)_{t\geq 0}$ of unitaries in $I^\dagger$ such that

$$\|u_t\phi(a)u_t^* - \psi(a)\| \to 0 \tag{5.56}$$

for all $a \in A$?

The first result in this direction is Dadarlat and Eilers' theorem ([62, Theorem 3.12]), giving a positive solution when $I = \mathcal{K}$. We point out the modifications needed in the treatment above to obtain this result, but we emphasize that the proof is essentially the same as that given in [62]. Paschke proved in [215] that the C^*-algebra $\mathcal{Q}(\mathcal{K}) \cap q_{\mathcal{K}}(\phi(A))'$ is K_1-injective. Apply this in the final line of the proof of Theorem 5.15(i) to conclude that u_0 is in the path component of the identify in $U(\mathcal{Q}(\mathcal{K}) \cap q_{\mathcal{K}}(\phi(A))')$, and then apply part (i) of Lemma 5.16 in place of (ii).

Dadarlat and Eilers' work provides a strategy for approaching the KK-uniqueness problem. Whenever $\mathcal{Q}(I) \cap q_I(\phi(A))'$ is K_1-injective for some (equivalently, any) absorbing *-homomorphism $\phi\colon A \to \mathcal{M}(I)$, the same proof shows that the KK-uniqueness problem has a positive solution (cf. [173, Theorem 2.11]). Using this strategy, a partial result was obtained in [194].

In the setting of the KK-uniqueness problem, the relative commutant $\mathcal{Q}(I) \cap q_I(\phi(A))'$ is well-known to be properly infinite (see Footnote 113). We would like to reiterate the following question from [18]. By the argument above, a positive answer would imply a positive answer to the KK-uniqueness problem.

Question 5.18 ([18, Question 2.9])**.** Is every properly infinite C^*-algebra K_1-injective?

We end this subsection by recording the easy implication of Theorem 1.4 — the hard direction follows from the KK-uniqueness theorem.

Proof of Theorem 1.4. The implication (i)⇒(ii) is Theorem 5.15(i). For (ii)⇒(i), note that (ii) implies that $[\phi \otimes 1_{\mathcal{Z}}, \psi \otimes 1_{\mathcal{Z}}]_{KK(A,I\otimes\mathcal{Z})} = 0$. The inclusion $\iota : I \to I \otimes \mathcal{Z}$ is a KK-equivalence by Proposition 4.2 and $[\phi \otimes 1_{\mathcal{Z}}, \psi \otimes 1_{\mathcal{Z}}]_{KK(A,I\otimes\mathcal{Z})} = KK(A, \iota)([\phi, \psi]_{KK(A,I)})$, by Proposition 5.4(iv), so it follows that $[\phi, \psi]_{KK(A,I)} = 0$. □

5.5. **$\mathcal{Z}$-stable KK-uniqueness for weakly nuclear maps.** In our subsequent work, we will require a version of the $\mathcal{Z}$-stable KK- and KL-uniqueness theorem in the setting of weakly nuclear maps (as a weakening of the hypothesis of nuclear domains). Since the proofs above work mutatis mutandis, we sketch the details here to avoid duplication. The material in this subsection is not used in the rest of this paper.

We start by recalling the relevant definitions from [258] (for KK_{nuc}) and [172, 110] (for KL_{nuc}).

Definition 5.19. Let A be a separable C^*-algebra and let be I be a σ-unital stable C^*-algebra.

(i) A *-homomorphism $\phi\colon A \to \mathcal{M}(I)$ is *weakly nuclear* if $x^*\phi(\,\cdot\,)x\colon A \to I$ is nuclear for all $x \in I$.[122]

(ii) A *nuclearly absorbing* *-homomorphism $\phi\colon A \to \mathcal{M}(I)$ (resp. $\theta\colon A \to Q(I)$) is defined as in Definition 5.8(i) (resp. (ii)), but replacing "for every *-homomorphism ψ" with "for every weakly nuclear *-homomorphism ψ".

(iii) The abelian group $KK_{\mathrm{nuc}}(A, I)$ is defined by requiring all *-homomorphisms (including the homotopies) in the definition of $KK(A, I)$ (Section 5.1) to be weakly nuclear.

(iv) Define

$$Z_{KK_{\mathrm{nuc}}(A,I)} \coloneqq \left\{ KK_{\mathrm{nuc}}(A, \mathrm{ev}_\infty)(\kappa)\colon \begin{array}{l} \kappa \in KK_{\mathrm{nuc}}(A, C(\overline{\mathbb{N}}, I)) \text{ and} \\ KK_{\mathrm{nuc}}(A, \mathrm{ev}_n)(\kappa) = 0 \,\forall n \in \mathbb{N} \end{array} \right\} \tag{5.57}$$

and

$$KL_{\mathrm{nuc}}(A, I) \coloneqq KK_{\mathrm{nuc}}(A, I)/Z_{KK_{\mathrm{nuc}}(A,I)}. \tag{5.58}$$

With these definitions in place, we record the following adaptation of the KK- and KL-uniqueness theorems (Theorem 5.15) to this setting.

Theorem 5.20. *Let A and I be C^*-algebras with A separable and I σ-unital and stable. Suppose that $\phi, \psi\colon A \to \mathcal{M}(I)$ are weakly nuclear, nuclearly absorbing *-homomorphisms such that (ϕ, ψ) is a Cuntz pair.*

(i) If $[\phi, \psi]_{KK_{\mathrm{nuc}}(A,I)} = 0$, then there is a norm-continuous path $(u_t)_{t\geq 0}$ of unitaries in $(I \otimes \mathcal{Z})^\dagger$ such that

$$\|u_t(\phi(a) \otimes 1_{\mathcal{Z}})u_t^* - \psi(a) \otimes 1_{\mathcal{Z}}\| \to 0, \quad a \in A. \tag{5.59}$$

(ii) If $[\phi, \psi]_{KL_{\mathrm{nuc}}(A,I)} = 0$, then there is a sequence $(u_n)_{n=1}^\infty$ of unitaries in $(I \otimes \mathcal{Z})^\dagger$ such that

$$\|u_n(\phi(a) \otimes 1_{\mathcal{Z}})u_n^* - \psi(a) \otimes 1_{\mathcal{Z}}\| \to 0, \quad a \in A. \tag{5.60}$$

Essentially, the proof follows that of Theorem 5.15, carefully verifying that all maps used in the proof are weakly nuclear, so that nuclear absorption can be used in place of absorption throughout. In a bit more detail, for a compact and Hausdorff space X, a completely positive map $\eta\colon A \to C(X, I)$ is nuclear exactly when $a \mapsto \eta(a)(x)$ is nuclear for every $x \in X$ by [110, Lemma 10.30]. Hence a *-homomorphism $\phi\colon A \to \mathcal{M}(C(X, I)) \cong C_\sigma(X, \mathcal{M}(I))$ is weakly nuclear exactly when it is pointwise weakly nuclear. This can be used with $X \coloneqq [0, 1]$ to deal with homotopies and with

[122] This is equivalent to saying that ϕ has the completely positive approximation property with respect to the strict topology on $\mathcal{M}(I)$ (see, for instance, [110, Proposition 5.11]). For this reason, weakly nuclear maps are sometimes called *strictly nuclear*.

$X := \overline{\mathbb{N}}$ to handle KL_{nuc}. In Proposition 5.9, we can additionally insist that ϕ is weakly nuclear, replace absorption by nuclear absorption in (i), and demand that ψ is weakly nuclear in (ii) and (iii).

With the observations above, the proof then works verbatim as the *-homomorphisms ψ_∞ and $(\psi_\infty)_\infty$ in (5.22) and (5.23) are both weakly nuclear. Accordingly, we obtain versions of Corollaries 5.10 and 5.11 for weakly nuclear, nuclearly absorbing maps. This enables us to commence following the proof of Theorem 5.15(i), working with KK_{nuc}-groups in place of KK-groups. In the second paragraph of the proof of Theorem 5.15, the calculation $[\phi, \mathrm{Ad}(u_0)\phi]_{KK_{\mathrm{nuc}}(A,I)} = [\phi, \psi]_{KK_{\mathrm{nuc}}(A,I)} = 0$ follows using the same argument since the homotopies in [62, Lemma 3.1] are pointwise weakly nuclear when ϕ and ψ are weakly nuclear. In addition, [62, Lemma 3.5] explicitly allows the map θ to be taken weakly nuclear. In the third paragraph, we can use nuclear absorption because $\theta|_A$ and ϕ are both weakly nuclear, and the rest of that paragraph goes through verbatim to make use of Lemma 5.16 and (5.37), proving Theorem 5.20(i). Theorem 5.20(ii) likewise follows the proof of Theorem 5.15(ii), replacing Theorem 5.14(i) with [253, Proposition 2.6].

6. The trace-kernel extension

We recall the trace-kernel extension in Section 6.1 and describe the background behind Theorem 1.3 in Section 6.2. In Section 6.3, we use this to compute the K-theory of the trace-kernel quotient (under the codomain hypotheses of Theorem A) and prove Theorem 1.5. We end with Section 6.4, in which we show that the trace-kernel ideal of a simple exact $\mathcal{Z}$-stable C^*-algebra is separably stable.

6.1. **The trace-kernel extension.** There is a long history of examining a C^*-algebra by transferring information from a suitable von Neumann algebraic closure. This technique gained particular prominence over the last decade, following Matui and Sato's breakthrough work ([200, 201]) on the Toms–Winter conjecture. Working primarily in the setting of simple nuclear C^*-algebras with finitely many extremal traces, they showed how to relate von Neumann and C^*-algebraic central sequence algebras, bringing to the forefront what we now call the trace-kernel ideal J_B in the sequence algebra B_∞. Later, the trace-kernel ideal was used to analyze the sequence algebra itself (along with its ultrapower), as opposed to the central sequence algebra. The papers [252, 253] demonstrate its power in extension and KK-theoretic arguments.

Let B be a C^*-algebra. The trace-kernel ideal in the sequence algebra B_∞ is obtained as the intersection of all the ideals associated to limit traces on B_∞ (recall that the collection of limit traces on B_∞ is denoted $T_\infty(B)$; see Definition 4.17). It appears implicitly in [200] and explicitly (in an ultrapower formulation) in [167] (which develops many of the fundamental properties of the trace-kernel ideal, particularly vis-à-vis central sequences) and [201].

Definition 6.1. Let B be a unital separable C^*-algebra satisfying $T(B) \neq \emptyset$. The *trace-kernel ideal* of B is

$$J_B := \{b \in B_\infty : \tau(b^*b) = 0 \text{ for all } \tau \in T_\infty(B)\}, \tag{6.1}$$

and the *trace-kernel extension* is the short exact sequence

$$0 \longrightarrow J_B \xrightarrow{j_B} B_\infty \xrightarrow{q_B} B^\infty \longrightarrow 0, \tag{6.2}$$

where j_B is the canonical inclusion, $B^\infty := B_\infty/J_B$, called the *trace-kernel quotient*, and q_B is the quotient map.

Note that given $x = (x_n)_{n=1}^\infty \in B_\infty$, we have $x \in J_B$ if and only if

$$\lim_{n\to\infty} \sup_{\tau\in T(B)} \tau(x_n^* x_n) = 0. \tag{6.3}$$

It is sometimes useful to view B^∞ as the quotient of $\ell^\infty(B)$ by the ideal of sequences $(x_n)_{n=1}^\infty$ satisfying (6.3). From this perspective, it is natural to use representative sequences in $\ell^\infty(B)$ to denote elements of B^∞.

Note too that the limit traces descend from B_∞ to give traces on B^∞; we will use the notation $T_\infty(B)$ in both cases. For a positive element $x \in B^\infty$, if $\tau(x) = 0$ for all $\tau \in T_\infty(B)$, then $x = 0$.

Remark 6.2. A unital *-homomorphism $\phi\colon B \to C$ induces a map $\phi_\infty\colon B_\infty \to C_\infty$. As ϕ is unital, any trace $\tau \in T_\infty(C) \subseteq T(C_\infty)$ induces a trace $\phi_\infty^*(\tau) \in T_\infty(B)$. Therefore, $\phi_\infty(J_B) \subseteq J_C$, and hence ϕ also induces a map $\phi^\infty\colon B^\infty \to C^\infty$. In this way, the assignment of the trace-kernel extension e_B to B is functorial for unital C^*-algebras with unital *-homomorphisms.

As recorded in Proposition 4.18, convex combinations of the limit traces are dense in $T(B_\infty)$ when B is $\mathcal{Z}$-stable and exact. When this holds, it descends to the trace-kernel quotient B^∞.[123]

Proposition 6.3. *Let B be a unital simple separable exact $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$.*

(i) *The map $T(q_B)\colon T(B^\infty) \to T(B_\infty)$ induced by the quotient map $q_B\colon B_\infty \to B^\infty$ is an affine homeomorphism. In particular, the convex hull of the limit traces $T_\infty(B)$ is weak*-dense in $T(B^\infty)$.*
(ii) *B^∞ has strict comparison of positive elements by bounded traces.*
(iii) *Let A be a C^*-algebra. A *-homomorphism $\theta : A \to B^\infty$ is full if and only if $\tau\circ\theta$ is faithful for all traces $\tau \in T(B^\infty)$.*

Proof. For (i), the affine map $T(q_B)$ is a continuous injection (because q_B is surjective). It is surjective as well — and therefore a homeomorphism — as its image is compact, convex, and contains the limit traces on B_∞, which have dense convex hull in $T(B_\infty)$ by Proposition 4.18.

Part (ii) follows since if $a, b \in M_k(B^\infty)$ are positive elements with $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(B^\infty)$, then taking positive lifts $\tilde a$ and $\tilde b$ of a and b in $M_k(B_\infty)$, the previous part gives $d_\tau(\tilde a) < d_\tau(\tilde b)$ for all $\tau \in T(B_\infty)$. As B_∞ has strict comparison of positive elements by bounded traces (Proposition 4.19(ii)), $\tilde a \precsim \tilde b$, and hence $a \precsim b$.

Part (iii) is an immediate consequence of (ii) and Lemma 4.12. □

We will need to work with matrix amplifications of the trace-kernel extension in Sections 6.3 and 7.4. The following observation is implicitly used (in the setting of ultrapowers) in [253, Section 4]. It boils down to the fact that $\tau \mapsto \tau_{M_k} \otimes \tau$ gives an affine homeomorphism $T(B) \to T(M_k \otimes B)$.

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[123] While we prove this using Proposition 4.18 here, it is possible for the convex combinations of the limit traces to be dense in $T(B^\infty)$ even when they are not dense in $T(B_\infty)$; see the discussion before [32, Proposition 2.5], which shows how to use the technology of complemented partitions of unity discussed in the next section to obtain results like Proposition 6.3(i) without passing through Proposition 4.18.

Remark 6.4. For a C^*-algebra D, write $\iota_D^{(k)}\colon D \to M_k(D)$ for the top-left corner embedding of D into $M_k(D)$. If B is a unital C^*-algebra and $T(B) \neq \emptyset$, then the following diagram commutes, with natural maps forming the isomorphism of extensions between the bottom two rows:

$$\begin{tikzcd}
0 \arrow[r] & J_B \arrow[r, "j_B"] \arrow[d, "\iota_{J_B}^{(k)}"] & B_\infty \arrow[r, "q_B"] \arrow[d, "\iota_{B_\infty}^{(k)}"] & B^\infty \arrow[r] \arrow[d, "\iota_{B^\infty}^{(k)}"] & 0 \\
0 \arrow[r] & M_k(J_B) \arrow[r] \arrow[d, "\cong"] & M_k(B_\infty) \arrow[r] \arrow[d, "\cong"] & M_k(B^\infty) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & J_{M_k(B)} \arrow[r, "j_{M_k(B)}"] & M_k(B)_\infty \arrow[r, "q_{M_k(B)}"] & M_k(B)^\infty \arrow[r] & 0.
\end{tikzcd} \tag{6.4}$$

6.2. **Classification into the trace-kernel quotient.** The intuition behind the classification of maps into the trace-kernel quotient is most readily seen with the ultrapower version of the trace-kernel extension $0 \to J_{B,\omega} \to B_\omega \to B^\omega \to 0$, for some free ultrafilter ω on $\mathbb{N}$. When B has a unique trace τ, the ultrapower trace-kernel quotient B^ω is the tracial von Neumann algebra ultrapower $(\pi_\tau(B)'', \tau)^\omega$, where π_τ is the GNS-representation associated to τ.[124] It is a well-known consequence of Connes' equivalence of injectivity and hyperfiniteness ([46]) that maps from separable nuclear C^*-algebras into type II_1 von Neumann algebras are classified up to strong*-approximate unitary equivalence by traces.[125] When A is separable and nuclear and B has a unique trace, it follows (using Kirchberg's ϵ-test) that maps from $A \to B^\omega$ are determined up to unitary equivalence by their trace.

In general, B^ω will not be a von Neumann algebra, but under the right conditions, classification of maps into B^ω (or B^∞) is possible by a tracial gluing technique. Fix a nuclear C^*-algebra A and maps $\phi, \psi\colon A \to B^\omega$ that agree on traces. For each $\tau \in T_\omega(B)$, Connes' theorem provides unitaries $u_\tau \in B^\omega$ that approximately conjugate ϕ onto ψ pointwise in the $\|\cdot\|_{2,\tau}$-norm. To prove that ϕ and ψ are unitarily equivalent, one must find a way to use nuclearity and $\mathcal{Z}$-stability of B to combine the u_τ's into a single unitary $u \in B^\omega$ that approximately conjugate ϕ onto ψ pointwise in the uniform trace norm.[126]

Techniques for solving this problem were developed in stages. When $T(B)$ is a Bauer simplex,[127] one can view B^ω as an ultrapower of a continuous W^*-bundle over this extreme boundary (see [21, Chapter 3]) — a notion introduced by Ozawa in [213]. When B is nuclear and $\mathcal{Z}$-stable, Ozawa's trivialization theorem ([213, Theorem 15]) provides suitable approximately central partitions of unity, which can be used to combine the u_τ above into a single u (see [21, Proposition 3.23]). Outside the Bauer simplex setting, the situation is more complicated. Here, one no

[124]The ultrapower trace-kernel quotient J_B is formed in a similar way by using limits along the ultrafilter instead of limits along $\mathbb{N}$. The isomorphism $B^\omega \cong (\pi_\tau(B)'', \tau)^\omega$ is an easy application of Kaplansky's density theorem. See [167], for example.

[125]This is most often stated when the codomain is a II_1 factor, or has separable predual ([69, Corollary 10, Theorem 5], for example); a more general statement is [39, Proposition 2.1], though note that the condition of countable decomposability there is not needed.

[126]That is, for a finite subset $\mathcal{F}$ of A and $\epsilon > 0$, the unitary u should satisfy

$$\max_{x \in \mathcal{F}} \sup_{\tau \in T_\omega(B)} \|u\phi(x)u^* - \psi(x)\|_{2,\tau} < \epsilon.$$

[127]$T(B)$ is a Bauer simplex if it is non-empty and the extreme boundary $\partial_e T(B)$ is compact.

longer has a W^*-bundle structure and must build partitions of unity in B^ω that take into account the affine structure of $T(B)$. A general technique for carrying this out using the nuclearity and $\mathcal{Z}$-stability of B was developed in [35], for use in the Toms–Winter conjecture. (See the second half of the introduction to [35] for more details.)

The exact form of the classification of maps by traces that we need was set out in the short sequel [32] to [35], which also handles the conversion from ultrapowers to the sequence algebra setup of this paper. Note that [32] contains a small error that is corrected in [34].

Theorem 6.5 (Classification of maps into B^∞ by traces; [32, Theorem A])**.** *Let A be a unital separable nuclear C^*-algebra and let B be a unital separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. For every positive unital linear map $\gamma\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B^\infty)$, there is a unital *-homomorphism $\theta\colon A \to B^\infty$ satisfying $\operatorname{Aff} T(\theta) = \gamma$. Moreover, θ is unique up to unitary equivalence.*

Proof. By Kadison duality (see the second paragraph of Section 2.1), there is a natural bijection between positive unital linear functions $\operatorname{Aff} T(A) \to \operatorname{Aff} T(B^\infty)$ and continuous affine functions $T(B^\infty) \to T(A)$. The existence and uniqueness of a not-necessarily-unital map θ follow from [32, Theorem A]. Further, if $\theta\colon A \to B^\infty$ is a *-homomorphism with $\tau \circ \theta \in T(A)$ for all $\tau \in T(B^\infty)$, (i.e., $\tau \circ \theta$ is a tracial state not just a tracial functional) then $1_{B_\infty} - \theta(1_A)$ is a positive element of B^∞ which vanishes on all traces on B^∞. Since a general positive element of B^∞ that vanishes on all traces is automatically zero, this implies θ is unital. □

6.3. **K-theory of the trace-kernel ideal and quotient.** We now turn to the computation of $K_1(J_B)$ when B is an allowed codomain in Theorem B. Firstly, the gluing procedures used to obtain the classification of maps into B^∞ also show that every unitary in B^∞ is an exponential.

Proposition 6.6 ([32, Proposition 2.1])**.** *Let B be a unital separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. If $u \in B^\infty$ is a unitary, then $u = e^{ih}$ for some self-adjoint $h \in B^\infty$ of norm at most π. In particular, every unitary in B^∞ is the image of a unitary in B_∞ under q_B.*

Proof. The first statement is a special case of the cited reference (taking $S :=$ $\{1_{B^\infty}\}$) and the lifting statement is then an immediate consequence. □

Combing Proposition 6.6 with the classification of maps into B^∞ by traces gives the following K-theory of B^∞. This is analogous to the fact that the K-theory of a II_1 factor is $(\mathbb{R}, 0)$.

Proposition 6.7. *Let B be a unital separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Then $(K_0(B^\infty), K_1(B^\infty)) \cong (\operatorname{Aff} T(B^\infty), 0)$, with the pairing map ρ_{B^∞} inducing the isomorphism in the first entry, and $K_*(B^\infty; \mathbb{Z}/n) = 0$ for all $n \geq 2$.*

Proof. The computation of $K_0(B^\infty)$ is a consequence of Theorem 6.5 with $A := \mathbb{C}^2$, as follows. Given $f \in \operatorname{Aff} T(B^\infty)_+$, fix $n \in \mathbb{N}$ such that $f \leq n$, which exists by the compactness of $T(B^\infty)$. Let $\tilde{f} \in \operatorname{Aff} T(M_n \otimes B^\infty)$ be given by $\tilde{f}(\tau_{M_n} \otimes \tau) := f(\tau)$ for $\tau \in T(B^\infty)$. Define $\gamma\colon \mathbb{R} \oplus \mathbb{R} \to \operatorname{Aff} T(M_n \otimes B^\infty)$ by $\gamma(s,t) := \frac{s}{n}\tilde{f} + t(1 - \frac{1}{n}\tilde{f})$. Then existence in Theorem 6.5 (with $M_n \otimes B$ in place of B, see Remark 4.7) gives us a unital *-homomorphism $\phi\colon \mathbb{C} \oplus \mathbb{C} \to M_n \otimes B^\infty$ realizing γ, and thus a projection $p := \phi(1,0) \in (M_n \otimes B)^\infty \cong M_n(B^\infty)$ (see Remark 6.4) such that

$(\tau_{M_n} \otimes \tau)(p) = \frac{1}{n} f(\tau)$ for all $\tau \in T(B^\infty)$. This means that $\rho_{B^\infty}([p]_0) = f$. Moreover, uniqueness in Theorem 6.5 tells us that any projection $q \in M_n \otimes B^\infty$ with the same property is unitarily equivalent to p, and thus $[p]_0 = [q]_0$ in $K_0(B^\infty)$. Taking linear combinations shows that the pairing map induces an isomorphism $K_0(B^\infty) \cong \operatorname{Aff} T(B^\infty)$.

Proposition 6.6, applied to $M_n \otimes B^\infty \cong (M_n \otimes B)^\infty$ (via Remarks 6.4 and 4.7), implies $K_1(B^\infty) = 0$. For the last statement, the map $K_0(B^\infty) \to K_0(B^\infty)$ of multiplication by n is an isomorphism for all $n \geq 2$ (as $K_0(B^\infty)$ is a real vector space). Using this, and $K_1(B^\infty) = 0$, in the exact sequence (2.29) implies $K_*(B^\infty; \mathbb{Z}/n) = 0$ for all $n \geq 2$. □

Now we turn both to the proof of Theorem 1.5 and the setup for Theorem 1.6. Note that when B is unital, simple, separable, exact, and $\mathcal{Z}$-stable, we have that $\operatorname{Aff} T(q_B)\colon \operatorname{Aff} T(B_\infty) \to \operatorname{Aff} T(B^\infty)$ is an isomorphism by Proposition 6.3(i). In the following diagram, the first row is an extract of the six-term exact sequence for the trace-kernel extension, while the second row is exact as it is Thomsen's extension for B_∞ (Proposition 4.20). Commutativity of the left-hand square is naturality of the pairing map.

(6.5)
$$\begin{tikzcd}
K_0(B_\infty) \arrow[r, "K_0(q_B)"] \arrow[d, equal] & K_0(B^\infty) \arrow[r, "\partial"] \arrow[d, "(\operatorname{Aff} T(q_B))^{-1} \circ \rho_{B^\infty}"] & K_1(J_B) \arrow[r, "K_1(j_B)"] \arrow[d, dashed, "\omega_B"] & K_1(B_\infty) \arrow[r, "K_1(q_B)"] \arrow[d, equal] & K_1(B^\infty) \arrow[d] \\
K_0(B_\infty) \arrow[r, "\rho_{B_\infty}"] & \operatorname{Aff} T(B_\infty) \arrow[r, "\mathrm{Th}_{B_\infty}"] & \overline{K}_1^{\mathrm{alg}}(B_\infty) \arrow[r, "\not{a}_{B_\infty}"] & K_1(B_\infty) \arrow[r] & 0
\end{tikzcd}$$

A diagram chase gives a natural map

$$\omega'_B\colon \operatorname{im} \partial \to \operatorname{im} \mathrm{Th}_{B_\infty} = \ker \not{a}_{B_\infty} \colon \partial([p]_0) \mapsto [e^{2\pi i a}]_{\mathrm{alg}}, \tag{6.6}$$

where $a \in M_k(B_\infty)$ is a self-adjoint lifting the projection $p \in M_k(B^\infty)$. For our calculation of $KL(A, J_B)$ in Theorem 8.9 (the precise version of Theorem 1.6), we need to know that ω'_B is an isomorphism and describe it explicitly in terms of elements of $\ker K_1(j_B) = \operatorname{im} \partial$. The explicit description we give extends to a natural map $\omega_B\colon K_1(J_B) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$. Moreover, both this and ω'_B are isomorphisms when B^∞ has the K-theory provided by Proposition 6.7.

Theorem 6.8 (Calculating $K_1(J_B)$)**.** *Let B be a unital separable exact $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$.*

(i) There is a natural map $\omega_B\colon K_1(J_B) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$ such that (6.5) commutes, given explicitly by

$$\omega_B([u]_1) := [u]_{\mathrm{alg}} - [s(u)]_{\mathrm{alg}}, \quad u \in U_\infty(J_B^\dagger), \tag{6.7}$$

where $s\colon J_B^\dagger \to \mathbb{C}1_{B_\infty} \subseteq B_\infty$ is the canonical character. This map extends the natural map ω'_B from (6.6)*.*

(ii) When B is also simple and nuclear, ω_B and ω'_B are isomorphisms.

Proof. (i): The map $\tilde{\omega}_B\colon U_\infty(J_B^\dagger) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$ given by $\tilde{\omega}_B(u) := [u]_{\mathrm{alg}} - [s(u)]_{\mathrm{alg}}$ for $u \in U_\infty(J_B^\dagger)$ is a group homomorphism. To show that this induces a well-defined map ω_B on $K_1(J_B)$, it suffices to check that $\tilde{\omega}_B(e^{2\pi i h}) = 0$ in $\overline{K}_1^{\mathrm{alg}}(B_\infty)$ for any self-adjoint $h \in M_n(J_B^\dagger)$. To this end, write $h = h_0 + s(h)$, where $h_0 = h_0^* \in M_n(J_B)$, so that $\tilde{\omega}_B(e^{2\pi i h}) = [e^{2\pi i h_0}]_{\mathrm{alg}} \in \ker \not\Delta_{B_\infty}$. Using exactness and $\mathcal{Z}$-stability of B, all traces on B_∞ vanish on J_B (Proposition 4.18), and so $\hat{h}_0 = 0$ in $\mathrm{Aff}\, T(B_\infty)$. Applying the Thomsen map from (2.15), $\tilde{\omega}_B(e^{2\pi i h}) = \mathrm{Th}_{B_\infty}(\hat{h}_0) = 0$, as required. The explicit formula (6.7) confirms that ω_B is natural in B.

By construction, the induced map ω_B makes the third square of (6.5) commute. For the second square, consider a projection $p \in M_n(B^\infty)$, which we lift to to a positive contraction $a \in M_n(B_\infty)$. This satisfies $\mathrm{Aff}\, T(q_B)(\hat{a}) = \hat{p}$, $e^{2\pi i a} \in U_n(J_B^\dagger)$ and $\partial([p]_0) = [e^{2\pi i a}]_1$. Noting that $s(e^{2\pi i a}) = 1_{M_n(B_\infty)}$, we have $(\omega_B \circ \partial)([p]_0) = [e^{2\pi i a}]_{\mathrm{alg}} = \mathrm{Th}_{B_\infty}(\hat{a})$ (from the definition of ω_B and that of the Thomsen map in (2.15)). As $(\mathrm{Aff}\, T(q_B)^{-1} \circ \rho_{B_\infty})([p]_0) = \mathrm{Aff}\, T(q_B)^{-1}(\hat{p}) = \hat{a}$, the commutativity follows. Note that commutativity of the second square shows that ω_B extends ω_B' from (6.6)

(ii): Proposition 6.3(i) ensures that $\mathrm{Aff}\, T(q_B)$ is an isomorphism and, using the nuclearity hypothesis, the K-theory computation of Proposition 6.7 shows that ρ_{B_∞} is an isomorphism and $K_1(B^\infty) = 0$. The second row of (6.5) is exact by Proposition 4.20, and as the four outermost vertical arrows of (6.5) are isomorphisms, the result follows from the five lemma. □

6.4. **Separable stability of the trace-kernel ideal J_B.** In order to work with $KK(A, J_B)$ in Sections 7 and 9, it would be most convenient if the trace-kernel ideal J_B was stable. However, provided $T(B) \neq \emptyset$ and $J_B \neq 0$, this is never the case.[128] This is similar to B_∞ not being $\mathcal{Z}$-stable (see Section 4.4), and the solution is the same: to separabilize (pass to separable subalgebras with the desired property).

Definition 6.9 (cf. [253, Definition 1.4])**.** A C^*-algebra D is *separably stable* if for every separable C^*-subalgebra $D_0 \subseteq D$, there exists a stable separable C^*-subalgebra $E \subseteq D$ with $D_0 \subseteq E$.

Our objective is to show in Lemma 6.11 that J_B is separably stable whenever B satisfies the codomain assumptions in the classification of embeddings. A version of this lemma was obtained in [253, Propositions 3.2 and 3.3] (in the ultrapower setting) for unital simple separable C^*-algebras B with a unique trace, which is also the unique quasitrace, and that are also stable under tensoring by the universal UHF algebra and have trivial K_1-group. These assumptions ensure that B_ω is real rank zero (as used in [253]), while in our setting of multiple traces, and working with $\mathcal{Z}$-stability, we must use Cuntz semigroup techniques rather than projections. To do this, we note that Hjelmborg and Rørdam's characterization of stability for σ-unital C^*-algebras via positive elements extends to characterize separable stability. We say that a C^*-algebra D satisfies the *Hjelmborg–Rørdam criterion* if, for every $a \in D_+$ and $\epsilon > 0$, there exists $x \in D$ satisfying $\|a - xx^*\| < \epsilon$ and $\|(x^*x)(xx^*)\| < \epsilon$. Section 2 of [141] shows that a σ-unital C^*-algebra D is stable if and only if it satisfies the Hjelmborg–Rørdam criterion.[129]

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[128] Since J_B is an SAW^*-algebra, this is a consequence of Ghasemi's result ([116]). To prove that J_B is indeed an SAW^*-algebra, given orthogonal positive contractions $e, f \in J_B$, one uses an ϵ-test argument (similar to [167, Proposition 4.6], which proves that J_B is a σ-ideal) with functional calculus on e to produce a positive contraction $h \in J_B$ such that $he = e$ and $hf = 0$.

[129] Note that what we call the Hjelmborg–Rørdam criterion is equivalent to condition [141, Proposition 2.2(b)] since the set $F(D)$ of [141] is dense in D_+ (as noted on [141, Page 154]).

Proposition 6.10. *Let D be a C^*-algebra. Then D is separably stable if and only if it satisfies the Hjelmborg–Rørdam criterion.*

Proof. It is immediate that separable stability implies the Hjelmborg–Rørdam criterion. To see the converse, it suffices to show that if D is a C^*-algebra satisfying the Hjelmborg–Rørdam criterion and D_0 is a separable subalgebra of D, then there exists a separable subalgebra $E \subseteq D$ satisfying the Hjelmborg–Rørdam criterion with $D_0 \subseteq E$. Fix a dense sequence $(d_l)_{l=1}^\infty$ in $(D_0)_+$. By the Hjelmborg–Rørdam criterion, for each $k, l \in \mathbb{N}$, there exists $x_{k,l} \in D$ such that $\|d_l - x_{k,l}^* x_{k,l}\| < 1/k$ and $\|(x_{k,l}^* x_{k,l})(x_{k,l} x_{k,l}^*)\| < 1/k$. Let $D_1 \coloneqq C^*(D_0, \{x_{k,l} : k, l \in \mathbb{N}\})$. This ensures that for any $a \in (D_0)_+$, and $\epsilon > 0$, there exists $x \in D_1$ with $\|a - x^*x\| < \epsilon$ and $\|(x^*x)(xx^*)\| < \epsilon$. Continuing in this way, we obtain a nested sequence $D_0 \subseteq D_1 \subseteq \cdots$ of separable C^*-subalgebras of D so that the Hjelmborg–Rørdam criterion for positive elements a in D_n can be witnessed by elements $x \in D_{n+1}$. Then $E \coloneqq \overline{\bigcup_{n=1}^\infty D_n}$ will satisfy the Hjelmborg–Rørdam criterion. □

With the Hjelmborg–Rørdam criterion in place, we now deduce separable stability of the relevant trace-kernel ideals.[130]

Lemma 6.11. *Let B be a unital simple separable exact $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Then J_B is separably stable.*

Proof. We check that J_B satisfies the Hjelmborg–Rørdam criterion of Proposition 6.10. Let $a \in (J_B)_+$ and $\epsilon > 0$. By perturbing a slightly and using functional calculus, we may assume that there is a contraction $e \in (J_B)_+$ such that $ea = a$. Set $b \coloneqq 1_{B_\infty} - e \in (B_\infty)_+$, and note that $ab = 0$. Since $e \in J_B$, we have $\tau(b) = 1$ for all $\tau \in T_\infty(B)$, and therefore, for all $\tau \in T(B_\infty)$ by density of the convex hull of the limit traces in $T(B_\infty)$ (Proposition 4.18). Hence $d_\tau(b) \geq 1$ for all $\tau \in T(B_\infty)$. On the other hand (again using Proposition 4.18), $d_\tau(a) = \lim_{r\to\infty} \tau(a^{1/r}) = 0$ for all $\tau \in T(B_\infty)$, since $a^{1/r} \in J_B$ for all r. As B_∞ has strict comparison of positive elements by bounded traces (Proposition 4.19(ii)), it follows that $a \precsim b$. By [165, Proposition 2.7(iii)] there exists $x \in B_\infty$ such that $x^*x = (a-\epsilon)_+$ and $xx^* \in \overline{bB_\infty b}$. Because $x^*x = (a-\epsilon)_+ \in J_B$, we get $x \in J_B$ and $\|x^*x - a\| < \epsilon$. Also, since $xx^* \in \overline{bB_\infty b}$ and $ab = 0$, we have $xx^*a = 0$, so $(xx^*)(x^*x) = 0$. □

Remark 6.12. As mentioned in the introduction, it is natural to consider the extension associated with the finite part of the bidual B^{**}_{fin}. However, when B has infinitely many extreme traces, the kernel of the surjection $B_\omega \to (B^{**}_{\mathrm{fin}})^\omega$ is not separably stable. To see this, we can take an element $b = (b_n)_{n=1}^\infty \in (B_\omega)_+$ such that $\|b_n\|_{2,\tau} \to 0$ for all τ yet $\sup_{\tau\in T(B)} \|b_n\|_{2,\tau} \to 1$ (i.e., the convergence is pointwise but not uniform). This means that the image of b in $(B^{**}_{\mathrm{fin}})^\omega$ is zero, but the trace condition makes it impossible to find $x^*x \in B_\omega$ close to b with $\|(xx^*)(x^*x)\|$ small.

7. Classifying lifts along the trace-kernel extension

The main objective of this section is to classify lifts from the trace-kernel quotient in terms of $KL(A, J_B)$. There are two versions of this result. We first establish

The equivalence of the Hjelmborg–Rørdam criterion and stability follows immediately from [141, Theorem 2.1 and Proposition 2.2].

[130] While we only need the result for nuclear B, we state it for B exact — the exactness only being used to apply Haagerup's result that quasitraces are traces ([131]).

Theorem 7.1, which classifies lifts of unitizably full maps $\theta\colon A \to B^\infty$, and then we follow a "de-unitization" procedure to convert this result into the classification of unital lifts (Theorem 7.10, which is the precise formulation of Theorem 1.2).

The classification of lifts has a long history, dating back to the theory of extensions of C^*-algebras developed by Brown, Douglas, and Fillmore ([27]). Our results build on more recent developments by CS ([252, 253]); see Remarks 7.3 and 7.4.

7.1. **Classifying lifts of unitizably full maps.** In the following theorem, we follow the definitions of KK- and KL-groups for non-σ-unital codomain C^*-algebras (such as J_B) from Section 5.1. We defer the proof to Section 7.3, allowing us to first address preliminaries.

Theorem 7.1 (Classification of lifts). *Let A be a separable nuclear C^*-algebra, let B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$, and suppose $\theta\colon A \to B^\infty$ is a unitizably full *-homomorphism.*

(i) For any $\kappa \in KK(A, B_\infty)$ with $KK(A, q_B)(\kappa) = [\theta]_{KK(A,B^\infty)}$, there exists a *-homomorphism $\phi\colon A \to B_\infty$ that lifts θ and such that $[\phi]_{KK(A,B_\infty)} = \kappa$.

(ii) Given a *-homomorphism $\psi\colon A \to B_\infty$ that lifts θ and any $\lambda \in KL(A, J_B)$, there exists a *-homomorphism $\phi\colon A \to B_\infty$ that also lifts θ and such that $[\phi, \psi]_{KL(A,J_B)} = \lambda$.

(iii) If ϕ_1 and ϕ_2 are *-homomorphisms that lift θ and satisfy $[\phi_1, \phi_2]_{KL(A,J_B)} = 0$, then they are unitarily equivalent. In particular, the map ϕ in (ii) is unique up to unitary equivalence.

The data from Theorem 7.1 is summarized in the following diagram: given θ and a KK-class κ we can lift θ to a map $A \to B_\infty$. Further, for a fixed lift ψ of θ, the lifts ϕ are classified up to unitary equivalence by the class $[\phi, \psi]_{KL(A,J_B)}$.

$$\begin{tikzcd}
 & & & A \arrow[dll, dashed, bend right, "\lambda=[\phi,\psi]_{KL}"'] \arrow[dl, dashed, bend right, "\kappa"'] \arrow[dl, "\phi,\psi"] \arrow[d, "\theta \text{ unitizably full}"] & \\
0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r] & B^\infty \arrow[r] & 0
\end{tikzcd} \tag{7.1}$$

Remark 7.2. It is important in Theorem 7.1(i) that we are given a KK-lift κ of θ. This is a genuine assumption — for example, it certainly fails when A has a non-trivial projection and B does not. When A satisfies the UCT, the existence of a KK-lift of θ is equivalent to the existence of K_0-lift of θ — that is, a map $\alpha_0\colon K_0(A) \to K_0(B_\infty)$ such that $K_0(q_B) \circ \alpha_0 = K_0(\theta)$ (by Proposition 8.6 and surjectivity of $\Gamma^{(A,B_\infty)}\colon KK(A, B_\infty) \to \mathrm{Hom}(K_*(A), K_*(B)))$. When we apply Theorem 7.1(i) in the classification of full approximate embeddings, we will use the UCT to produce a KK-lift of θ compatible with a specified morphism in total K-theory.

Remark 7.3. CS's proof in [252] of the quasidiagonality theorem of [269] (that every faithful trace on a separable nuclear C^*-algebra satisfying the UCT is quasidiagonal) can be viewed as a special case of Theorem 7.1(i).[131] Indeed, a standard computation (see Section 3 of [252], for example) shows that $K_0(q_{\mathcal{Q}})\colon K_0(\mathcal{Q}_\infty) \to K_0(\mathcal{Q}^\infty)$ is a surjective linear map of rational vector spaces and hence splits. Now, if A is a nuclear C^*-algebra with a faithful trace, one obtains a unitizably full *-homomorphism $\theta\colon A \to \mathcal{Q}^\infty$ from the amenability of this trace. This necessarily

[131] [252] works with ultrapowers rather than the sequence algebras we use here. This is more natural in the unique trace setting because the tracial ultrapower is a von Neumann algebra.

admits a K_0-lift since $K_0(q_{\mathcal{Q}})$ is a split surjection, so when A satisfies the UCT, θ admits a KK-lift as well by Remark 7.2. Now, Theorem 7.1(i) implies there is an embedding $A \hookrightarrow \mathcal{Q}_\infty$, which allows access to Voiculescu's local characterization of quasidiagonality ([284, Theorem 1]) (since A is nuclear, and so this map has a c.p.c. lift to $\ell^\infty(\mathcal{Q})$ by the Choi–Effros lifting theorem).

Remark 7.4. CS implicitly establishes versions of Theorem 7.1(ii) and (iii) in the proofs of [253, Propositions 4.2 and 4.3], in the case that B has a unique (quasi)-trace and is $\mathcal{Q}$-stable (but is not necessarily nuclear). These results, and CS's approach to the quasidiagonality theorem in [252], allow for exact domains with appropriate amenability conditions on the trace induced by θ and on ϕ and ψ (in [252], this recaptures the more general version of the quasidiagonality theorem from [112]). In comparison, Theorem 7.1 relaxes $\mathcal{Q}$-stability to $\mathcal{Z}$-stability and allows for general trace simplices but only covers nuclear domains and codomains. The additional machinery to allow for exact domains and general $\mathcal{Z}$-stable codomains (whose quasitraces are traces) will be found in the second paper of this series.

Remark 7.5. The uniqueness in Theorem 7.1(iii) means that, given ψ as in (ii), any two lifts $\phi_1, \phi_2 \colon A \to B_\infty$ of θ with $[\phi_1, \psi]_{KL(A,J_B)} = [\phi_2, \psi]_{KL(A,J_B)}$ are unitarily equivalent. In order to view this as a classification of lifts up to unitary equivalence by $KL(A, J_B)$, we should check that $[\,\cdot\,, \psi]_{KK(A,J_B)}$ (and hence $[\,\cdot\,, \psi]_{KL(A,J_B)}$) is invariant under unitary equivalence: a priori, it is only invariant under conjugation by unitaries in $J_B^\dagger$ (see Proposition 5.4). Given $\phi_1 \colon A \to B_\infty$ lifting θ and a unitary $u \in B_\infty$, set $\phi_2 \coloneqq \mathrm{Ad}(u) \circ \phi_1$. By [32, Proposition 2.1], the unitary group of $B^\infty \cap q_B(\phi_1(A))'$ is path-connected, and hence Lemma 5.16(i) implies that there is a continuous path of unitaries $(v_t)_{t\geq 0} \subseteq J_B^\dagger$ such that $\mathrm{Ad}(v_t) \circ \phi_1 \to \phi_2$ point-norm (taking the continuous path $(u_t)_{t\geq 0}$ of the lemma to be the constant path u). Then a standard application of Kirchberg's ϵ-test (Lemma 3.14) provides a unitary $v \in J_B^\dagger$ with $\mathrm{Ad}(v) \circ \phi_1 = \phi_2$. This implies $[\phi_1, \psi]_{KK(A,J_B)} = [\phi_2, \psi]_{KK(A,J_B)}$, as claimed.

Pullback arguments relating extensions to the standard multiplier-corona extension are repeatedly used in the proof of Theorem 7.1. We set out our notation for this below, which will be used freely throughout the rest of the section.

Notation 7.6. Given an extension $\mathsf{e} \colon 0 \to I \xrightarrow{j_\mathsf{e}} E \xrightarrow{q_\mathsf{e}} D \to 0$, we let $\mu_\mathsf{e} \colon E \to \mathcal{M}(I)$ be the canonical map of E into the multiplier algebra of I, and write $\bar{\mu}_\mathsf{e} \colon D \to \mathcal{Q}(I)$ for the Busby map of e. Recall that these maps fit into the following commuting diagram, in which the right-hand square is a pullback:

$$\begin{array}{ccccccccccc} \mathsf{e}\colon & 0 & \longrightarrow & I & \xrightarrow{j_\mathsf{e}} & E & \xrightarrow{q_\mathsf{e}} & D & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle \mu_\mathsf{e}} & & \downarrow{\scriptstyle \bar{\mu}_\mathsf{e}} & & \\ \bar{\mathsf{e}}\colon & 0 & \longrightarrow & I & \xrightarrow{\bar{j}_\mathsf{e}} & \mathcal{M}(I) & \xrightarrow{\bar{q}_\mathsf{e}} & \mathcal{Q}(I) & \longrightarrow & 0. \end{array} \tag{7.2}$$

That is, *-homomorphisms $\theta \colon A \to D$ and $\bar{\phi} \colon A \to \mathcal{M}(I)$ with $\bar{q}_\mathsf{e} \circ \bar{\phi} = \bar{\mu}_\mathsf{e} \circ \theta$ induce a unique *-homomorphism $\phi \colon A \to E$ with $\mu_\mathsf{e} \circ \phi = \bar{\phi}$ and $q_\mathsf{e} \circ \phi = \theta$. More concretely, $E \cong \{(x, d) \in \mathcal{M}(I) \oplus D : \bar{q}_\mathsf{e}(x) = \bar{\mu}_\mathsf{e}(d)\}$ via the isomorphism $e \mapsto (\mu_\mathsf{e}(e), q_\mathsf{e}(e))$.

Lemma 7.7. *Let A be a separable nuclear C^*-algebra and let*

$$\mathsf{e} \colon 0 \longrightarrow I \longrightarrow E \longrightarrow D \longrightarrow 0 \tag{7.3}$$

be a unital separable extension such that I is stable and E is $\mathcal{Z}$-stable. If $\theta\colon A \to D$ is a unitizably full *-homomorphism, then $\bar{\mu}_{\mathsf{e}} \circ \theta\colon A \to \mathcal{Q}(I)$ is an absorbing *-homomorphism, and if $\phi\colon A \to E$ is a *-homomorphism that lifts θ, then $\mu_{\mathsf{e}} \circ \phi\colon A \to \mathcal{M}(I)$ is an absorbing *-homomorphism.

Proof. Since $\bar{\mu}_{\mathsf{e}} \circ \theta$ is the composition of a unitizably full *-homomorphism followed by a unital *-homomorphism, it is unitizably full, so absorbing by Theorem 5.13. The second part follows from the first and Corollary 5.10. $\square$

7.2. **Separabilization.** Theorem 7.1 will be proved using the KK-existence and uniqueness results of Section 5. As such, it is vital to separabilize — pass to a separable subextension — so that one can obtain the absorption needed to apply these results. We do this in Lemma 7.8. The idea is to use the separable stability of J_B and separable $\mathcal{Z}$-stability of B_∞ to factorize the data in the diagram (7.1) through a unital separable subextension. This is illustrated in the diagram below, where E is $\mathcal{Z}$-stable and I is stable:[132] The proof of [253, Proposition 1.9], used below, employs separably inheritable properties (in the sense of Blackadar, [11, II.8.5]), and the fact that they are closed under countable intersections.

(7.4)
$$
\begin{tikzcd}
 & & & A \arrow[d, "\theta|^D"] \arrow[dl, shift left, "{\phi|^E,\psi|^E}"'] \arrow[dl, shift right] \arrow[dl, dashed, bend right, "\kappa'"'] \arrow[dll, dash dot, bend right, "{\lambda'=[\phi|^E,\psi|^E]_{KL}}"'] & \\
0 \arrow[r] & I \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, hook] & D \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r] & B^\infty \arrow[r] & 0.
\end{tikzcd}
$$

Lemma 7.8. *Let A, B, and θ be as in Theorem 7.1. There is a unital separable subextension*

$$\mathsf{e}\colon 0 \to I \xrightarrow{j_{\mathsf{e}}} E \xrightarrow{q_{\mathsf{e}}} D \to 0 \tag{7.5}$$

of the trace-kernel extension e_B such that θ corestricts to a unitizably full map $\theta|^D\colon A \to D$, E is unital and $\mathcal{Z}$-stable, and I is stable. Furthermore:

(i) Given $\kappa \in KK(A, B_\infty)$ such that $KK(A, q_B)(\kappa) = [\theta]_{KK(A,B^\infty)}$, e can be chosen so that κ factorizes as $KK(A, \iota_{E\subseteq B_\infty})(\kappa')$ with $\kappa' \in KK(A, E)$ and $KK(A, q_{\mathsf{e}})(\kappa') = [\theta|^D]_{KK(A,D)}$.

(ii) Given $\psi\colon A \to B_\infty$ and $\kappa \in KK(A, J_B)$, e can be chosen so that both $\psi(A) \subseteq E$ and κ factorizes as $KK(A, \iota_{I\subseteq J_B})(\kappa')$ for some $\kappa' \in KK(A, I)$.

(iii) Given two maps $\phi_1, \phi_2\colon A \to B_\infty$, e can be chosen so that $\phi_1(A) \cup \phi_2(A) \subseteq E$. Moreover, if each of ϕ_1 and ϕ_2 lift θ and $[\phi_1, \phi_2]_{KL(A,J_B)} = 0$ and $\phi_1|^E$ and $\phi_2|^E$ denote the corestrictions of ϕ_1 and ϕ_2 to E, then e can be chosen so that $\left[\phi_1|^E, \phi_2|^E\right]_{KL(A,I)} = 0$.

Proof. Note that if e is a unital separable subextension of e_B satisfying (i), (ii), or (iii), then the same holds for any larger subextension of e_B. Similarly, if e is such that $\theta|^D$ is unitizably full, then the same property is satisfied for any larger

[132] There are slight differences between the data in this diagram and in the lemma, as in order to correct the KK-class of a lift in the proof Theorem 7.1(i), we will need to factorize a KK-class rather than a KL-class using part (ii) of the lemma. To prove the uniqueness part of Theorem 7.1, we only need to factorize maps inducing the trivial KL-class, leading to part (iii) of the lemma.

subextension of e_B since $1_{B^\infty} \in D$. We start with any separable subextension $\mathsf{e}_0\colon 0 \to I_0 \to E_0 \to D_0 \to 0$ such that E_0 contains 1_{B_∞} and then enlarge the algebras while keeping them separable as follows.

Applying [253, Proposition 1.9] to the forced unitization $\theta^\dagger\colon A^\dagger \to B^\infty$, D_0 may be enlarged to arrange that θ corestricts to a unitizably full map $A \to D_0$. We make the corresponding enlargement of E_0 so that it contains lifts of a countable dense subset of D_0, and enlarge I_0 to $J_B \cap E_0$.

For conditions (i), (ii), and (iii), we use the characterizations of $KK(A,C)$ and $KL(A,C)$ as inductive limits of $KK(A,C_0)$ and $KL(A,C_0)$ over separable subalgebras C_0 of C (this is Definition 5.3 for KK and Proposition 5.6 for KL). In particular, given κ as in (i), considering the identity $KK(A,q_B)(\kappa) = [\theta]_{KK(A,B^\infty)}$, we can find suitable separable enlargements of E_0 and D_0 and $\kappa' \in KK(A,E_0)$ so that $q_B(E_0) = D_0$, $\kappa = KK(A,\iota_{E_0\subseteq B_\infty})(\kappa')$, and $KK(A,q_B|_{E_0})(\kappa') = [\theta|^{D_0}]_{KK(A,D_0)}$.

Given ψ and κ as in (ii), I_0 may be enlarged so that $\psi(A) \subseteq E_0$ and κ factorizes through I_0; this forces corresponding enlargements to E_0 and D_0. Likewise, given ϕ_1 and ϕ_2 as in (iii), we can enlarge E_0 (and hence also I_0 and D_0) so that $\phi_1(A) \cup \phi_2(A) \subseteq E_0$. As $[\phi_1,\phi_2]_{KL(A,J_B)} = 0$, by the characterization of KL as an inductive limit, a suitable separable enlargement of I_0 and E_0 satisfies $[\phi_1|^{E_0},\phi_2|^{E_0}]_{KL(A,I_0)} = 0$. We enlarge D_0 to the image of $q_B(E_0)$.

Finally, the $\mathcal{Z}$-stability hypothesis on B ensures that B_∞ is separably $\mathcal{Z}$-stable by Proposition 4.19(i), and, together with nuclearity, this hypothesis also gives that J_B is separably stable by Lemma 6.11. As both stability and $\mathcal{Z}$-stability are preserved under sequential inductive limits of separable C^*-algebras with injective connecting maps ([141, Corollary 4.1] and [274, Corollary 3.4]), we can enlarge $0 \to I_0 \to E_0 \to D_0 \to 0$ to e as in (7.5) in such a way that I is stable and E is $\mathcal{Z}$-stable by [253, Proposition 1.6]. □

7.3. **Proof of Theorem 7.1.** We start the proof of Theorem 7.1 with the following strengthening of part (ii) of that theorem, which modifies lifts to realize classes in $KK(A,J_B)$ (rather than just classes in $KL(A,J_B)$); we will also use this to correct the KK-class of a lift in the proof of part (i).

Proposition 7.9. *In the notation of Theorem 7.1, if $\psi\colon A \to B_\infty$ is a lift of θ and $\kappa \in KK(A,J_B)$, then there exists a *-homomorphism $\phi\colon A \to B_\infty$ that also lifts θ and such that $[\phi,\psi]_{KK(A,J_B)} = \kappa$.*

Proof. Use Lemma 7.8(ii) to find a unital separable subextension e of the trace-kernel extension such that E is $\mathcal{Z}$-stable, I is stable, $\psi(A) \subseteq E$, $\theta|^D$ is unitizably full, and $\kappa = KK(A,\iota_{I\subseteq J_B})(\kappa')$ for some $\kappa' \in KK(A,I)$. By Lemma 7.7, $\mu_{\mathsf{e}} \circ \psi$ is absorbing. Therefore, by KK-existence (Theorem 5.14(i)), κ' can be realized as $[\bar\phi,\mu_{\mathsf{e}}\circ\psi]_{KK(A,I)}$ for some absorbing *-homomorphism $\bar\phi\colon A \to \mathcal{M}(I)$ so that $(\bar\phi,\mu_{\mathsf{e}}\circ\psi)$ forms a Cuntz pair. In particular, $\bar\phi$ lifts $\bar\mu_{\mathsf{e}} \circ \theta|^D$, and so the pullback square in (7.2) induces a *-homomorphism $\phi\colon A \to E$ lifting $\theta|^D$ and satisfying $\bar\phi = \mu_{\mathsf{e}} \circ \phi$. Then, by definition (see also Footnote 100),

$$\begin{aligned}[\phi,\psi]_{KK(A,I)} &= [\mu_{\mathsf{e}}\circ\phi,\mu_{\mathsf{e}}\circ\psi]_{KK(A,I)} \\ &= [\bar\phi,\mu_{\mathsf{e}}\circ\psi]_{KK(A,I)} \\ &= \kappa'.\end{aligned} \tag{7.6}$$

Therefore, viewing ϕ as a map into B_∞, $[\phi,\psi]_{KK(A,J_B)} = \kappa$. □

Proof of Theorem 7.1. (i): Use Lemma 7.8(i) to produce a unital separable subextension e of the trace-kernel extension e_B such that I is stable, E is $\mathcal{Z}$-stable, $\theta(A) \subseteq D$, $\theta|^D$ is unitizably full, and so that κ factorizes as $\kappa = KK(A, \iota_{E\subseteq B_\infty})(\kappa')$ for some $\kappa' \in KK(A, E)$ with $KK(A, q_{\mathsf{e}})([\theta|^D]) = \kappa'$. Since κ' is a KK-lift of $\theta|^D$ and the pullback square in (7.2) commutes, it follows that $[\bar{\mu}_{\mathsf{e}} \circ \theta|^D]_{KK(A,\mathcal{Q}(I))}$ factorizes through $KK(A, \mathcal{M}(I))$. As I is stable, $KK(A, \mathcal{M}(I)) = 0$ by [59, Proposition 4.1], so $[\bar{\mu}_{\mathsf{e}} \circ \theta|^D]_{KK(A,\mathcal{Q}(I))} = 0$. By the KK-existence theorem (Theorem 5.14(ii)), $\bar{\mu}_{\mathsf{e}} \circ \theta|^D \colon A \to \mathcal{Q}(I)$ lifts to a *-homomorphism $\bar{\psi}\colon A \to \mathcal{M}(I)$. By the pullback (7.2), $(\bar{\psi}, \theta|^D)$ induces a *-homomorphism $\psi\colon A \to E$. Regarding ψ as taking values in B_∞, it follows that ψ lifts θ.

There is no reason to expect that ψ provides the KK-class we need. This must be corrected. By construction, in $KK(A, B^\infty)$, we have

$$\begin{aligned} KK(A, q_B)(\kappa) &= [\theta]_{KK(A,B^\infty)} \\ &= [q_B \circ \psi]_{KK(A,B^\infty)} \\ &= KK(A, q_B)([\psi]_{KK(A,B_\infty)}), \end{aligned} \tag{7.7}$$

(the last equality is recorded in Proposition 5.4(iv)). Accordingly, $\kappa - [\psi]_{KK(A,B_\infty)}$ lies in the kernel of $KK(A, q_B)$. By half-exactness of $KK(A, \cdot\,)$ when A is nuclear (see Proposition 5.4(v)), there exists $\kappa \in KK(A, J_B)$ with

$$KK(A, j_B)(\kappa') = \kappa - [\psi]_{KK(A,B_\infty)}. \tag{7.8}$$

By Proposition 7.9, there exists $\phi\colon A \to B_\infty$ also lifting θ such that $[\phi, \psi]_{KK(A,J_B)} = \kappa'$. Using Proposition 5.4(iv), we have

$$\begin{aligned} [\phi]_{KK(A,B_\infty)} - [\psi]_{KK(A,B_\infty)} &= KK(A, j_B)(\kappa') \\ &\overset{(7.8)}{=} \kappa - [\psi]_{KK(A,B_\infty)}, \end{aligned} \tag{7.9}$$

and so $[\phi]_{KK(A,B_\infty)} = \kappa$.

(ii): This is an immediate consequence of Proposition 7.9, since $KL(A, J_B)$ is a quotient of $KK(A, J_B)$.

(iii): Fix a subextension e as in Lemma 7.8(iii). Then both the maps

$$\mu_{\mathsf{e}} \circ \phi_1|^E, \mu_{\mathsf{e}} \circ \phi_2|^E \colon A \to \mathcal{M}(I) \tag{7.10}$$

are absorbing by Lemma 7.7. The extension e is constructed so that the Cuntz pair $(\mu_{\mathsf{e}}\circ\phi_1|^E, \mu_{\mathsf{e}}\circ\phi_2|^E)$ represents the trivial class in $KL(A, I)$. Therefore, the $\mathcal{Z}$-stable KL-uniqueness theorem (Theorem 5.15(ii)) gives a sequence $(u_n)_{n=1}^\infty$ of unitaries in $(I \otimes \mathcal{Z})^\dagger \subseteq E \otimes \mathcal{Z}$ such that

$$\|\mathrm{Ad}(u_n)\big((\mu_{\mathsf{e}} \otimes \mathrm{id}_{\mathcal{Z}})(\phi_1(a) \otimes 1_{\mathcal{Z}})\big) - (\mu_{\mathsf{e}} \otimes \mathrm{id}_{\mathcal{Z}})(\phi_2(a) \otimes 1_{\mathcal{Z}})\| \to 0 \tag{7.11}$$

for all $a \in A$. Since $\mathrm{Ad}(u_n)$ commutes with $\mu_{\mathsf{e}}\otimes\mathrm{id}_{\mathcal{Z}}$, and $(\mu_{\mathsf{e}}\otimes\mathrm{id}_{\mathcal{Z}})|_{I\otimes\mathcal{Z}}$ is injective, it follows that $\mathrm{Ad}(u_n)(\phi_1(a) \otimes 1_{\mathcal{Z}}) \to \phi_2(a) \otimes 1_{\mathcal{Z}}$.

We now have that $\phi|^E\otimes 1_{\mathcal{Z}}$ and $\psi|^E\otimes 1_{\mathcal{Z}}$ are approximately unitarily equivalent. Since E is $\mathcal{Z}$-stable, Proposition 4.5 implies ϕ and ψ are approximately unitarily equivalent. Therefore, ϕ and ψ are unitarily equivalent by Lemma 3.15. □

7.4. **Classification of unital lifts.** We end this section by proving Theorem 1.2, a unital version of Theorem 7.1 which is needed in the proof of the classification of full approximate embeddings (Theorem 1.1). The map $\Gamma_0^{(A,B_\infty)}\colon KK(A, B_\infty) \to \mathrm{Hom}\big(K_0(A), K_0(B_\infty)\big)$ below was defined in (5.11).

Theorem 7.10 (Classification of unital lifts)**.** *Let A be a unital separable nuclear C^*-algebra, let B be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$, and suppose $\theta\colon A \to B^\infty$ is a unital *-homomorphism such that $\tau \circ \theta$ is a faithful trace on A for all $\tau \in T(B^\infty)$.*

(i) For any $\kappa \in KK(A, B_\infty)$ satisfying $KK(A, q_B)(\kappa) = [\theta]_{KK(A,B^\infty)}$ and $\Gamma_0^{(A,B_\infty)}(\kappa)([1_A]_0) = [1_{B_\infty}]_0$, there exists a unital *-homomorphism $\phi\colon A \to B_\infty$ that lifts θ and satisfies $[\phi]_{KK(A,B_\infty)} = \kappa$.

(ii) Given any unital *-homomorphism $\psi\colon A \to B_\infty$ that lifts θ and any $\lambda \in \ker KL(A, j_B)$, there exists a unital *-homomorphism $\phi\colon A \to B_\infty$ that also lifts θ and satisfies $[\phi, \psi]_{KL(A,J_B)} = \lambda$.

(iii) Any two unital *-homomorphisms ϕ_1 and ϕ_2 that both lift θ and satisfy $[\phi_1, \phi_2]_{KL(A,J_B)} = 0$ are unitarily equivalent.

A unital *-homomorphism can never be unitizably full, so Theorem 7.1 cannot be applied directly to such maps. Instead, we use a standard "de-unitization" trick (seen in [252] and [253], for example), working in a 2×2 matrix amplification to provide additional room for maps to become unitizably full. Recall that $\iota_D^{(2)}\colon D \to M_2(D)$ denotes the top left-hand corner embedding (see Remark 6.4).

Proposition 7.11. *Let A and D be unital C^*-algebras. If $\theta\colon A \to D$ is a unital and full *-homomorphism, then $\iota_D^{(2)} \circ \theta$ is unitizably full.*

Proof. An element $x \in A^\dagger$ can be written as $x = a + \lambda(1_{A^\dagger} - 1_A)$ for some $a \in A$ and $\lambda \in \mathbb{C}$. Thus

$$(\iota_D^{(2)} \circ \theta)^\dagger(x) = \begin{pmatrix} \theta(a) & 0 \\ 0 & \lambda 1_D \end{pmatrix}. \tag{7.12}$$

If $a \neq 0$ or $\lambda \neq 0$, then this generates $M_2(D)$ as an ideal since θ is full. □

Given a unital map $\theta\colon A \to B^\infty$, we will apply Theorem 7.1 to $\iota_{B^\infty}^{(2)} \circ \theta$.

Proof of Theorem 7.10. (i): The hypothesis that $\tau \circ \theta$ is faithful for every $\tau \in T(A)$ ensures that θ is full by Lemma 4.12. Accordingly, $\iota_{B^\infty}^{(2)} \circ \theta\colon A \to M_2(B^\infty)$ is unitizably full by Proposition 7.11. Applying commutativity of the top-right square of (6.4) to $KK(A, q_B)(\kappa) = [\theta]_{KK(A,B^\infty)}$ gives

$$KK(A, q_{M_2(B)})\big(KK(A, \iota_{B_\infty}^{(2)})(\kappa)\big) = [\iota_{B^\infty}^{(2)} \circ \theta]_{KK(A,M_2(B)^\infty)}; \tag{7.13}$$

i.e., $KK(A, \iota_{B_\infty}^{(2)})(\kappa)$ is a KK-lift of $\iota_{B^\infty}^{(2)} \circ \theta$. Therefore, Theorem 7.1(i), applied to $M_2(B)$ in place of B (using Remark 4.7), gives a map $\phi^{(2)}\colon A \to M_2(B_\infty) \cong M_2(B)_\infty$ (see Remark 6.4) lifting $\iota_{B^\infty}^{(2)} \circ \theta$ with

$$[\phi^{(2)}]_{KK(A,M_2(B_\infty))} = KK(A, \iota_{B_\infty}^{(2)})(\kappa). \tag{7.14}$$

As $\Gamma_0^{(A,B_\infty)}(\kappa)([1_A]_0) = [1_{B_\infty}]_0$, it follows from naturality of $\Gamma_0^{(A,\,\cdot\,)}$ and (5.12) that $[\phi^{(2)}(1_A)]_0 = [\iota_{B_\infty}^{(2)}(1_{B_\infty})]_0$. Applying Proposition 4.19(i) with $M_2(B)$ in place of B (and using Remark 4.7), we see that $M_2(B_\infty) \cong M_2(B)_\infty$ has stable rank one, and hence $\phi^{(2)}(1_A)$ is unitarily equivalent to $\iota^{(2)}_{B_\infty}(1_{B_\infty})$.[133] Therefore, $\phi^{(2)}$ is unitarily equivalent to a map of the form $\iota^{(2)}_{B_\infty} \circ \tilde{\phi}$ for a unital *-homomorphism $\tilde{\phi}\colon A \to B_\infty$.

Fix a trace $\tau \in T(B^\infty)$. Recall that τ_2 is the canonical non-normalized tracial functional on $M_2(B^\infty)$ extending τ (so, $\tau_2(1_{M_2(B^\infty)}) = 2$). Then

$$\begin{aligned}
\tau \circ q_B \circ \tilde{\phi} &= \tau_2 \circ \iota^{(2)}_{B^\infty} \circ q_B \circ \tilde{\phi} \\
&\overset{\text{Rem. 6.4}}{=} \tau_2 \circ q_{M_2(B)} \circ \iota^{(2)}_{B_\infty} \circ \tilde{\phi} \\
&= \tau_2 \circ q_{M_2(B)} \circ \phi^{(2)} \\
&= \tau_2 \circ \iota^{(2)}_{B_\infty} \circ \theta \\
&= \tau \circ \theta,
\end{aligned} \tag{7.15}$$

where the third equality follows from unitary invariance of the trace. As this holds for all $\tau \in T(B^\infty)$, we have $\mathrm{Aff}\,T(q_B \circ \tilde{\phi}) = \mathrm{Aff}\,T(\theta)$. Therefore, by the classification of maps into B^∞ (Theorem 6.5), $q_B \circ \tilde{\phi}$ is unitarily equivalent to θ. The von Neumann-like behavior of B^∞ ensures that unitaries in B^∞ lift to unitaries in B_∞ (Proposition 6.6), so we can find a unitary $u \in B_\infty$ such that $\mathrm{Ad}(q_B(u)) \circ q_B \circ \tilde{\phi} = \theta$. Define $\phi \coloneqq \mathrm{Ad}(u) \circ \tilde{\phi}$. Since ϕ is unitarily equivalent to $\iota^{(2)}_{B_\infty} \circ \phi$, (7.14) gives

$$KK(A, \iota^{(2)}_{B_\infty})([\phi]_{KK(A,B_\infty)}) = KK(A, \iota^{(2)}_{B_\infty})(\kappa). \tag{7.16}$$

But $KK(A, \iota^{(2)}_{B_\infty})$ is an isomorphism (Proposition 5.4(i)), and so $[\phi]_{KK(A,B_\infty)} = \kappa$.

(ii): Using the commutativity of the top-right square of (6.4) and making the identification $M_2(B^\infty) \cong M_2(B)^\infty$ (see Remark 6.4) it follows that $\iota^{(2)}_{B_\infty} \circ \psi$ lifts $\iota^{(2)}_{B^\infty} \circ \theta$, and the latter map is unitizably full, as noted in the proof of part (i).

Applying Theorem 7.1(ii) to $M_2(B)$, $\iota^{(2)}_{B_\infty} \circ \psi$, $\iota^{(2)}_{B^\infty} \circ \theta$, and $KL(A, \iota^{(2)}_{J_B})(\lambda)$, there is a *-homomorphism $\phi^{(2)}\colon A \to M_2(B_\infty) \cong M_2(B)_\infty$ lifting $\iota^{(2)}_{B^\infty} \circ \theta$ and with $[\phi^{(2)}, \iota^{(2)}_{B_\infty} \circ \psi]_{KL(A,M_2(J_B))} = KL(A, \iota^{(2)}_{J_B})(\lambda)$. Then using Proposition 5.4(iv) in the first step, and commutativity of the top-left square of (6.4) at the third step, we have

$$\begin{aligned}
[\phi^{(2)}]_{KL(A,M_2(B_\infty))} &- [\iota^{(2)}_{B_\infty} \circ \psi]_{KL(A,M_2(B_\infty))} \\
&= KL(A, j_{M_2(B)})\big([\phi^{(2)}, \iota^{(2)}_{B_\infty} \circ \psi]_{KL(A,M_2(J_B))}\big) \\
&= KL(A, j_{M_2(B)} \circ \iota^{(2)}_{J_B})(\lambda) \\
&= KL(A, \iota^{(2)}_{B_\infty}) \circ KL(A, j_B)(\lambda) \\
&= 0.
\end{aligned} \tag{7.17}$$

Therefore, the induced map on K_0 vanishes, so $[\phi^{(2)}(1_A)]_0 = [(\iota^{(2)}_{B_\infty} \circ \psi)(1_A)]_0 = [\iota^{(2)}_{B_\infty}(1_{B_\infty})]_0$.

Using that $M_2(B_\infty) \cong M_2(B)_\infty$ has stable rank one (by Remark 4.7 and 4.19(i)), there exists a unitary $u \in M_2(B_\infty)$ such that $\mathrm{Ad}(u) \circ \phi^{(2)}(1_A) = \iota^{(2)}_{B_\infty}(1_{B_\infty})$. As $\phi^{(2)}$ lifts $\iota^{(2)}_{B^\infty} \circ \theta$, we have

$$\big(\mathrm{Ad}(q_{M_2(B)}(u)) \circ \iota^{(2)}_{B^\infty}\big)(1_{B^\infty}) = \iota^{(2)}_{B^\infty}(1_{B^\infty}), \tag{7.18}$$

[133] This uses the well-known fact that stable rank one implies cancellation in K_0, and this, in turn, implies that projections which agree in K_0 are unitarily equivalent; see [10, Proposition 6.5.1], for example.

so we can write

$$q_{M_2(B)}(u) = \overline{v} \oplus \overline{w} := \begin{pmatrix} \overline{v} & 0 \\ 0 & \overline{w} \end{pmatrix} \tag{7.19}$$

for some unitaries $\overline{v}, \overline{w} \in B^\infty$. These lift to unitaries $v, w \in B_\infty$ respectively by Proposition 6.6. Since $\big(\mathrm{Ad}((v \oplus w)^* u) \circ \phi^{(2)}\big)(1_A) = \iota^{(2)}_{B_\infty}(1_{B_\infty})$, we can write $\mathrm{Ad}((v \oplus w)^* u) \circ \phi^{(2)} = \iota^{(2)}_{B_\infty} \circ \phi$ for some unital *-homomorphism $\phi\colon A \to B_\infty$. By construction, $q_{M_2(B)}((v \oplus w)^* u) = 1_{M_2(B_\infty)}$, so ϕ lifts θ. As $(v \oplus w)^* u \in J^\dagger_{M_2(B)}$, we have

$$[\iota^{(2)}_{B_\infty} \circ \phi, \iota^{(2)}_{B_\infty} \circ \psi]_{KL(A, M_2(J_B))} = [\phi^{(2)}, \iota^{(2)}_{B_\infty} \circ \psi]_{KL(A, M_2(J_B))} \tag{7.20}$$

(by Proposition 5.4(ii)). Therefore,

$$\begin{aligned} KL(A, \iota^{(2)}_{J_B})([\phi, \psi]_{KL(A, J_B)}) &= [\iota^{(2)}_{B_\infty} \circ \phi, \iota^{(2)}_{B_\infty} \circ \psi]_{KL(A, M_2(J_B))} \\ &= KL(A, \iota^{(2)}_{J_B})(\lambda). \end{aligned} \tag{7.21}$$

Since $KL(A, \iota^{(2)}_{J_B})$ is an isomorphism (Proposition 5.7(i)), $[\phi, \psi]_{KL(A, J_B)} = \lambda$.

(iii): Given two unital lifts $\phi_1, \phi_2\colon A \to B_\infty$ of θ such that $[\phi_1, \phi_2]_{KL(A, J_B)} = 0$, Proposition 5.7(iv) yields

$$[\iota^{(2)}_{B_\infty} \circ \phi_1, \iota^{(2)}_{B_\infty} \circ \phi_2]_{KL(A, M_2(J_B))} = KL(A, \iota^{(2)}_{J_B})([\phi_1, \phi_2]_{KL(A, J_B)}) = 0. \tag{7.22}$$

Just as at the beginning of (i), $\iota^{(2)}_{B^\infty} \circ \theta$ is unitizably full, so we can apply Theorem 7.1(iii) with $M_2(B)$ in place of B, $\iota^{(2)}_{B^\infty} \circ \theta$ in place of θ, $\iota^{(2)}_{B_\infty} \circ \phi_2$ in place of ψ, $\iota^{(2)}_{B_\infty} \circ \phi_1$ in place of ϕ, and $\lambda = 0$ to learn that $\iota^{(2)}_{B_\infty} \circ \phi_1$ and $\iota^{(2)}_{B_\infty} \circ \phi_2$ are unitarily equivalent. Since ϕ_1 and ϕ_2 are unital, a unitary $u \in M_2(B_\infty)$ with $\mathrm{Ad}(u) \circ \iota^{(2)}_{B_\infty} \circ \phi_1 = \iota^{(2)}_{B_\infty} \circ \phi_2$ must be of the form $u = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ for unitaries $v, w \in B_\infty$. Hence $\mathrm{Ad}(v) \circ \phi_1 = \phi_2$. □

8. The universal (multi)coefficient theorem and computing $KL(A, J_B)$

Our main goal in this section is to prove Theorem 1.6, computing $KL(A, J_B)$ in terms of the invariant $\underline{K}T_u$. This is the point in the argument at which the universal (multi)coefficient theorem becomes crucial. We review the UCT and UMCT in Sections 8.1 and 8.2 and turn to the calculation of $KL(A, J_B)$ in Section 8.3.

8.1. The universal coefficient theorem.

Inspired by Brown's earlier UCT for Brown–Douglas–Fillmore theory ([26]), Rosenberg and Schochet's UCT ([245]) establishes a class of C^*-algebras A for which $KK(A, I)$ can be computed using K-theory.

Definition 8.1. A separable C^*-algebra A is said to *satisfy the universal coefficient theorem (UCT)*, or *belong to the UCT class* if, for every σ-unital C^*-algebra I, the map $\Gamma^{(A,I)}$ from (5.11) is surjective and a natural map

$$\ker \Gamma^{(A,I)} \to \mathrm{Ext}(K_*(A), K_{1-*}(I)). \tag{8.1}$$

is bijective.[134]

[134] The definition of this map will not be needed here; it is described as the map κ in [10, Section 23.1].

Accordingly, when A satisfies the UCT, there is a short exact sequence[135]

$$0 \to \mathrm{Ext}\big(K_*(A), K_{1-*}(I)\big) \to KK(A,I) \xrightarrow{\Gamma^{(A,I)}} \mathrm{Hom}\big(K_*(A), K_*(I)\big) \to 0 \tag{8.2}$$

where the first map is the inverse of the map in (8.1). It is through this sequence that $KK(A,I)$ can be calculated using K-theory.

The fundamental question is: which C^*-algebras satisfy the UCT? Rosenberg and Schochet tackle this through closure properties: they consider the smallest class (often called the *bootstrap class*) of separable nuclear C^*-algebras containing all separable type I C^*-algebras closed under Morita equivalence, inductive limits, the 2-out-of-3-property for extensions, and crossed products by $\mathbb{R}$ and $\mathbb{Z}$. It follows from [245] that the UCT holds for every C^*-algebra in this class, and, moreover, they observe that a separable C^*-algebra satisfies the UCT if and only if it is KK-equivalent to an abelian C^*-algebra (see the text following [245, Remark 7.6]). Today, this last statement is often taken as the definition of the UCT class.

At the time Rosenberg and Schochet's paper was submitted, it was unknown whether all separable C^*-algebras satisfy the UCT. Skandalis showed this is not the case ([258, Théorème 4.1 and Corollaire 4.2]),[136] but it remains a major open problem whether all separable nuclear C^*-algebras satisfy the UCT. Building on Higson and Kasparov's deep work on the Baum–Connes conjecture ([137]), Tu established the UCT for C^*-algebras associated to second countable locally compact Hausdorff amenable groupoids ([277, Proposition 10.7 and Lemme 3.5]). This was utilized by Barlak and Li ([6]) to show that all nuclear C^*-algebras containing a Cartan masa satisfy the UCT, using Renault's reconstruction theorem ([231]) for Cartan masas in terms of twisted étale locally compact Hausdorff effective groupoid C^*-algebras.

Remark 8.2. The problem of whether the crossed product $A \rtimes G$ satisfies the UCT has a very different status depending on whether G has torsion.

(i) Let A be a separable C^*-algebra which satisfies the UCT and let G be a countable discrete amenable group acting on A. When G is torsion-free, it follows from [203, Corollary 9.4] that $A \rtimes G$ satisfies the UCT. In fact, the same result implies $A \rtimes G$ satisfies the UCT if $A \rtimes H$ does for every finite subgroup H of G. To see this, note that the notation $\langle \star \rangle$ in [203] denotes the class of separable C^*-algebras which satisfy the UCT (since it is closed under KK-equivalence and contains the bootstrap class described above, and hence all separable abelian C^*-algebras). The dual Dirac morphism hypothesis of [203, Corollary 9.4] follows from [137], and when G is discrete, all compact subgroups of G are finite.

(ii) For finite groups, the UCT problem (whether every separable nuclear C^*-algebra satisfies the UCT) is equivalent to whether the UCT is preserved by crossed products of nuclear C^*-algebras by finite groups. More precisely, [7, Theorem 4.17] shows that the UCT problem is equivalent to whether

[135] Often, the statement of the UCT is graded, in that it also involves a map $\Gamma^{(A,I),1}\colon KK^1(A,I) := KK(A, I \otimes C_0(0,1)) \to \mathrm{Hom}(K_*(A), K_{1-*}(I))$ and its kernel. Since $I \otimes C_0(0,1)$ is σ-unital when I is, these two definitions are equivalent.

[136] Skandalis showed that if G is an infinite lattice in a rank one connected simple Lie group then $C^*_r(G)$ is not KK-equivalent to a nuclear C^*-algebra, and, in particular, not KK-equivalent to an abelian one. So every such $C^*_r(G)$ fails the UCT.

every crossed product of $\mathcal{O}_2$ by $\mathbb{Z}/2$ or $\mathbb{Z}/3$ satisfies the UCT. This is related to results announced by Kirchberg (cf. [10, Exercise 23.15.12]).

We record the following explicit computation of $\Gamma_1^{(A,I)}$ for use in the calculation of $\ker KL(A, j_B)$ in Section 8.3. This is well-known to experts, but isn't easily pinned down in the literature; the approach we take borrows ideas from [133].

Proposition 8.3. *Suppose A is a unital separable C^*-algebra, I is a C^*-algebra, and $(\phi,\psi)\colon A \rightrightarrows E \rhd I$ is a unital Cuntz pair (i.e., E, ϕ, and ψ are all unital). Then $\Gamma_1^{(A,I)}([\phi,\psi]_{KK(A,I)})\colon K_1(A) \to K_1(I)$ is the map*

$$[u]_{K_1(A)} \mapsto [\phi(u)\psi(u)^*]_{K_1(I)}, \tag{8.3}$$

for all unitaries $u \in U_n(A)$ and $n \in \mathbb{N}$.

Note that the right side of (8.3) is indeed in $K_1(I)$, since $\phi(u)\psi(u)^* \in U_n(I^\dagger)$ by the hypothesis that ϕ and ψ are unital.

Proof. Consider the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\tilde{\jmath}} & C & \xrightarrow{\tilde{q}} & A & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle q\circ\phi = q\circ\psi} & & \\ 0 & \longrightarrow & I & \xrightarrow{j} & E & \xrightarrow{q} & E/I & \longrightarrow & 0. \end{array} \tag{8.4}$$

By the pullback property, there are unital *-homomorphisms $\tilde{\phi}, \tilde{\psi}\colon A \to C$ so that

$$\pi \circ \tilde{\phi} = \phi,\ \ \pi \circ \tilde{\psi} = \psi,\ \text{ and } \tilde{q} \circ \tilde{\phi} = \tilde{q} \circ \tilde{\psi} = \mathrm{id}_A. \tag{8.5}$$

By Proposition 5.4(iv) (with $J \coloneqq I$, $F \coloneqq C$, $\theta \coloneqq \mathrm{id}_I$, and $\bar{\theta} \coloneqq \pi$), $[\phi,\psi]_{KK(A,I)} = [\tilde{\phi},\tilde{\psi}]_{KK(A,I)}$. Using Proposition 5.4(iv) again, we have

$$KK(A,\tilde{\jmath})([\phi,\psi]_{KK(A,I)}) = [\tilde{\phi}]_{KK(A,C)} - [\tilde{\psi}]_{KK(A,C)}. \tag{8.6}$$

Using (8.6) and naturality of $\Gamma_1^{(A,\,\cdot\,)}$, we obtain

$$\begin{aligned} K_1(\tilde{\jmath}) \circ \Gamma_1^{(A,I)}([\phi,\psi]_{KK(A,I)}) &= \Gamma_1^{(A,C)}([\tilde{\phi}]_{KK(A,C)} - [\tilde{\psi}]_{KK(A,C)}) \\ &\overset{(5.12)}{=} K_1(\tilde{\phi}) - K_1(\tilde{\psi}). \end{aligned} \tag{8.7}$$

For $n \in \mathbb{N}$, and a unitary $u \in U_n(A)$, $\pi(\tilde{\phi}(u)\tilde{\psi}(u)^*) = \phi(u)\psi(u)^*$, and so in $U_n(I^\dagger)$,

$$\tilde{\phi}(u)\tilde{\psi}(u)^* = \phi(u)\psi(u)^*. \tag{8.8}$$

Consequently, we get

$$(K_1(\tilde{\phi}) - K_1(\tilde{\psi}))([u]_{K_1(A)}) = [\tilde{\phi}(u)\tilde{\psi}(u)^*]_{K_1(C)} = K_1(\tilde{\jmath})([\phi(u)\psi(u)^*]_{K_1(I)}). \tag{8.9}$$

As the top row of (8.4) splits by (8.5), it follows that $K_1(\tilde{\jmath})$ is injective. Combining this with (8.7) and (8.9) we get

$$\Gamma_1^{(A,I)}([\phi,\psi]_{KK(A,I)})([u]_{K_1(A)}) = [\phi(u)\psi(u)^*]_{K_1(I)}. \tag{8.10}$$

□

8.2. **The universal multicoefficient theorem.** We now turn to the universal multicoefficient theorem obtained by Dadarlat and Loring in [66]. From a modern viewpoint, this computes KL as homomorphisms of total K-theory.

At this point we need to change pictures of total K-theory from the definition $K_i(D;\mathbb{Z}/n) \coloneqq K_{1-i}(D\otimes\mathbb{I}_n)$ we gave in Section 2.3 to the definition $K_i(D;\mathbb{Z}/n) \coloneqq KK(\mathbb{I}_n, S^iD)$ used by Dadarlat and Loring.[137] As noted in [66], it is not so difficult to provide a natural isomorphism between $K_{1-i}(D\otimes\mathbb{I}_n)$ and $KK(\mathbb{I}_n, S^iD)$; we do so for completeness in Proposition A.1 (the only proof we know of in the literature is somewhat indirect, given in [150] as a consequence of Spanier–Whitehead duality). However, as Dadarlat and Loring also observe, it would be awkward to explicitly transfer the Bockstein operations through such an isomorphism, so they construct a family of Bockstein operations directly in their picture $KK(\mathbb{I}_n, D)$. As experts are undoubtedly aware, any two such families generate the same ring[138] and so give rise to the same morphisms in total K-theory (see Lemma A.4(i)). We formalize this in the following proposition (proved in Appendix A.2), the upshot of which is that it is legitimate for us to apply the universal multicoefficient theorem with either definition of total K-theory.

Proposition 8.4. *The definition of total K-theory from* [66] *is naturally isomorphic to the definition in Section 2.3. Precisely, given any natural isomorphism of the functors $KK^i(\mathbb{I}_n,\,\cdot\,)$ with $K_i(\,\cdot\,;\mathbb{Z}/n)$, the resulting morphisms of total K-theory coincide.*

For C^*-algebras A and I with A separable, working in the $KK(\mathbb{I}_n, S^iD)$ picture of total K-theory, the Kasparov product provides a map

$$\Gamma_\Lambda^{(A,I)}\colon KK(A,I)\to \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(I)) \tag{8.11}$$

that is natural in both variables (in this picture, the Bockstein operations are given by various Kasparov products, as set out in [255], so associativity of the Kasparov product shows that the induced maps do intertwine the Bockstein operations).[139] Moreover, as the Kasparov product descends by continuity to KL, $\Gamma_\Lambda^{(A,I)}$ factors through $KL(A,I)$ (for the non-separable case, this is shown in the proof of Theorem 8.5 in Appendix B).

Assuming that A satisfies the UCT (and assuming I is separable), Dadarlat and Loring's universal multicoefficient theorem identifies the kernel of the map in (8.11) with the subgroup $\mathrm{PExt}(K_*(A), K_{1-*}(I))$ of $\mathrm{Ext}(K_*(A), K_{1-*}(I))$ consisting of *pure extensions*.[140] Using his work on topologies on KK-theory, Dadarlat subsequently identified $\mathrm{PExt}(K_*(A), K_{1-*}(I))$ naturally with $Z_{KK(A,I)}\subseteq KK(A,I)$ from (5.13) when A satisfies the UCT and I is σ-unital ([61], following work by Schochet in the nuclear setting, [256]). Combining these results gives the modern viewpoint on the Dadarlat–Loring universal multicoefficient theorem (which is

[137] Here $S^0D = D$, and S^1D is the suspension $C_0(0,1)\otimes D$.

[138] The Bockstein operations need not, however, agree on the nose.

[139] As with $\Gamma^{(A,I)}$ defined in (5.11), $\Gamma_\Lambda^{(A,I)}$ is defined first using the Kasparov product with I separable, and then for general I, as a limit over separable subalgebras I_0 of I. See (B.34).

[140] That is, group extensions where every element of finite order in the quotient lifts to an element of the same order.

equivalent to the UCT for separable nuclear C^*-algebras).[141] In Appendix B.4 we note how to extend the universal multicoefficient theorem to general I.

Theorem 8.5 (The universal multicoefficient theorem)**.** *Let A and I be C^*-algebras with A separable. Then the map $\Gamma_\Lambda^{(A,I)}$ in (8.11) induces a map*

$$\widetilde{\Gamma}_\Lambda^{(A,I)}\colon KL(A,I) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I)). \tag{8.12}$$

This map is natural in A and I and is an isomorphism when A satisfies the UCT. For a *-homomorphism $\phi : A \to I$, the map satisfies

$$\widetilde{\Gamma}_\Lambda^{(A,I)}([\phi]_{KL(A,I)}) = \underline{K}(\phi). \tag{8.13}$$

In the special case when the codomain is the trace-kernel quotient of one of our allowed codomains, the K-theory computations of Section 6.3 combine with the universal (multi)coefficient theorem to show that both $\Gamma_0^{(A,B^\infty)}$ and $\Gamma_\Lambda^{(A,B^\infty)}$ compute $KK(A, B^\infty)$ (and not just $KL(A, B^\infty)$).

Proposition 8.6. *Let A be a separable C^*-algebra satisfying the UCT and let B be a unital simple separable nuclear finite $\mathcal{Z}$-stable C^*-algebra. Then the canonical maps*

$$\begin{aligned} &\Gamma_0^{(A,B^\infty)}\colon KK(A,B^\infty) \to \mathrm{Hom}(K_0(A), K_0(B^\infty)), \;\; \textit{and} \\ &\Gamma_\Lambda^{(A,B^\infty)}\colon KK(A,B^\infty) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B^\infty)) \end{aligned} \tag{8.14}$$

are isomorphisms.

Proof. The K-theory of B^∞ was computed in Proposition 6.7: we have $K_0(B^\infty) \cong \mathrm{Aff}\, T(B^\infty)$ and $K_1(B^\infty) = 0$. In particular, $K_0(B^\infty)$ is divisible. It follows that $\mathrm{Ext}(K_*(A), K_{1-*}(B^\infty)) = 0$ (by [285, Corollary 2.3.2 and Exercise 2.5.1], for example). Therefore, as A satisfies the UCT, $\Gamma_0^{(A,B^\infty)}$ is an isomorphism by the exactness of (8.2).

As $\Gamma_0^{(A,B^\infty)}$ factors through $\Gamma_\Lambda^{(A,B^\infty)}$, the latter is necessarily injective. Combining this with the surjectivity of $\Gamma_\Lambda^{(A,B^\infty)}$ from the universal multicoefficient theorem gives the result. $\square$

8.3. **Computing $KL(A, J_B)$.** Theorem 8.7 below relates the KL-class of a unital (A, J_B)-Cuntz pair $(\phi,\psi)\colon A \rightrightarrows B_\infty \rhd J_B$ satisfying $[\phi]_{KK(A,B_\infty)} = [\psi]_{KK(A,B_\infty)}$ and the homomorphisms they induce on $\overline{K}_1^{\mathrm{alg}}(A)$. In this case, the map $\overline{K}_1^{\mathrm{alg}}(\phi) - \overline{K}_1^{\mathrm{alg}}(\psi)$ yields a homomorphism

$$K_1(A)/\mathrm{Tor}(K_1(A)) \to \ker \not{a}_{B_\infty} \subseteq \overline{K}_1^{\mathrm{alg}}(B_\infty). \tag{8.15}$$

The following says that, under the UCT, the map in (8.15) encodes the class $[\phi,\psi]_{KL(A,J_B)}$. The connection between the map $R_{A,B}$ in the following theorem and

[141] Lin introduced the notion that a separable nuclear C^*-algebra A satisfies the *approximate universal coefficient theorem* in [181] to mean that the maps $\widetilde{\Gamma}_\Lambda^{(A,I)}$ are isomorphisms for all separable C^*-algebras I. (Lin uses a formally different topology on $KK(A,I)$ when he takes the closure of $\{0\}$, but as explained by Dadarlat in [61, Section 5], this is equivalent to the topology set out in Definition 5.5 used to define $KL(A,I)$). In [61, Theorem 5.5], Dadarlat shows that for separable nuclear C^*-algebras A, Lin's approximate universal coefficient theorem and the universal coefficient theorem are equivalent. That is, a separable nuclear C^*-algebra A satisfies the UCT if and only if the maps in (8.12) are always isomorphisms.

the “rotation maps” $R_{\phi,\psi}$ appearing in the Gong–Lin–Niu classification ([125, 126]) is discussed in Remark 8.10.

Theorem 8.7. *Let A be a unital separable C^*-algebra, and let B be a unital simple separable nuclear finite $\mathcal{Z}$-stable C^*-algebra. Then there is a natural group homomorphism*

$$R_{A,B}\colon \ker KL(A, j_B) \to \mathrm{Hom}\big(K_1(A)/\mathrm{Tor}(K_1(A)), \ker \not{a}_{B_\infty}\big) \tag{8.16}$$

with the following property. For a unital (A, J_B)-Cuntz pair $(\phi, \psi)\colon A \rightrightarrows B_\infty \rhd J_B$ for which $[\phi]_{KL(A,B_\infty)} = [\psi]_{KL(A,B_\infty)}$, the diagram

$$\begin{tikzcd} \overline{K}_1^{\mathrm{alg}}(A) \arrow[rr, "\overline{K}_1^{\mathrm{alg}}(\phi)-\overline{K}_1^{\mathrm{alg}}(\psi)"] \arrow[d, two heads, "\not{a}_A"] & & \overline{K}_1^{\mathrm{alg}}(B_\infty) \\ K_1(A) \arrow[r, "t_A"] & K_1(A)/\mathrm{Tor}(K_1(A)) \arrow[r, "R_{A,B}([\phi,\psi]_{KL(A,J_B)})"] & \ker \not{a}_{B_\infty} \arrow[u, hook] \end{tikzcd} \tag{8.17}$$

commutes, where $t_A\colon K_1(A) \to K_1(A)/\mathrm{Tor}(K_1(A))$ is the quotient map. If, in addition, A satisfies the UCT, then $R_{A,B}$ is an isomorphism.

Proof. Applying naturality of the maps $\widetilde{\Gamma}_\Lambda^{(A,\,\cdot\,)}$ in the universal multicoefficient theorem (Theorem 8.5) to j_B induces $\xi\colon \ker KL(A, j_B) \to \ker \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(j_B))$ as in the following commutative diagram:

$$\begin{tikzcd} \ker KL(A, j_B) \arrow[r, "\xi"] \arrow[d, hook] & \ker \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(j_B)) \arrow[d, hook] \\ KL(A, J_B) \arrow[r, "\widetilde{\Gamma}_\Lambda^{(A,J_B)}"] \arrow[d, "KL(A,j_B)"'] & \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(J_B)\big) \arrow[d, "\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(j_B))"] \\ KL(A, B_\infty) \arrow[r, "\widetilde{\Gamma}_\Lambda^{(A,B_\infty)}"'] & \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B_\infty)\big) \end{tikzcd} \tag{8.18}$$

The universal multicoefficient theorem tells us that the horizontal maps $\widetilde{\Gamma}_\Lambda^{(A,J_B)}$ and $\widetilde{\Gamma}_\Lambda^{(A,B_\infty)}$ are isomorphisms when A satisfies the UCT. Therefore, ξ is also an isomorphism in this case.

As B is nuclear and $\mathcal{Z}$-stable, the K-theory computation of Proposition 6.7 ensures that $K_1(B^\infty) = 0$ and $K_0(B^\infty) \cong \mathrm{Aff}\, T(B^\infty)$, so that $K_0(B^\infty)$ is uniquely divisible. Thus Lemma 2.16 applies to the trace-kernel extension and gives a natural isomorphism

$$\ker \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(j_B)\big) \xrightarrow{\cong} \mathrm{Hom}\big(K_1(A)/\mathrm{Tor}\big(K_1(A)\big), \ker K_1(j_B)\big) \tag{8.19}$$

sending $\underline{\alpha}$ to the map $\alpha_1'\colon K_1(A)/\mathrm{Tor}(K_1(A)) \to \ker K_1(j_B)$ induced by α_1. Theorem 6.8 provides a natural isomorphism $\omega_B'\colon \ker K_1(j_B) \to \ker \not{a}_{B_\infty}$ which is given explicitly by (6.7). We define $R_{A,B}$ to be the composition of these three natural maps, as follows:

$$
\begin{tikzcd}
\ker KL(A, j_B) \arrow[r, "\xi"] \arrow[ddr, bend right, "R_{A,B}"'] & \ker \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(j_B)\big) \arrow[d, "\underline{\alpha} \mapsto \alpha_1'"] \\
& \mathrm{Hom}\big(K_1(A)/\mathrm{Tor}\big(K_1(A)\big), \ker K_1(j_B)\big) \arrow[d, "\mathrm{Hom}(K_1(A)/\mathrm{Tor}(K_1(A)), \omega_B')"] \\
& \mathrm{Hom}(K_1(A)/\mathrm{Tor}(K_1(A)), \ker \not a_{B_\infty}).
\end{tikzcd}
\tag{8.20}
$$

As the vertical maps above are isomorphisms, $R_{A,B}$ is an isomorphism when ξ is, and so $R_{A,B}$ is an isomorphism when A satisfies the UCT.

Now let $(\phi, \psi)\colon A \rightrightarrows B_\infty \rhd J_B$ be a unital Cuntz pair such that $[\phi, \psi]_{KK(A, J_B)}$ belongs to $\ker KK(A, j_B)$. The computation of $R_{A,B}([\phi, \psi]_{KL(A,J_B)}) \circ t_A \circ \not a_A$ is summarized in the following commutative diagram (commutativity of the middle square is the definition of $R_{A,B}$):

$$
\begin{tikzcd}
\overline{K}_1^{\mathrm{alg}}(A) \arrow[r, "t_A \circ \not a_A"] \arrow[dr, "\not a_A"'] & K_1(A)/\mathrm{Tor}(K_1(A)) \arrow[r, "R_{A,B}([\phi,\psi]_{KL(A,J_B)})"] & \ker \not a_{B_\infty} & \subseteq & \overline{K}_1^{\mathrm{alg}}(B_\infty) \\
& K_1(A) \arrow[u, "t_A"] \arrow[r, "\tilde{\Gamma}_1^{(A,J_B)}([\phi,\psi]_{KL(A,J_B)})"'] & \ker K_1(j_B) \arrow[u, "\omega_B'"] & \subseteq & K_1(J_B). \arrow[u, "\omega_B"]
\end{tikzcd}
\tag{8.21}
$$

It remains to check that

$$
\omega_B \circ \tilde{\Gamma}_1^{(A,J_B)}\big([\phi, \psi]_{KL(A,J_B)}\big) \circ \not a_A = \overline{K}_1^{\mathrm{alg}}(\phi) - \overline{K}_1^{\mathrm{alg}}(\psi). \tag{8.22}
$$

Let $u \in U_n(A)$. By Proposition 8.3,

$$
\tilde{\Gamma}_1^{(A,J_B)}([\phi, \psi]_{KL(A,J_B)})([u]_1) = [\phi(u)\psi(u)^*]_1. \tag{8.23}
$$

Letting $s : J_B^\dagger \to \mathbb{C}1_{B_\infty} \subseteq B_\infty$ be the canonical scalar map, so that $s(\phi(u)\psi(u)^*) = 1_{M_n(B_\infty)}$ since (ϕ, ψ) is a unital Cuntz pair. Therefore, the explicit formula for ω_B from Theorem 6.8(i) gives

$$
\begin{aligned}
\omega_B\big(\tilde{\Gamma}_1^{(A,J_B)}([\phi, \psi]_{KL(A,J_B)})([u]_1)\big) &= [\phi(u)\psi(u)^*]_{\mathrm{alg}} - [1_{B_\infty}]_{\mathrm{alg}} \\
&= [\phi(u)]_{\mathrm{alg}} - [\psi(u)]_{\mathrm{alg}} \\
&= (\overline{K}_1^{\mathrm{alg}}(\phi) - \overline{K}_1^{\mathrm{alg}}(\psi))([u]_{\mathrm{alg}}),
\end{aligned}
\tag{8.24}
$$

as required. □

The map in Theorem 8.7 can be directly related to algebraic K_1.

Remark 8.8. Let A be a unital separable C^*-algebra and B a unital separable exact $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Then there is a natural homomorphism

$$
KL(A, J_B) \to \mathrm{Hom}(\overline{K}_1^{\mathrm{alg}}(A), \overline{K}_1^{\mathrm{alg}}(B_\infty)), \quad \lambda \mapsto \omega_B \circ \tilde{\Gamma}_1^{(A,J_B)}(\lambda) \circ \not a_A, \tag{8.25}
$$

where $\omega_B\colon K_1(J_B) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$ is the map from Theorem 6.8. The same computation as in the end of the proof of Theorem 8.7 shows that if $(\phi, \psi)\colon A \rightrightarrows B_\infty \rhd J_B$ is a unital (A, J_B)-Cuntz pair, then

$$
[\phi, \psi]_{KL(A,J_B)} \mapsto \overline{K}_1^{\mathrm{alg}}(\phi) - \overline{K}_1^{\mathrm{alg}}(\psi) \tag{8.26}
$$

is the map in (8.25).

We conclude this section by proving Theorem 1.6, explicitly describing $KL(A, J_B)$ in terms of $\underline{K}T_u(\,\cdot\,)$.

Theorem 8.9. *Let A be a unital separable C^*-algebra satisfying the UCT and let B be a unital simple separable nuclear finite $\mathcal{Z}$-stable C^*-algebra. The map*

$$\Theta \coloneqq \tilde{\Gamma}_\Lambda^{(A,B_\infty)} \circ KL(A, j_B)\colon KL(A, J_B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \tag{8.27}$$

fits into the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(j_B)\big) \arrow[r] & KL(A, J_B) \arrow[dll, rounded corners, "\Theta"'] \\ \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B_\infty)\big) \arrow[r] & \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B^\infty)\big), & \end{tikzcd} \tag{8.28}$$

where the second arrow is the restriction of $(\tilde{\Gamma}_\Lambda^{(A,J_B)})^{-1}$,[142] *and the final arrow is given by $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(q_B)\big)$. Moreover:*

- *(i) The range of Θ consists of all $\underline{\alpha}$ for which $\rho_{B_\infty} \circ \alpha_0 = 0$ and the kernel of Θ is $\ker KL(A, j_B)$.*
- *(ii) Let $\psi\colon A \to B_\infty$ be a unital full *-homomorphism. There is a bijection Ω taking $\ker\Theta \subseteq KL(A, J_B)$ to the set of $\beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$ such that $(\underline{K}(\psi), \beta, \mathrm{Aff}\, T(\psi))$ is a $\underline{K}T_u(\,\cdot\,)$-morphism, given by*

$$\Omega(\lambda) \coloneqq \overline{K}_1^{\mathrm{alg}}(\psi) + R_{A,B}(\lambda) \circ t_A \circ \not{a}_A, \qquad \lambda \in \ker\Theta. \tag{8.29}$$

Proof. The exactness at the first two entries of (8.28) follows from the naturality of $\tilde{\Gamma}_\Lambda^{(A,\,\cdot\,)}$. Indeed, after using $\tilde{\Gamma}_\Lambda^{(A,J_B)}$ to identify $KL(A, J_B)$ with $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(J_B)\big)$ via the universal multicoefficient theorem, the second arrow becomes the inclusion, Θ becomes $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(j_B)\big)$ and (8.28) is certainly exact at this entry. For exactness at $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B_\infty))$, it suffices to show that $\mathrm{im}\,\Theta$ is contained in $\ker \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(q_B)\big)$ since the reverse inclusion follows from $q_B \circ j_B = 0$. The issue is that, in general, neither $KL(A, \,\cdot\,)$ nor $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(\,\cdot\,)\big)$ is half-exact,[143] so we instead use half-exactness of $KK(A, \,\cdot\,)$ together with the control on $KK(A, B^\infty)$ given by the von Neumann-like behavior of B^∞.

Consider the commutative diagram

$$\begin{tikzcd} KK(A, J_B) \arrow[r] \arrow[d, "\Gamma_\Lambda^{(A,J_B)}"] & KK(A, B_\infty) \arrow[r] \arrow[d, "\Gamma_\Lambda^{(A,B_\infty)}"] & KK(A, B^\infty) \arrow[d, "\Gamma_\Lambda^{(A,B^\infty)}"] \\ \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(J_B)\big) \arrow[r] & \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B_\infty)\big) \arrow[r] & \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B^\infty)\big) \end{tikzcd} \tag{8.30}$$

---

[142] Recall that $\tilde{\Gamma}_\Lambda^{(A,J_B)}$ is an isomorphism by the universal multicoefficient theorem (Theorem 8.5).

[143] For instance, let $SQ \coloneqq C_0((0,1), Q)$ be the suspension of the universal UHF algebra Q. Then $KL(SQ, \,\cdot\,) \cong \mathrm{Hom}_\Lambda\big(\underline{K}(SQ), \underline{K}(\,\cdot\,)\big) \cong \mathrm{Hom}\big(\mathbb{Q}, K_1(\,\cdot\,)\big)$. Consider $C \coloneqq \{f \in C_0([0,1), Q) : f(0) \in \mathbb{C}1_Q\}$, which contains SQ as an ideal with $C/SQ \cong \mathbb{C}$. Applying K_1 to $SQ \to C \to \mathbb{C}$ gives $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, and thus applying $KL(SQ, \,\cdot\,)$ or $\mathrm{Hom}_\Lambda\big(\underline{K}(SQ), \underline{K}(\,\cdot\,)\big)$ to this sequence produces $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to 0$. This is not exact since the cokernel of the left map is $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \neq 0$.

associated to the trace-kernel extension. Since A satisfies the UCT, A is KK-equivalent to an abelian C^*-algebra (see [10, Theorem 23.10.5]). Then half-exactness of $KK(A,\,\cdot\,)$ (recorded as Proposition 5.4(v)) gives exactness of the top row of (8.30). A diagram chase using the surjectivity of $\Gamma_\Lambda^{(A,B_\infty)}$ (from the UMCT) and the injectivity of $\Gamma_\Lambda^{(A,B^\infty)}$ (from Proposition 8.6) proves that $\ker \mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(q_B)\big)$ is contained in $\mathrm{im}\,\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(j_B)\big)$.[144] By naturality of $\tilde{\Gamma}_\Lambda^{(A,\,\cdot\,)}$,

$$\Theta = \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(j_B)) \circ \tilde{\Gamma}_\Lambda^{(A,J_B)}, \tag{8.31}$$

so that (using that $\tilde{\Gamma}_\Lambda^{(A,J_B)}$ is an isomorphism by the universal multicoefficient theorem) $\mathrm{im}\,\Theta = \mathrm{im}\,\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(j_B))$. Therefore, (8.28) is exact.

(i): The exactness of (8.28) implies that the image of Θ consists of the set of all $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(B_\infty)$ for which $\underline{K}(q_B)\circ\underline{\alpha} = 0$. By Proposition 6.7, this condition is equivalent to $\rho_{B_\infty}\circ\alpha_0 = 0$. The kernel of Θ is $\ker KL(A,j_B)$ as $\tilde{\Gamma}_\Lambda^{(A,B_\infty)}$ is an isomorphism.

(ii): Fix $\lambda \in \ker\Theta$ and define $\beta \coloneqq \Omega(\lambda)$. We first check that $(\underline{K}(\psi),\beta,\mathrm{Aff}\,T(\psi))$ is a morphism $\underline{K}T_u(A) \to \underline{K}T_u(B_\infty)$. This boils down to the fact that $\underline{K}T_u(\psi)$ is a morphism (Proposition 3.6) and the correcting term $R_{A,B}(\lambda)\circ t_A\circ \not{a}_A$ in (8.29) is annihilated in the $\underline{K}T_u(\,\cdot\,)$-compatibility relations.

Precisely, the first square of (3.14) commutes for free since it does not involve the map β. For the second square, since $\not{a}_A \circ \mathrm{Th}_A = 0$, we have

$$\beta\circ\mathrm{Th}_A \overset{(8.29)}{=} \overline{K}_1^{\mathrm{alg}}(\psi)\circ\mathrm{Th}_A \overset{\text{Prop. 3.6}}{=} \mathrm{Th}_{B_\infty}\circ\mathrm{Aff}\,T(\psi), \tag{8.32}$$

For the third square of (3.14), since $\mathrm{im}\,R_{A,B}(\lambda) \subseteq \ker \not{a}_{B_\infty}$, we similarly have

$$\not{a}_{B_\infty}\circ\beta \overset{(8.29)}{=} \not{a}_{B_\infty}\circ\overline{K}_1^{\mathrm{alg}}(\psi) \overset{\text{Prop. 3.6}}{=} K_1(\psi)\circ\not{a}_A. \tag{8.33}$$

Finally, for commutativity of (3.15), note that by (3.8), we have $\not{a}_A\circ\zeta_A^{(n)} = \nu_{0,A}^{(n)}$, so the range of this composition is contained in $\mathrm{Tor}(K_1(A),\mathbb{Z}/n)$. Consequently, $t_A\circ\not{a}_A\circ\zeta_A^{(n)} = 0$, so similar to the previous calculations, we have

$$\beta\circ\zeta_A^{(n)} \overset{(8.29)}{=} \overline{K}_1^{\mathrm{alg}}(\psi)\circ\zeta_A^{(n)} \overset{\text{Prop. 3.6}}{=} \zeta_B^{(n)}\circ K_0(\psi;\mathbb{Z}/n). \tag{8.34}$$

This shows that $(\underline{K}(\psi),\beta,\mathrm{Aff}\,T(\psi))$ is a $\underline{K}T_u(\,\cdot\,)$-morphism.

Suppose now that $\lambda,\lambda' \in \ker\Theta$ satisfy $\Omega(\lambda) = \Omega(\lambda')$. By (8.29), we get $R_{A,B}(\lambda-\lambda')\circ t_A\circ\not{a}_A = 0$. Since $t_A\circ\not{a}_A$ is surjective, and $R_{A,B}$ is injective (from Theorem 8.7), it follows that $\lambda = \lambda'$. This shows that Ω is injective.

Finally, let us show that Ω is surjective; that is, if $\beta : \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$ is such that $(\underline{K}(\psi),\beta,\mathrm{Aff}\,T(\psi))$ is a morphism $\underline{K}T_u(A) \to \underline{K}T_u(B_\infty)$, then there exists $\lambda \in \ker\Theta$ such that $\Omega(\lambda) = \beta$. Using Lemma 3.11 with $\beta' \coloneqq \overline{K}_1^{\mathrm{alg}}(\psi)$, there

[144] Any $\underline{\alpha} \in \ker\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(q_B))$ is induced by some $\kappa \in KK(A,B_\infty)$, which is in the kernel of $KK(A,q_B)$ (by injectivity of $\Gamma_\Lambda^{(A,B^\infty)}$). Half-exactness shows that $\kappa = KK(A,j_B)(\kappa')$ for some $\kappa' \in KK(A,J_B)$. Commutativity of the left square of (8.30) shows that $\Gamma_\Lambda^{(A,J_B)}(\kappa')$ is mapped onto $\underline{\alpha}$ via $\mathrm{Hom}(\underline{K}(A),\underline{K}(j_B))$

is a homomorphism $r\colon K_1(A)/\mathrm{Tor}(K_1(A)) \to \ker \not{a}_{B_\infty}$ such that

$$
\begin{tikzcd}
\overline{K}_1^{\mathrm{alg}}(A) \arrow[rr, "\beta - \overline{K}_1^{\mathrm{alg}}(\psi)"] \arrow[d, two heads, "\not{a}_A"] & & \overline{K}_1^{\mathrm{alg}}(B_\infty) \\
K_1(A) \arrow[r, two heads, "t_A"] & K_1(A)/\mathrm{Tor}(K_1(A)) \arrow[r, "r"] & \ker \not{a}_{B_\infty} \arrow[u, hook]
\end{tikzcd}
\tag{8.35}
$$

commutes. By surjectivity of $R_{A,B}$ (from Theorem 8.7), it follows that there exists $\lambda \in \ker KL(A, j_B) = \ker \Theta$ such that $r = R_{A,B}(\lambda)$. Hence,

$$
\begin{aligned}
\Omega(\lambda) &\overset{(8.29)}{=} \overline{K}_1^{\mathrm{alg}}(\psi) + R_{A,B}(\lambda) \circ t_A \circ \not{a}_A \\
&= \overline{K}_1^{\mathrm{alg}}(\psi) + r \circ t_A \circ \not{a}_A \\
&\overset{(8.35)}{=} \overline{K}_1^{\mathrm{alg}}(\psi) + (\beta - \overline{K}_1^{\mathrm{alg}}(\psi)) \\
&= \beta,
\end{aligned}
\tag{8.36}
$$

as required. □

Remark 8.10. The map $R_{A,B}$ in Theorem 8.7 is related to the rotation maps appearing in early classification results of *-homomorphism up to asymptotic unitary equivalence, dating back to [170]. For the sake of comparison, we recall the version of the rotation map from [125, Definition 2.21].

Suppose A and B are unital C^*-algebras with A separable and $\phi, \psi\colon A \to B$ are *-homomorphisms with $[\phi]_{KK(A,B)} = [\psi]_{KK(A,B)}$ and $\mathrm{Aff}\, T(\phi) = \mathrm{Aff}\, T(\psi)$. Define the *mapping torus* of ϕ and ψ by

$$M_{\phi,\psi} \coloneqq \{(f, a) \in C([0,1], B) \oplus A : \phi(a) = f(0) \text{ and } \psi(a) = f(1)\}. \tag{8.37}$$

There is a corresponding extension

$$0 \longrightarrow SB \longrightarrow M_{\phi,\psi} \longrightarrow A \longrightarrow 0 \tag{8.38}$$

with the first map being the inclusion into the first factor and the second map being the projection onto the second factor. Using that $[\phi]_{KK(A,B)} = [\psi]_{KK(A,B)}$, this extension admits a KK-splitting $\kappa \in KK(A, M_{\phi,\psi})$,[145] and hence there is an induced split exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & K_0(B) \arrow[r] & K_1(M_{\phi,\psi}) \arrow[r, shift right] & K_1(A) \arrow[l, shift right, "\kappa_1"'] \arrow[r] & 0,
\end{tikzcd}
\tag{8.39}
$$

where $\kappa_1 \coloneqq \Gamma_1^{(A, M_{\phi,\psi})}(\kappa)$.

Using that $\mathrm{Aff}\, T(\phi) = \mathrm{Aff}\, T(\psi)$, the de la Harpe–Skandalis determinant produces well-defined map $R_{\phi,\psi}\colon K_1(M_{\phi,\psi}) \to \mathrm{Aff}\, T(B)$ by

$$R_{\phi,\psi}([(u,v)]_1) \coloneqq \tilde{\Delta}_B(u), \qquad (u,v) \in U_\infty(M_{\phi,\psi}), \tag{8.40}$$

called the *rotation map* corresponding to ϕ and ψ. The composition

$$K_1(A) \xrightarrow{\kappa_1} K_1(M_{\phi,\psi}) \xrightarrow{R_{\phi,\psi}} \mathrm{Aff}\, T(B) \twoheadrightarrow \mathrm{Aff}\, T(B)/\overline{\mathrm{im}\, \rho_B}, \tag{8.41}$$

[145] In the six-term exact sequence obtained by applying the functor $KK(A, \cdot\,)$ to (8.38), the boundary map is naturally identified with $KK(A,\phi) - KK(A,\psi)$, and hence it vanishes since ϕ and ψ agree in $KK(A,B)$. The KK-splitting κ is obtained as a lift of $[\mathrm{id}_A]_{KK(A,A)}$ along the surjective map $KK(A, M_{\phi,\psi}) \to KK(A,A)$.

which is denoted $\tilde{R}_{\phi,\psi}$, is independent of the choice of splitting κ.[146]

If $(\phi,\psi)\colon A \rightrightarrows B_\infty \rhd J_B$ is a Cuntz pair as in Theorem 8.7, and, in addition, $[\phi]_{KK(A,B_\infty)} = [\psi]_{KK(A,B_\infty)}$, so that both $\tilde{R}_{\phi,\psi}$ and $R_{A,B}([\phi,\psi]_{KL(A,J_B)})$ are defined, then they are related by the equation

$$\tilde{R}_{\phi,\psi} = -\overline{\det}_{B_\infty} \circ R_{A,B}([\phi,\psi]_{KL(A,J_B)}) \circ t_A. \tag{8.42}$$

To see this, by (2.23) and (8.41), it suffices to show that

$$\mathrm{Th}_{B_\infty} \circ R_{\phi,\psi} \circ \kappa_1 = -R_{A,B}([\phi,\psi]_{KL(A,J_B)}) \circ t_A. \tag{8.43}$$

For $v \in U_\infty(A)$, write $\kappa_1([v]_1) = [(u,v')]_1$, so that $[v']_1 = [v]_1$ and u is a path in $U_\infty(B_\infty)$ from $\phi(v')$ to $\psi(v')$. Then

$$\begin{aligned}
\mathrm{Th}_{B_\infty}(R_{\phi,\psi}(\kappa_1([v]_1))) &= \mathrm{Th}_{B_\infty}(R_{\phi,\psi}([(u,v')]_1)) \\
&\overset{(8.40)}{=} \mathrm{Th}_{B_\infty}(\tilde{\Delta}_{B_\infty}(u)) \\
&\overset{(2.19),(2.15)}{=} [u(1)u(0)^*]_{\mathrm{alg}} \\
&= [\psi(v')]_{\mathrm{alg}} - [\phi(v')]_{\mathrm{alg}} \\
&= (\overline{K}_1^{\mathrm{alg}}(\psi) - \overline{K}_1^{\mathrm{alg}}(\phi))([v']_{\mathrm{alg}}) \\
&\overset{(8.17)}{=} -R_{A,B}([\phi,\psi]_{KL(A,J_B)})(t_A([v']_1)) \\
&= -R_{A,B}([\phi,\psi]_{KL(A,J_B)})(t_A([v]_1)),
\end{aligned} \tag{8.44}$$

establishing (8.42).

9. Classification of unital *-homomorphisms

We now have all the pieces in place to establish the main classification theorems: Theorems A, B, and 1.1. First we prove the classification of unital full approximate embeddings, and then we use intertwining arguments to deduce Theorem B in Section 9.2 and Theorem A in Section 9.3.

9.1. **Classification of unital full approximate embeddings.**

We start with the precise statement of Theorem 1.1.

Theorem 9.1 (Classification of unital full approximate embeddings)**.** *Let A and B be unital separable nuclear C^*-algebras such that A satisfies the UCT and B is simple and $\mathcal{Z}$-stable. If $(\underline{\alpha},\beta,\gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B_\infty)$ is a faithful morphism, then there exists a unital full *-homomorphism $\phi\colon A \to B_\infty$ such that $\underline{K}T_u(\phi) = (\underline{\alpha},\beta,\gamma)$, and this ϕ is unique up to unitary equivalence.*

As mentioned in the introduction, by means of Kirchberg's dichotomy (Theorem 4.6), B is either purely infinite or stably finite. While the stably finite case is our main result, let us first discuss the purely infinite case, which is a refined version of the Kirchberg–Phillips Theorem. This could be proved using Kirchberg's (unpublished) techniques from the 1990s ([159]); we will deduce the result from the purely infinite classification theorem in [110].

Proof of Theorem 9.1 when $T(B) = \emptyset$. B is purely infinite and hence is $\mathcal{O}_\infty$-stable by [164, Theorem 3.15]. As noted in Remark 3.7, $\underline{K}T_u$-morphisms from $\underline{K}T_u(A)$ to $\underline{K}T_u(B_\infty)$ (all of which are vacuously faithful) correspond to $\underline{K}$-morphisms from $\underline{K}(A)$ to $\underline{K}(B_\infty)$ sending $[1_A]_0$ to $[1_{B_\infty}]_0$. Hence by [110, Theorem 8.12], these

[146] Two different splitting κ and κ' induce a map $\kappa_1 - \kappa_1'\colon K_1(A) \to K_0(B)$. Well-definedness follows from [183, Lemma 3.3].

correspond bijectively to full $\mathcal{O}_\infty$-stable (in the sense of [110, Definition 4.1]) unital *-homomorphisms $A \to B_\infty$, up to approximate unitary equivalence.

By [253, Proposition 1.12], B_∞ is separably $\mathcal{O}_\infty$-stable, and consequently any *-homomorphism $A \to B_\infty$ factorizes through a separable $\mathcal{O}_\infty$-stable subalgebra of B_∞, and is therefore $\mathcal{O}_\infty$-stable. Note also that approximate unitary equivalence is the same as unitary equivalence (as recorded in Lemma 3.15). This proves the theorem in this case. $\square$

Now we return to the stably finite case, beginning with the existence portion of Theorem 9.1. The strategy is to first realize γ by a map into B^∞, obtained from the classification of maps into B^∞, and lift this to a map into B_∞ realizing $\underline{\alpha}$ using the classification of unital lifts. Finally, we adjust this lift by combining the second part of the classification of unital lifts with the KL computations in Section 8.

Proof of existence in Theorem 9.1 when $T(B) \neq \emptyset$. Suppose $(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B)$ is a faithful morphism. By the classification of maps into B^∞ by traces (Theorem 6.5), there is a unital *-homomorphism $\theta\colon A \to B^\infty$ such that

$$\operatorname{Aff} T(\theta) = \operatorname{Aff} T(q_B) \circ \gamma\colon\ \operatorname{Aff} T(A) \to \operatorname{Aff} T(B^\infty). \tag{9.1}$$

Since $(\underline{\alpha}, \beta, \gamma)$ is faithful, every trace $\tau \in T(B^\infty)$ gives rise to a faithful trace $\tau \circ \theta = \gamma^*(\tau \circ q_B)$ on A.

Since A satisfies the UCT, Dadarlat and Loring's universal multicoefficient theorem (Theorem 8.5) provides $\kappa \in KK(A, B_\infty)$ inducing $\underline{\alpha}$. We claim that κ is a KK-lift of θ, i.e., that

$$KK(A, q_B)(\kappa) = [\theta]_{KK(A,B^\infty)}. \tag{9.2}$$

To see this, first note that $\Gamma_0^{(A,B^\infty)}(KK(A,q_B)(\kappa)) = K_0(q_B) \circ \alpha_0$ using (5.12). Using naturality of the pairing map ρ in the first and last line, and the compatibility of α_0 and γ (i.e., commutativity of the first square in (3.14) from the conditions on $\underline{K}T_u$-morphisms) in the second, we have

$$\begin{aligned} \rho_{B^\infty} \circ K_0(q_B) \circ \alpha_0 &= \operatorname{Aff} T(q_B) \circ \rho_{B_\infty} \circ \alpha_0 \\ &= \operatorname{Aff} T(q_B) \circ \gamma \circ \rho_A \\ &\overset{(9.1)}{=} \operatorname{Aff} T(\theta) \circ \rho_A \\ &= \rho_{B^\infty} \circ K_0(\theta). \end{aligned} \tag{9.3}$$

Since ρ_{B^∞} is an isomorphism (Proposition 6.7), $K_0(\theta) = K_0(q_B) \circ \alpha_0$, and so $\Gamma_0^{(A,B^\infty)}(KK(A,q_B)(\kappa)) = \Gamma_0^{(A,B^\infty)}([\theta]_{KK(A,B^\infty)})$. The claim in (9.2) now follows as $\Gamma_0^{(A,B^\infty)}$ is an isomorphism by Proposition 8.6 (using that A satisfies the UCT).

Note also that $\Gamma_0^{(A,B_\infty)}(\kappa)([1_A]_0) = \alpha_0([1_A]_0) = [1_{B_\infty}]_0$, as $(\underline{\alpha}, \beta, \gamma)$ is a $\underline{K}T_u$-morphism. Therefore, the classification of unital lifts (Theorem 7.10(i)) implies the existence of a unital *-homomorphism $\psi\colon A \to B_\infty$ lifting θ and satisfying $[\psi]_{KK(A,B_\infty)} = \kappa$. Since ψ lifts θ and $\operatorname{Aff} T(q_B)$ is an isomorphism (by Proposition 6.3(i)), we have

$$\operatorname{Aff} T(\psi) = \gamma. \tag{9.4}$$

Likewise, since κ induces $\underline{\alpha}$, we have $\underline{K}(\psi) = \underline{\alpha}$. However, there is no reason why $\overline{K}_1^{\mathrm{alg}}(\psi)$ should be equal to β.

By Theorem 8.9(ii), there exists $\lambda \in \ker KL(A, j_B) \subseteq KL(A, J_B)$ such that[147]

$$\beta = \overline{K}_1^{\mathrm{alg}}(\psi) + R_{A,B}(\lambda) \circ t_A \circ \not{a}_A. \tag{9.5}$$

By the classification of unital lifts (Theorem 7.10(ii)), there exists a unital *-homomorphism $\phi\colon A \to B_\infty$ lifting θ with $[\phi, \psi]_{KL(A,J_B)} = \lambda$. Since this class lies in $\ker KL(A, j_B)$, we have $[\phi]_{KL(A,B_\infty)} = [\psi]_{KL(A,B_\infty)}$ by Proposition 5.7(iv). Therefore, by applying $\tilde{\Gamma}_\Lambda^{(A,B_\infty)}$, we have

$$\underline{K}(\phi) = \underline{K}(\psi) = \underline{\alpha}. \tag{9.6}$$

By Theorem 8.7, we have

$$\begin{aligned} \overline{K}_1^{\mathrm{alg}}(\phi) &= \overline{K}_1^{\mathrm{alg}}(\psi) + R_{A,B}([\phi, \psi]_{KL(A,J_B)}) \circ t_A \circ \not{a}_A \\ &= \overline{K}_1^{\mathrm{alg}}(\psi) + R_{A,B}(\lambda) \circ t_A \circ \not{a}_A \\ &\overset{(9.5)}{=} \beta. \end{aligned} \tag{9.7}$$

Finally, since ϕ lifts θ and $\mathrm{Aff}\, T(q_B)$ is an isomorphism, we have $\mathrm{Aff}\, T(\phi) = \gamma$. The three previous equations combine to show that $\underline{K}T_u(\phi) = (\underline{\alpha}, \beta, \gamma)$, completing the proof. □

Now we turn to uniqueness. The pairing maps $\zeta^{(n)}$ of Section 3.1 are not needed for this part of the theorem.

Proof of uniqueness in Theorem 9.1 when $T(B) \neq \emptyset$. Let $\phi, \psi\colon A \to B_\infty$ be two unital full *-homomorphisms such that $\underline{K}T_u(\phi) = \underline{K}T_u(\psi)$. Then, in particular, $\mathrm{Aff}\, T(\phi) = \mathrm{Aff}\, T(\psi)$, and we have

$$\mathrm{Aff}\, T(q_B \circ \phi) = \mathrm{Aff}\, T(q_B \circ \psi)\colon\ \mathrm{Aff}\, T(A) \to \mathrm{Aff}\, T(B^\infty). \tag{9.8}$$

Hence $q_B \circ \phi$ and $q_B \circ \psi$ are unitarily equivalent by the classification of maps into B^∞ by traces (Theorem 6.5). Since every unitary in B^∞ can be lifted to a unitary in B_∞ by Proposition 6.6, we have that $q_B \circ \phi = q_B \circ \mathrm{Ad}(u) \circ \psi$ for some unitary $u \in B_\infty$. By replacing ψ with $\mathrm{Ad}(u) \circ \psi$ (which leaves $\underline{K}T_u(\,\cdot\,)$ unchanged — see Proposition 3.6), we may assume that $q_B \circ \phi = q_B \circ \psi$. Therefore, (ϕ, ψ) forms an (A, J_B)-Cuntz pair, and defines an element $[\phi, \psi]_{KL(A,J_B)}$ in $KL(A, J_B)$.

Letting Θ be as in Theorem 8.9(i), we have

$$\begin{aligned} \Theta([\phi, \psi]_{KL(A,J_B)}) &\overset{\text{Thm. 8.9(i)}}{=} \big(\tilde{\Gamma}_\Lambda^{(A,B_\infty)} \circ KL(A, j_B)\big)([\phi, \psi]_{KL(A,J_B)}) \\ &\overset{\text{Prop. 5.7(iv)}}{=} \tilde{\Gamma}_\Lambda^{(A,B_\infty)}([\phi]_{KL(A,B_\infty)} - [\psi]_{KL(A,B_\infty)}) \\ &\overset{(8.13)}{=} \underline{K}(\phi) - \underline{K}(\psi) \\ &= 0. \end{aligned} \tag{9.9}$$

This shows that $[\phi, \psi]_{KL(A,J_B)} \in \ker \Theta = \ker KL(A, j_B)$.

Since $\overline{K}_1^{\mathrm{alg}}(\phi) = \overline{K}_1^{\mathrm{alg}}(\psi)$ and both t_A and $\not{a}_A$ are surjective, it follows by Theorem 8.9 that $R_{A,B}([\phi, \psi]_{KL(A,J_B)}) = 0$. Since $R_{A,B}$ is injective (by Theorem 8.9), we conclude that $[\phi, \psi]_{KL(A,J_B)} = 0$.

Hence by the classification of unital lifts (Theorem 7.10(iii)), it follows that ϕ and ψ are unitarily equivalent. □

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[147]It is in Theorem 8.9(ii), and the application to Lemma 3.11 within, that we use the fact that β is compatible with the maps $\zeta^{(n)}$.

9.2. **Classification of embeddings.** Next we use the classification of unital full approximate embeddings to prove the classification of embeddings theorem (Theorem B). Whereas uniqueness up to approximate unitary equivalence in Theorem B is an immediate consequence of the uniqueness up to unitary equivalence in Theorem 9.1, an intertwining argument is needed for existence. In the literature (such as [176, 182, 125, 126]), this intertwining argument is often applied in an approximate form, by reformulating the approximate classification theorem in terms of approximately multiplicative maps, requiring the use of approximately defined maps on the invariant. In contrast, we opt for an approach that uses intertwining by reparameterization to identify those maps $A \to B_\infty$ that are unitarily equivalent to maps factoring through B. To the best of our knowledge, the idea first appears (albeit in a continuous form) in Phillip's approach to the Kirchberg–Phillips theorem in [221, Proposition 1.3.7], where it is attributed to Kirchberg.

To set this up, given a function $r\colon \mathbb{N} \to \mathbb{N}$ define a reparameterization map $r^*\colon \ell^\infty(B) \to \ell^\infty(B)$ by $r^*((b_n)_{n=1}^\infty) := (b_{r(n)})_{n=1}^\infty$. When $\lim_{n\to\infty} r(n) = \infty$, this also induces a map $B_\infty \to B_\infty$, which we continue to denote by r^*. Recall from (3.44) that $\iota_B\colon B \to B_\infty$ denotes the canonical embedding.

Proposition 9.2 (Intertwining by reparameterization, [113, Theorem 4.3])**.** *Let A and B be C^*-algebras, with A separable and B unital. Let $\phi_\infty\colon A \to B_\infty$ be a *-homomorphism. There exists a *-homomorphism $\phi\colon A \to B$ such that ϕ_∞ and $\iota_B \circ \phi$ are unitarily equivalent if and only if for every $r\colon \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} r(n) = \infty$, the maps ϕ_∞ and $r^* \circ \phi_\infty$ are unitarily equivalent.*

With this additional ingredient in place we can prove the classification of embeddings (Theorem B).

Theorem 9.3 (Classification of embeddings)**.** *Let A and B be unital separable nuclear C^*-algebras such that A satisfies the UCT and B is simple and $\mathcal{Z}$-stable. If $(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B)$ is a faithful morphism, then there exists a unital injective *-homomorphism $\phi\colon A \to B$ such that $\underline{K}T_u(\phi) = (\underline{\alpha}, \beta, \gamma)$, and this ϕ is unique up to approximate unitary equivalence.*

Proof. Uniqueness is essentially an immediate consequence of Theorem 9.1. Given unital injective *-homomorphisms $\phi, \psi\colon A \to B$, the compositions $\iota_B \circ \phi, \iota_B \circ \psi\colon A \to B_\infty$ are full as B is simple. Accordingly, if $\underline{K}T_u(\phi) = \underline{K}T_u(\psi)$, then $\underline{K}T_u(\iota_B \circ \phi) = \underline{K}T_u(\iota_B \circ \psi)$ and so $\iota_B \circ \phi$ and $\iota_B \circ \psi$ are unitarily equivalent by Theorem 9.1. As unitaries in B_∞ can be lifted to unitaries in $\ell^\infty(B)$, ϕ and ψ are approximately unitarily equivalent.

For existence, consider a faithful $\underline{K}T_u$-morphism

$$(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B). \tag{9.10}$$

Then $\underline{K}T_u(\iota_B) \circ (\underline{\alpha}, \beta, \gamma)$ is also faithful, so the existence part of Theorem 9.1 gives a unital full *-homomorphism $\phi_\infty\colon A \to B_\infty$ such that

$$\underline{K}T_u(\phi_\infty) = \underline{K}T_u(\iota_B) \circ (\underline{\alpha}, \beta, \gamma). \tag{9.11}$$

Fix $r\colon \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} r(n) = \infty$. Since $r^* \circ \iota_B = \iota_B$, it follows that

$$\underline{K}T_u(r^* \circ \phi_\infty) = \big(\underline{K}T_u(r^*) \circ \underline{K}T_u(\iota_B)\big)(\underline{\alpha}, \beta, \gamma) = \underline{K}T_u(\phi_\infty). \tag{9.12}$$

Moreover, $r^* \circ \phi_\infty$ is full as r^* is unital and ϕ_∞ is full. Therefore, $r^* \circ \phi_\infty$ and ϕ_∞ are unitarily equivalent by the uniqueness portion of Theorem 9.1. Then intertwining by reparameterizations (Proposition 9.2) gives a *-homomorphism $\phi\colon A \to B$ such that $\iota_B \circ \phi$ and ϕ_∞ are unitarily equivalent. As ϕ_∞ is unital and faithful, so is ϕ. Note that

$$\underline{K}T_u(\iota_B) \circ \underline{K}T_u(\phi) = \underline{K}T_u(\iota_B \circ \phi) \overset{\text{Prop. 3.6}}{=} \underline{K}T_u(\phi_\infty) = \underline{K}T_u(\iota_B) \circ (\underline{\alpha}, \beta, \gamma). \tag{9.13}$$

By Lemma 3.16, $\underline{K}T_u(\iota_B)$ is a monomorphism, so $\underline{K}T_u(\phi) = (\underline{\alpha}, \beta, \gamma)$. ☐

Remark 9.4. We use sequence algebras B_∞ rather that ultrapowers B_ω in our classification of full approximate embeddings (Theorem 1.1) in order to access the intertwining by reparameterization technique; reparameterizations do not make sense for ultrapowers. Prompted by the discrepancy between sequence algebras and ultrapowers, Farah proved that an existence result into the ultrapower is equivalent to an existence result into the sequence algebra ([107, Theorem A]).

Corollary C is an immediate consequence of the classification of embeddings.

Corollary 9.5. *Let A and B be unital separable nuclear C^*-algebras with nonempty tracial state spaces, and suppose that A satisfies the UCT and that B is simple and $\mathcal{Z}$-stable. Given a faithful unital compatible pair*

$$(\alpha_*, \gamma)\colon (K_*(A), \operatorname{Aff} T(A)) \to (K_*(B), \operatorname{Aff} T(B)), \tag{9.14}$$

there is a unital embedding $\phi\colon A \to B$ such that $K_*(\phi) = \alpha_*$ and $\operatorname{Aff} T(\phi) = \gamma$.

Proof. By Proposition 2.10, we can extend the pair (α_*, γ) to a compatible triple $(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B)$. The classification of embeddings theorem (Theorem 9.3) then gives a unital embedding $\phi\colon A \to B$ with $\underline{K}T_u(\phi) = (\underline{\alpha}, \beta, \gamma)$. Hence $K_*(\phi) = \alpha_*$ and $\operatorname{Aff} T(\phi) = \gamma$. ☐

Note that in the previous corollary the embedding ϕ is not necessarily unique up to approximate unitary equivalence. In general, uniqueness requires the total invariant. See Example 9.11 below.

Remark 9.6. Let us note how Theorem 9.3 relates to other classification results for embeddings. In [197, Corollary 6.8 and Theorem 7.1], Matui proves uniqueness results for embeddings $A \to B$, where A is either a unital AH algebra (including $C(X)$) or a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra that is rationally TAF and satisfies the UCT. Matui allows unital simple separable stably finite $\mathcal{Z}$-stable codomains B that are either rationally TAF or exact with finitely many extreme traces that are separated by projections. For comparing maps $\phi, \psi\colon A \to B$, Matui uses an invariant consisting of $\underline{K}(\,\cdot\,)$, $\operatorname{Aff} T(\,\cdot\,)$ and (vanishing of) an induced homomorphism $\Theta_{\phi,\psi}\colon K_1(A) \to \operatorname{Aff} T(B)/\overline{\operatorname{im} \rho_B}$ given by

$$\Theta_{\phi,\psi}([u]_1) = \det{}_B(\phi(u)^*\psi(u)), \qquad u \in U_\infty(A).^{148} \tag{9.15}$$

Using this, one can compute that $\overline{\mathrm{Th}}_B \circ \Theta_{\phi,\psi} \circ \not{a}_A = \overline{K}_1^{\mathrm{alg}}(\psi) - \overline{K}_1^{\mathrm{alg}}(\phi)$ so (since $\not{a}_A$ is surjective and $\overline{\mathrm{Th}}_B$ is injective) $\Theta_{\phi,\psi}$ vanishes if and only if $\overline{K}_1^{\mathrm{alg}}(\phi) = \overline{K}_1^{\mathrm{alg}}(\psi)$. Note that this map Θ coincides with the map $\tilde{R}_{\phi,\psi}$ of Remark 8.10 in the case $[\phi]_{KK(A,B)} = [\psi]_{KK(A,B)}$ (so that $\tilde{R}_{\phi,\psi}$ is defined). Accordingly, in the case when the codomains are additionally assumed to be nuclear, Matui's uniqueness theorems are encompassed within Theorem 9.3 above.

[148] The well-definedness of $\Theta_{\phi,\psi}$ requires $K_1(\phi) = K_1(\psi)$ and $T(\phi) = T(\psi)$; see [197, Lemma 3.1].

While this paper was being prepared, Gong, Lin, and Niu independently obtained a classification of embeddings ([124, Theorems 4.3 and 6.5]), building on prior work of Lin and Niu in the rationally tracial rank one setting ([190]). Compared with Theorem 9.3, the main difference in their result is a simplicity hypothesis on the domain algebra.[149]

Whereas our approach establishes Theorem 9.3 as an ingredient towards classification of C^*-algebras, Gong, Lin, and Niu use their earlier classification theorems ([125, 126]) and the additional simplicity hypothesis on A to obtain internal tracial approximation structure on its UHF-stabilizations. They also use a variation of Winter's localization technique.

9.3. **The unital classification theorem.** The stably finite half of the unital classification theorem is now obtained by the standard strategy of symmetrizing the assumptions on the domain and codomain in the classification of embeddings and using the following form of Elliott's two-sided intertwining argument (see [240, Corollary 2.3.4], for example).

Proposition 9.7 (Elliott's two-sided intertwining)**.** *Let A and B be unital separable C^*-algebras, and let $\phi_0\colon A \to B$ and $\psi_0\colon B \to A$ be *-homomorphisms. Suppose that $\psi_0 \circ \phi_0$ is approximately unitarily equivalent to id_A and $\phi_0 \circ \psi_0$ is approximately unitarily equivalent to id_B. Then ϕ_0 is approximately unitarily equivalent to an isomorphism between A and B.*

When A and B are as in the unital classification theorem (Theorem A), an isomorphism of the total invariant can be lifted to an isomorphism of the algebra.

Theorem 9.8. *Let A and B be unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebras satisfying the UCT. Suppose*

$$(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B) \tag{9.16}$$

is a $\underline{K}T_u$-isomorphism. Then there exists a *-isomorphism $\phi\colon A \to B$, unique up to approximate unitary equivalence, such that $\underline{K}T_u(\phi) = (\underline{\alpha}, \beta, \gamma)$.

Proof. Uniqueness follows from the uniqueness portion of the classification of embeddings theorem (Theorem 9.3), noting that faithfulness of the invariant is automatic since all traces on A are faithful.

The existence portion of the classification of embeddings theorem (Theorem 9.3) gives unital *-homomorphisms $\phi_0\colon A \to B$ and $\psi_0\colon A \to B$ such that $\underline{K}T_u(\phi_0) = (\underline{\alpha}, \beta, \gamma)$ and $\underline{K}T_u(\psi_0) = \underline{K}T_u(\phi_0)^{-1}$. By the uniqueness aspect of the classification of embeddings theorem (Theorem 9.3), $\phi_0 \circ \psi_0$ is approximately unitarily equivalent to id_B and $\psi_0 \circ \phi_0$ is approximately unitarily equivalent to id_A. Therefore, Elliott's two-sided intertwining argument (Proposition 9.7) provides a *-isomorphism $\phi\colon A \to B$ which is approximately unitarily equivalent to ϕ_0. Hence $\underline{K}T_u(\phi) = \underline{K}T_u(\phi_0) = (\underline{\alpha}, \beta, \gamma)$ since $\underline{K}T_u$ is invariant under approximate unitary equivalence (Proposition 3.6). ☐

We now establish Theorem A as a consequence.

[149]The codomain hypotheses differ in their existence and uniqueness theorem but largely overlap with our codomain hypotheses.

Theorem 9.9. *Let A and B be unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebras satisfying the UCT. Suppose*

$$(\alpha_*, \gamma)\colon (K_*(A), \mathrm{Aff}\, T(A)) \to (K_*(B), \mathrm{Aff}\, T(B)) \tag{9.17}$$

is a KT_u-isomorphism. Then there exists a *-isomorphism $\phi\colon A \to B$ such that $K_*(\phi) = \alpha_*$ and $\mathrm{Aff}\, T(\phi) = \gamma$.

Proof. By Proposition 2.10, there is a unital faithful invertible compatible triple $(\underline{\alpha}, \beta, \gamma)\colon \underline{K}T_u(A) \to \underline{K}T_u(B)$ extending (α_*, γ). By Proposition 9.8, there is an isomorphism $\phi\colon A \to B$ with $\underline{K}T_u(\phi) = (\underline{\alpha}, \beta, \gamma)$. Hence $K_*(\phi) = \alpha_*$ and $\mathrm{Aff}\, T(\phi) = \gamma$. □

The classification of automorphisms modulo approximate unitary equivalence has been well-studied in operator algebras. For example, Connes' theorem builds on his prior work on this problem for II_1 factors (e.g., [45, 43]). Given a unital C^*-algebra A, write $\mathrm{Aut}(A)$ for the automorphism group of A, and $\overline{\mathrm{Inn}(A)}$ for the subgroup of approximately inner automorphisms. When A is a unital Kirchberg C^*-algebra satisfying the UCT, the quotient group $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ is isomorphic to the group of unital automorphisms of $\underline{K}(A)$ (cf. [104, Theorem 5.6]). For unital finite classifiable C^*-algebras, unital automorphisms of the total invariant encode this information.

Corollary 9.10. *Let A be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra satisfying the universal coefficient theorem. Then the quotient $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ is isomorphic to the group of automorphisms of $\underline{K}T_u(A)$.*

Proof. This follows immediately from Theorem 9.8. □

We found the following Bernoulli shift example, where the $\overline{K}_1^{\mathrm{alg}}$-class of an automorphism is particularly discernible, informative during our work on this paper.

Example 9.11 (Automorphisms of $\mathcal{Z} \wr \mathbb{Z}$)**.** Consider the infinite tensor product $\mathcal{Z}^{\otimes\mathbb{Z}}$, let $\mathbb{Z}$ act on this algebra by shifting, and let $A \coloneqq \mathcal{Z}^{\otimes\mathbb{Z}} \rtimes \mathbb{Z}$ denote the crossed product. As we review below, it is well-known that such a construction gives rise to a C^*-algebra covered by Theorem A.

It is immediate that A is separable and unital. Since $\mathcal{Z}^{\otimes\mathbb{Z}}$ is isomorphic to $\mathcal{Z}$ ([144, Corollary 8.8]), it is nuclear and satisfies the UCT. Both properties are preserved by crossed products by $\mathbb{Z}$, and therefore A is also nuclear and satisfies the UCT. The shift action on $\mathcal{R}^{\bar{\otimes}\mathbb{Z}}$ is outer,[150] which means that the shift action on $\mathcal{Z}^{\otimes\mathbb{Z}}$ is strongly outer. Therefore, A is simple by [169, Theorem 3.1], $\mathcal{Z}$-stable by [199, Corollary 4.10], and has unique trace by [266, Theorem 4.3(4)⇒(1)] (cf. [279, Theorem 1.11]).

Now we turn to the invariant of A. Using $K_0(\mathcal{Z}) \cong \mathbb{Z}$ with unit corresponding to $1 \in \mathbb{Z}$ and $K_1(\mathcal{Z}) = 0$, the Pimsner–Voiculescu sequence ([224]) implies $K_0(A) \cong \mathbb{Z}$ with the unit corresponding to $1 \in \mathbb{Z}$ and $K_1(A) \cong \mathbb{Z}$. Combining this with the fact that A has unique trace, we may therefore (by (2.10)) identify

$$\overline{K}_1^{\mathrm{alg}}(A) \cong \mathbb{T} \oplus \mathbb{Z}, \tag{9.18}$$

and so (as explained below) Corollary 9.10 gives

$$\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \cong \mathbb{T} \rtimes \mathbb{Z}/2. \tag{9.19}$$

[150] This is well-known; a proof can be found in [217, Lemma 2.5].

To do this concretely, let $u \in A$ denote the canonical unitary implementing the group action. Then $[u]_{\mathrm{alg}}$ is a generator of the copy of $\mathbb{Z}$. We get a group monomorphism from $\mathbb{T}$ to $\overline{K}_1^{\mathrm{alg}}(A)$ given by

$$z \mapsto [z1_A]_{\mathrm{alg}}, \tag{9.20}$$

giving a complementary copy of $\mathbb{T}$.

Since $K_*(A)$ is torsion-free, we have, by (2.35), that $K_i(A;\mathbb{Z}/n) \cong K_i(A) \otimes \mathbb{Z}/n = \mathbb{Z}/n$ for all n. We also see that for an automorphism ϕ of A, its action on K-theory determines its action on total K-theory; since ϕ also fixes $[1_A]_0$, $\underline{K}T_u(\phi)$ is entirely determined by $\overline{K}_1^{\mathrm{alg}}(\phi)$.

An automorphism of $\overline{K}_1^{\mathrm{alg}}(\phi)$ is compatible with the Thomsen map Th_A if and only if it acts as the identity on $\mathbb{T}$. Hence, by Corollary 9.10, $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ identifies with the group of automorphisms of $\overline{K}_1^{\mathrm{alg}}(A) \cong \mathbb{T} \oplus \mathbb{Z}$ that fix $\mathbb{T}$. Each such automorphism is determined by its action on the generator $[u]_{\mathrm{alg}}$ of the copy of $\mathbb{Z}$. This group of automorphisms is $\mathbb{T} \rtimes \mathbb{Z}/2$, where an element $z \in \mathbb{T}$ corresponds to the automorphism α_z taking $[u]_{\mathrm{alg}}$ to $[zu]_{\mathrm{alg}}$, and the generator of $\mathbb{Z}/2$ corresponds to the automorphism ω taking $[u]_{\mathrm{alg}}$ to $-[u]_{\mathrm{alg}}$.

We can find explicit representatives for α_z and ω as follows:

(i) For $z \in \mathbb{T}$, let $\phi_z \colon A \to A$ denote the gauge action. In other words, ϕ_z fixes $\mathcal{Z}^{\otimes\mathbb{Z}}$ and

$$\phi_z(u) := zu. \tag{9.21}$$

Then ϕ_z induces the automorphism α_z.

(ii) Let $\psi \colon A \to A$ be the *-homomorphism determined by

$$\begin{aligned} \psi(u) &:= u^*, \\ \psi(\cdots \otimes a_{-1} \otimes a_0 \otimes a_1 \otimes \cdots) &:= \cdots \otimes a_1 \otimes a_0 \otimes a_{-1} \otimes \cdots , \end{aligned} \tag{9.22}$$

where $\cdots \otimes a_{-1} \otimes a_0 \otimes a_1 \otimes \cdots$ denotes an elementary tensor in $\mathcal{Z}^{\otimes\mathbb{Z}}$. Then we can see that $[\psi(u)]_{\mathrm{alg}} = [u^*]_{\mathrm{alg}} = -[u]_{\mathrm{alg}}$, so ψ induces the automorphism ω.

Since the gauge actions act as the identity on $KT_u(A)$, their non-innerness is not detected by that invariant.

Interestingly, the automorphisms ϕ_z and ψ just described generate a copy of $\mathbb{T} \rtimes \mathbb{Z}/2$ inside $\mathrm{Aut}(A)$ (that is, without taking the quotient by $\overline{\mathrm{Inn}(A)}$). Consequently,

$$\mathrm{Aut}(A) \cong \overline{\mathrm{Inn}(A)} \rtimes (\mathbb{T} \rtimes \mathbb{Z}/2) \cong \overline{\mathrm{Inn}(A)} \rtimes \mathrm{Aut}(\underline{K}T_u(A). \tag{9.23}$$

In particular, if G is a discrete group acting on $\underline{K}T_u(A)$, then the action lifts to an action on A.

Equipped with the classification of automorphisms modulo approximately inner automorphisms (Corollary 9.10), the following question becomes pertinent.

Question 9.12. Let A be a classifiable C^*-algebra (that is, one satisfying the hypotheses of Theorem A) and let G be a discrete group. When does a group action $G \curvearrowright \underline{K}T_u(A)$ lift to an action $G \curvearrowright A$?

For finite cyclic groups acting on AF algebras, this question is already open (see [10, Exercise 10.11.3]). State-of-the-art results for Kirchberg algebras can be found in [155] — in particular, Katsura shows that actions of $\mathbb{Z}/n$ on $KT_u(A)$ lift to actions on A when A is a unital Kirchberg algebra satisfying the UCT. Combining Corollary 9.10 with a result of Barlak and Szabó ([7, Corollary 2.7]), an interesting case of this problem in the stably finite case can be obtained as follows.

Corollary 9.13. *If A is a unital separable simple nuclear C^*-algebra, G is a finite group, and $A \otimes M_{|G|^\infty} \cong A$, then every group action $G \curvearrowright \underline{K}T_u(A)$ lifts to an action $G \curvearrowright A$.*

In particular, Question 9.12 has a positive answer when A is $\mathcal{Q}$-stable and G is finite.

Appendix A. Total K-theory

In this appendix we collect some technical facts regarding total K-theory that are not easily available in the literature.

A.1. **Two definitions of $\mathbb{Z}/n$-coefficients.**

Our first objective is to give a direct proof of the following proposition which relates the two pictures of total K-theory we use in this paper. To the best of our knowledge, the only proof in the literature is found in the recent paper [150, Theorem 8.1], where it is derived from a Spanier–Whitehead duality theory developed therein.

Proposition A.1. *There exists a natural isomorphism between the two functors $K_0(\,\cdot\,;\mathbb{Z}/n) := K_1(\mathbb{I}_n \otimes \,\cdot\,)$ and $KK(\mathbb{I}_n, \,\cdot\,)$ (defined on the category of separable C^*-algebras).*

Before proving the proposition, we isolate some notation. For separable C^*-algebras C, D, and E there is a natural map[151]

$$\mathcal{T}_E^{(C,D)} \colon KK(C,D) \to KK(E \otimes C, E \otimes D), \tag{A.1}$$

defined by

$$\mathcal{T}_E^{(C,D)}\big([\phi,\psi]_{KK(C,D)}\big) = [\mathrm{id}_E \otimes \phi, \mathrm{id}_E \otimes \psi]_{KK(E\otimes C, E\otimes D)} \tag{A.2}$$

for a Cuntz pair $(\phi,\psi)\colon C \rightrightarrows \mathcal{M}(D\otimes\mathcal{K}) \rhd D \otimes \mathcal{K}$. The following standard lemma is a KK-theoretic analogue of the fact that that $\theta \otimes \mathrm{id}_C$ and $\mathrm{id}_E \otimes \phi$ commute for *-homomorphisms $\theta\colon E \to F$ and $\phi\colon C \to D$.[152]

Lemma A.2. *Let C, D, E and F be separable C^*-algebras and let $\theta\colon E \to F$ be a *-homomorphism. Then the diagram*

$$\begin{tikzcd} KK(C,D) \arrow[r, "\mathcal{T}_E^{(C,D)}"] \arrow[d, "\mathcal{T}_F^{(C,D)}"] & KK(E\otimes C, E\otimes D) \arrow[d, "KK(E\otimes C, \theta\otimes \mathrm{id}_D)"] \\ KK(F\otimes C, F\otimes D) \arrow[r, "KK(\theta\otimes\mathrm{id}_C, F\otimes D)"] & KK(E\otimes C, F\otimes D) \end{tikzcd} \tag{A.3}$$

commutes.

[151] Here, and throughout this section, all tensor products are spatial.

[152] To see the analogy, consider the effect of the diagram (A.3) on $[\phi]_{KK(C,D)}$.

Proof. Without loss of generality, we may assume D and F are stable. By results of Thomsen, after replacing θ with a homotopic *-homomorphism, we may assume that θ extends to a strictly continuous map $\mathcal{M}(\theta)\colon \mathcal{M}(E) \to \mathcal{M}(F)$, which also ensures that $\theta \otimes \mathrm{id}_D\colon E \otimes D \to F \otimes D$ extends to a strictly continuous map $\mathcal{M}(\theta \otimes \mathrm{id}_D)\colon \mathcal{M}(E \otimes D) \to \mathcal{M}(F \otimes D)$.[153] For a Cuntz pair $(\phi, \psi)\colon C \rightrightarrows \mathcal{M}(D) \rhd D$, there is a commutative diagram

$$\begin{array}{ccccc} E \otimes C & \overset{\mathrm{id}_E\otimes\phi}{\underset{\mathrm{id}_E\otimes\psi}{\rightrightarrows}} & \mathcal{M}(E \otimes D) & \rhd & E \otimes D \\ \big\downarrow{\scriptstyle \theta\otimes\mathrm{id}_C} & & \big\downarrow{\scriptstyle \mathcal{M}(\theta\otimes\mathrm{id}_D)} & & \big\downarrow{\scriptstyle \theta\otimes\mathrm{id}_D} \\ F \otimes C & \overset{\mathrm{id}_F\otimes\phi}{\underset{\mathrm{id}_F\otimes\psi}{\rightrightarrows}} & \mathcal{M}(F \otimes D) & \rhd & F \otimes D, \end{array} \tag{A.4}$$

and the result follows. $\square$

Proof of Proposition A.1. It suffices to show $K_0(\mathbb{I}_n \otimes\, .\,) \cong KK^1(\mathbb{I}_n, \,\cdot\,)$ as the result follows by taking suspensions. By Bott periodicity and the stability of KK-theory, it is enough to show

$$KK(\mathbb{C}, \mathbb{I}_n \otimes \,\cdot\,) \cong KK(\mathbb{I}_n, SM_n \otimes \,\cdot\,). \tag{A.5}$$

This is what we will show.

Recall that for $n \geq 2$,

$$\mathbb{I}_n \coloneqq \{f \in C([0,1], M_n) : f(0) \in \mathbb{C}1_{M_n}, f(1) = 0\}. \tag{A.6}$$

Let $\mu_n\colon SM_n \to \mathbb{I}_n$ and $\nu_n\colon \mathbb{I}_n \to \mathbb{C}$ denote the canonical *-homomorphisms given by the inclusion and evaluation at 0, respectively. Define

$$\theta\colon \mathbb{I}_n \otimes \mathbb{I}_n \longrightarrow SM_n \tag{A.7}$$

to be the *-homomorphism given on elementary tensors by

$$\theta(f \otimes g)(t) \coloneqq \begin{cases} f(1-2t)\nu_n(g), & t \in [0, \frac{1}{2}]; \\ g(2t-1)\nu_n(f), & t \in [\frac{1}{2}, 1] \end{cases} \tag{A.8}$$

for $f, g \in \mathbb{I}_n$ and $t \in [0,1]$. Define a natural transformation

$$\Omega\colon KK(\mathbb{C}, \mathbb{I}_n \otimes \,\cdot\,) \to KK(\mathbb{I}_n, SM_n \otimes \,\cdot\,) \tag{A.9}$$

by

$$\Omega_D \coloneqq KK(\mathbb{I}_n, \theta \otimes \mathrm{id}_D) \circ \mathcal{T}_{\mathbb{I}_n}^{(\mathbb{C}, \mathbb{I}_n \otimes D)} \tag{A.10}$$

for a separable C^*-algebra D. We will show Ω_D is an isomorphism.

The composition $\theta \circ (\mathrm{id}_{\mathbb{I}_n} \otimes \mu_n)\colon \mathbb{I}_n \otimes SM_n \to SM_n$ is given on elementary tensors by

$$\theta\big((\mathrm{id}_{\mathbb{I}_n} \otimes \mu_n)(f \otimes g)\big)(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}]; \\ g(2t-1)\nu_n(f), & t \in [\frac{1}{2}, 1] \end{cases} \tag{A.11}$$

[153]By [142, Lemma 1.3.19], the stability of B implies θ is homotopic to a quasi-unital *-homomorphism ([142, Definition 1.3.13]). Replace θ by this quasi-unital *-homomorphism. Then for an approximate unit for $E \otimes D$ of the form $(e_\lambda \otimes d_\lambda)$, we have $(\theta \otimes \mathrm{id}_D)(e_\lambda \otimes d_\lambda) \to \mathcal{M}(\theta)(1_{\mathcal{M}(E)}) \otimes 1_{\mathcal{M}(D)}$ strictly in $\mathcal{M}(E \otimes D)$. It follows that $\overline{(\theta \otimes \mathrm{id}_D)(E \otimes D)(F \otimes D)} = \mathcal{M}(\theta)(1_{\mathcal{M}(E)} \otimes 1_{\mathcal{M}(D)})(F \otimes D)$, so that $\theta \otimes \mathrm{id}_D$ has the specified strictly continuous extension to $\mathcal{M}(\theta \otimes \mathrm{id}_D)\colon \mathcal{M}(E \otimes D) \to \mathcal{M}(F \otimes D)$ by [142, Corollary 1.1.15]. These results are originally found in [264].

for $f \in \mathbb{I}_n$, $g \in SM_n$, and $t \in [0,1]$. It follows that $\theta \circ (\mathrm{id}_{\mathbb{I}_n} \otimes \mu_n)$ is homotopic to

$$\nu_n \otimes \mathrm{id}_{SM_n} \colon \mathbb{I}_n \otimes SM_n \longrightarrow \mathbb{C} \otimes SM_n = SM_n. \tag{A.12}$$

In particular, if D is a separable C^*-algebra, then homotopy invariance of KK-theory implies

$$KK(\mathbb{I}_n, (\theta \circ (\mathrm{id}_{\mathbb{I}_n} \otimes \mu_n)) \otimes \mathrm{id}_D) = KK(\mathbb{I}_n, \nu_n \otimes \mathrm{id}_{SM_n} \otimes \mathrm{id}_D). \tag{A.13}$$

For any separable C^*-algebra D, we get

$$\begin{aligned} &\Omega_D \circ KK(\mathbb{C}, \mu_n \otimes \mathrm{id}_D) \\ &\overset{(A.10)}{=} KK(\mathbb{I}_n, \theta \otimes \mathrm{id}_D) \circ \mathcal{T}_{\mathbb{I}_n}^{(\mathbb{C}, \mathbb{I}_n \otimes D)} \circ KK(\mathbb{C}, \mu_n \otimes \mathrm{id}_D) \\ &= KK(\mathbb{I}_n, \theta \otimes \mathrm{id}_D) \circ KK(\mathbb{I}_n, \mathrm{id}_{\mathbb{I}_n} \otimes \mu_n \otimes \mathrm{id}_D) \circ \mathcal{T}_{\mathbb{I}_n}^{(\mathbb{C}, SM_n \otimes D)} \\ &= KK(\mathbb{I}_n, (\theta \circ (\mathrm{id}_{\mathbb{I}_n} \otimes \mu_n)) \otimes \mathrm{id}_D) \circ \mathcal{T}_{\mathbb{I}_n}^{(\mathbb{C}, SM_n \otimes D)} \\ &\overset{(A.13)}{=} KK(\mathbb{I}_n, \nu_n \otimes \mathrm{id}_{SM_n} \otimes \mathrm{id}_D) \circ \mathcal{T}_{\mathbb{I}_n}^{(\mathbb{C}, SM_n \otimes D)} \\ &\overset{\text{Lem. A.2}}{=} KK(\mathrm{id}_{\mathbb{C}} \otimes \nu_n, \mathbb{C} \otimes SM_n \otimes D) \circ \mathcal{T}_{\mathbb{C}}^{(\mathbb{C}, SM_n \otimes D)} \\ &= KK(\nu_n, SM_n \otimes D), \end{aligned} \tag{A.14}$$

using the naturality of $\mathcal{T}_{\mathbb{I}_n}^{(\mathbb{C}, \,\cdot\,)}$ for the second inequality and the functoriality of $KK(\mathbb{I}_n, \,\cdot\,)$ for the third. It follows that that the diagram

$$\begin{array}{ccc} KK(\mathbb{C}, SM_n \otimes D) & = & KK(\mathbb{C}, SM_n \otimes D) \\ \big\downarrow{\scriptstyle \times n} & & \big\downarrow{\scriptstyle \times n} \\ KK(\mathbb{C}, SM_n \otimes D) & = & KK(\mathbb{C}, SM_n \otimes D) \\ \big\downarrow{\scriptstyle KK(\mathbb{C}, \mu_n \otimes \mathrm{id}_D)} & & \big\downarrow{\scriptstyle KK(\nu_n SM_n \otimes D)} \\ KK(\mathbb{C}, \mathbb{I}_n \otimes D) & \xrightarrow{\Omega_D} & KK(\mathbb{I}_n, SM_n \otimes D) \end{array} \tag{A.15}$$

commutes.

Similarly, $\theta \circ (\mu_n \otimes \mathrm{id}_{\mathbb{I}_n})$ is homotopic to

$$\sigma \otimes \nu_n \colon SM_n \otimes \mathbb{I}_n \to SM_n \otimes \mathbb{C} = SM_n, \tag{A.16}$$

where $\sigma \colon SM_n \to SM_n$ is the *-homomorphism $\sigma(f)(t) \coloneqq f(1-t)$ for $t \in [0,1]$ and $f \in SM_n$. This yields

$$KK(SM_n, (\theta \circ (\mu_n \otimes \mathrm{id}_{\mathbb{I}_n})) \otimes \mathrm{id}_D) = KK(SM_n, \sigma \otimes \nu_n \otimes \mathrm{id}_D) \tag{A.17}$$

for every separable C^*-algebra D. Further, since $KK(SM_n, \sigma) = -\mathrm{id}_{KK(SM_n, SM_n)}$, Bott periodicity implies that tensoring with σ induces a natural isomorphism in KK-theory; i.e.,

$$\eta^{(C,D)} \coloneqq KK(SM_n \otimes C, \sigma \otimes \mathrm{id}_D) \circ \mathcal{T}_{SM_n}^{(C,D)} \tag{A.18}$$

is an isomorphism $KK(C, D) \xrightarrow{\cong} KK(SM_n \otimes C, SM_n \otimes D)$ for all C^*-algebras C and D.

Fix a separable C^*-algebra D. We compute

(A.19)

$$
\begin{aligned}
&KK(\mu_n, SM_n \otimes D) \circ \Omega_D \\
&\overset{(A.10)}{=} KK(\mu_n, SM_n \otimes D) \circ KK(\mathbb{I}_n, \theta \otimes \mathrm{id}_D) \circ \mathcal{T}_{\mathbb{I}_n}^{(\mathbb{C}, \mathbb{I}_n \otimes D)} \\
&= KK(SM_n, \theta \otimes \mathrm{id}_D) \circ KK(\mu_n, \mathbb{I}_n \otimes \mathbb{I}_n \otimes D) \circ \mathcal{T}_{\mathbb{I}_n}^{(\mathbb{C}, \mathbb{I}_n \otimes D)} \\
&\overset{\text{Lem. A.2}}{=} KK(SM_n, \theta \otimes \mathrm{id}_D) \circ KK(SM_n, \mu_n \otimes \mathrm{id}_{\mathbb{I}_n} \otimes \mathrm{id}_D) \circ \mathcal{T}_{SM_n}^{(\mathbb{C}, \mathbb{I}_n \otimes D)} \\
&= KK(SM_n, (\theta \circ (\mu_n \otimes \mathrm{id}_{\mathbb{I}_n})) \otimes \mathrm{id}_D) \circ \mathcal{T}_{SM_n}^{(\mathbb{C}, \mathbb{I}_n \otimes D)} \\
&\overset{(A.17)}{=} KK(SM_n, \sigma \otimes \nu_n \otimes \mathrm{id}_D) \circ \mathcal{T}_{SM_n}^{(\mathbb{C}, \mathbb{I}_n \otimes D)} \\
&= KK(SM_n, \sigma \otimes \mathrm{id}_D) \circ KK(SM_n, \mathrm{id}_{SM_n} \otimes \nu_n \otimes \mathrm{id}_D) \circ \mathcal{T}_{SM_n}^{(\mathbb{C}, \mathbb{I}_n \otimes D)} \\
&= KK(SM_n, \sigma \otimes \mathrm{id}_D) \circ \mathcal{T}_{SM_n}^{(\mathbb{C}, D)} \circ KK(\mathbb{C}, \nu_n \otimes \mathrm{id}_D) \\
&\overset{(A.18)}{=} \eta^{(\mathbb{C}, D)} \circ KK(\mathbb{C}, \nu_n \otimes \mathrm{id}_D),
\end{aligned}
$$

where the second equality follows from bifunctoriality of $KK(\,\cdot\,,\,\cdot\,)$, the fourth and sixth equality follow from the functoriality of $KK(SM_n,\,\cdot\,)$, and the seventh equality follows from the naturality of $\mathcal{T}_{SM_n}^{(\mathbb{C},\,\cdot\,)}$. This implies

$$
\text{(A.20)} \qquad
\begin{tikzcd}
KK(\mathbb{C}, \mathbb{I}_n \otimes D) \arrow[r, "\Omega_D"] \arrow[d, "KK(\mathbb{C}, \nu_n \otimes \mathrm{id}_D)"] & KK(\mathbb{I}_n, SM_n \otimes D) \arrow[d, "KK(\mu_n, SM_n \otimes D)"] \\
KK(\mathbb{C}, D) \arrow[r, "\cong", "\eta^{(\mathbb{C},D)}"'] \arrow[d, "\times n"] & KK(SM_n, SM_n \otimes D) \arrow[d, "\times n"] \\
KK(\mathbb{C}, D) \arrow[r, "\cong", "\eta^{(\mathbb{C},D)}"'] & KK(SM_n, SM_n \otimes D)
\end{tikzcd}
$$

commutes.

When concatenating the left columns of (A.15) and (A.20), this becomes part of the six-term exact sequence (2.29) for the Bockstein maps, and thus this concatenation is exact. Similarly, concatenating the right columns gives an exact sequence, using the Puppe exact sequence in $KK(\,\cdot\,, SM_n \otimes D)$ applied to the diagonal inclusion $\mathbb{C} \to M_n$ (see [11, Theorem 19.4.3]).[154] By the five lemma it follows that Ω_A is an isomorphism. $\square$

A.2. **Two definitions of total K-theory.** In this section, we prove Proposition 8.4. This amounts to showing that for any two sets of Bockstein operations, the corresponding maps are integer multiples of each other. To that end, we consider an arbitrary collection of natural transformations

$$
\text{(A.21)} \qquad K_i(\,\cdot\,) \xrightarrow{\mu_i'^{(n)}} K_i(\,\cdot\,;\mathbb{Z}/n) \xrightarrow{\nu_i'^{(n)}} K_{1-i}(\,\cdot\,)
$$

[154] This follows by considering the six-term exact sequences in $KK(\,\cdot\,, SM_n \otimes D)$ corresponding to the rows of

$$
\begin{tikzcd}
0 \arrow[r] & SM_n \arrow[r] \arrow[d, equal] & \mathbb{I}_n \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & SM_n \arrow[r] & CM_n \arrow[r] & M_n \arrow[r] & 0.
\end{tikzcd}
$$

Using that the boundary map in the second row in an isomorphism, since the cone is contractible, we may identify the boundary map in the top row with multiplication by n.

and

$$K_i(\,\cdot\,;\mathbb{Z}/n) \xrightarrow{\kappa_i'^{(nm,n)}} K_i(\,\cdot\,;\mathbb{Z}/nm) \xrightarrow{\kappa_i'^{(n,nm)}} K_i(\,\cdot\,;\mathbb{Z}/n) \tag{A.22}$$

indexed by $i = 0, 1$ and $n, m \geq 2$.

Remark A.3. If natural transformations as above are only defined on the functors restricted to the category of separable C^*-algebras or σ-unital C^*-algebras, then they can canonically be extended to the functors defined on the category of all C^*-algebras since the functors K_i and $K_i(\,\cdot\,;\mathbb{Z}/n)$ preserve inductive limits. Hence we may assume, for example, that the Bockstein operations in [66] (which are only defined there for σ-unital C^*-algebras) are defined on all C^*-algebras.

Lemma A.4. *Given a collection of natural transformations as in* (A.21) *and* (A.22)*, suppose that $k\mu_i'^{(n)}$, $k\nu_i'^{(n)}$, $k\kappa_i'^{(nm,n)}$, and $k\kappa_i'^{(n,nm)}$ are non-zero for all $i = 0, 1$, $n, m \geq 2$, and $k = 1, \ldots, n-1$.*

(i) Each map $\mu_i'^{(n)}$, $\nu_i'^{(n)}$, $\kappa_i'^{(nm,n)}$, and $\kappa_i'^{(n,nm)}$ and its corresponding Bockstein map $\mu_i^{(n)}$, $\nu_i^{(n)}$, $\kappa_i^{(nm,n)}$, and $\kappa_i^{(n,nm)}$ are integer multiples of each other.

(ii) For any C^*-algebras D and E, a collection of homomorphisms $\alpha_i\colon K_i(D) \to K_i(E)$ and $\alpha_i^{(n)}\colon K_i(D;\mathbb{Z}/n) \to K_i(E;\mathbb{Z}/n)$ forms a Λ-homomorphism if and only if these homomorphisms intertwine all the natural transformations $\mu_i'^{(n)}$, $\nu_i'^{(n)}$, $\kappa_i'^{(nm,n)}$, and $\kappa_i'^{(n,nm)}$.

(iii) These natural transformations induce natural exact sequences as in (2.29)*,* (2.32)*, and* (2.31)*.*

Proof. Part (ii) follows from (i). For example, to show that if α_0 and $\alpha_1^{(n)}$ intertwine $\nu_1'^{(n)}$ then they intertwine $\nu_1^{(n)}$, we start with $\alpha_0 \circ \nu_{1,D}'^{(n)} = \nu_{1,E}'^{(n)} \circ \alpha_1^{(n)}$, and, from (i), obtain some $j \in \mathbb{Z}$ such that $\nu_1^{(n)} = j\nu_1'^{(n)}$. Then

$$\alpha_0 \circ \nu_{1,D}^{(n)} = j\alpha_0 \circ \nu_{1,D}'^{(n)} = j\nu_{1,E}'^{(n)} \circ \alpha_1^{(n)} = \nu_{1,E}^{(n)} \circ \alpha_1^{(n)}. \tag{A.23}$$

The other cases are all similar.

Similarly, part (iii) follows from (i). For example, as above, if $\nu_1^{(n)} = j\nu_1'^{(n)}$ for some $j \in \mathbb{Z}$, then for any C^*-algebra D we get $\ker \nu_{i,D}'^{(n)} \subseteq \ker \nu_{i,D}^{(n)}$ and $\operatorname{im} \nu_{i,D}'^{(n)} \subseteq \operatorname{im} \nu_{i,D}^{(n)}$. By symmetry, we obtain equalities. The same type of argument works for the other Bockstein operations, and also for the composition $\mu_{1-i}'^{(n)} \circ \nu_i'^{(n)}$ in (2.32).

Now, to prove (i), we focus on showing that $\nu_1^{(n)}$ is an integer multiple of $\nu_1'^{(n)}$; a similar argument proves the other direction.[155]

By continuity of the functors K_0 and $K_1(\,\cdot\,;\mathbb{Z}/n)$, it suffices to find $j \in \mathbb{N}$ such that $\nu_{1,D}^{(n)} = j\nu_{1,D}'^{(n)}$ when D is a separable C^*-algebra. When restricted to this setting, by associativity of the Kasparov product, the functors K_0 and $K_1(\,\cdot\,;\mathbb{Z}/n)$ factorize through the KK-category,[156] in which we have natural isomorphisms

[155]For this, we need to know that the hypothesis holds for $\nu_1^{(n)}$ in place of $\nu_1'^{(n)}$; this follows by noting that

$$\nu_{1,S\mathbb{I}_n}^{(n)}\colon K_1(S\mathbb{I}_n;\mathbb{Z}/n) \to K_0(S\mathbb{I}_n) \cong \mathbb{Z}/n$$

is an isomorphism (by exactness of (2.29), since $K_1(\mathbb{I}_n) \cong \mathbb{Z}/n$ and $K_0(\mathbb{I}_n) = 0$).

[156]The KK-category is the category having as objects all separable C^*-algebras and with $KK(D, E)$ as the set of morphisms from D to E.

$K_1(\,\cdot\,;\mathbb{Z}/n) \cong KK(S\mathbb{I}_n,\,\cdot\,)$ given in Proposition A.1. For any separable C^*-algebra D, the functor $KK(D,\,\cdot\,)$ is a Hom-functor in the KK-category, so by the Yoneda lemma, the group of natural transformations from $KK(D,\,\cdot\,)$ to F is naturally isomorphic to $F(D)$ for any additive functor F from the KK-category to the category of abelian groups. Hence the group of natural transformations from $K_1(\,\cdot\,;\mathbb{Z}/n)$ to K_0 is naturally isomorphic to $K_0(S\mathbb{I}_n) \cong \mathbb{Z}/n$. In particular, a natural transformation $\eta\colon K_1(\,\cdot\,;\mathbb{Z}/n) \to K_0$ generates the entire group of natural transformation between these functors if and only if $k\eta \neq 0$ for every $k = 1, \ldots, n-1$. This is true for $\nu_1'^{(n)}$ by assumption. Hence $\nu_1^{(n)}$ is an integer multiple of $\nu_1'^{(n)}$. □

Proof of Proposition 8.4. After identifying $KK^i(\mathbb{I}_n,\,\cdot\,)$ with $K_i(\,\cdot\,;\mathbb{Z}/n)$ via natural isomorphisms (which exist by Proposition A.1), the Bockstein operations in [66] induce natural transformations satisfying the criterion of Lemma A.4. □

A.3. **A split exact sequence relating ordinary and total K-theory.** We end this appendix by showing how to use Bödigheimer's work ([19, 20]) to prove Proposition 2.15 by means of an unnatural splitting of an exact sequence relating total K-theory with ordinary K-theory.

Recall that for an abelian group G and $n \geq 2$, we identify $\mathrm{Tor}(G, \mathbb{Z}/n)$ with the n-torsion group, $\{g \in G : ng = 0\}$. In particular, for any m, n, we have a natural inclusion $\mathrm{Tor}(G, \mathbb{Z}/n) \to \mathrm{Tor}(G, \mathbb{Z}/mn)$, as well as a natural map $\mathrm{Tor}(G, \mathbb{Z}/mn) \to \mathrm{Tor}(G, \mathbb{Z}/n)$.

Let $n \geq 2$, and let $n = p_1^{r_1} \cdots p_k^{r_k}$ be its prime factorization. For any C^*-algebra D, $i = 0, 1$, and $j = 1, \ldots, k$, using commutativity of (2.31), we get a commuting diagram[157]

(A.24)
$$\begin{tikzcd}
K_i(D)/p_j^{r_j}K_i(D) \arrow[r, tail, "\overline{\mu}_{i,D}^{(p_j^{r_j})}"] \arrow[d, two heads, "\times n/p_j^{r_j}"] & K_i(D;\mathbb{Z}/p_j^{r_j}) \arrow[r, two heads, "\overline{\nu}_{i,D}^{(p_j^{r_j})}"] \arrow[d, "\kappa_{i,D}^{(n,p_j^{r_j})}"] & \mathrm{Tor}(K_{1-i}(D), \mathbb{Z}/p_j^{r_j}) \arrow[d, tail] \\
K_i(D)/nK_i(D) \arrow[r, tail, "\overline{\mu}_{i,D}^{(n)}"] & K_i(D;\mathbb{Z}/n) \arrow[r, two heads, "\overline{\nu}_{i,D}^{(n)}"] & \mathrm{Tor}(K_{1-i}(D), \mathbb{Z}/n).
\end{tikzcd}$$

where the rows are as in (2.35). Taking the direct sum of the top row for $j = 1, \ldots, k$, we get a commutative diagram

(A.25)
$$\begin{tikzcd}
\bigoplus_j K_i(D)/p_j^{r_j}K_i(D) \arrow[r, tail, "\bigoplus_j \overline{\mu}_{i,D}^{(p_j^{r_j})}"] \arrow[d, "\cong"] & \bigoplus_j K_i(D;\mathbb{Z}/p_j^{r_j}) \arrow[r, two heads, "\bigoplus_j \overline{\nu}_{i,D}^{(p_j^{r_j})}"] \arrow[d, "\bigoplus_j \kappa_{i,D}^{(n,p_j^{r_j})}", "\cong"'] & \bigoplus_j \mathrm{Tor}(K_{1-i}(D), \mathbb{Z}/p_j^{r_j}) \arrow[d, "\cong"] \\
K_i(D)/nK_i(D) \arrow[r, tail, "\overline{\mu}_{i,D}^{(n)}"] & K_i(D;\mathbb{Z}/n) \arrow[r, two heads, "\overline{\nu}_{i,D}^{(n)}"] & \mathrm{Tor}(K_{1-i}(D), \mathbb{Z}/n),
\end{tikzcd}$$

where the left and right vertical maps are isomorphisms by the Chinese remainder theorem ($\mathbb{Z}/n \cong \bigoplus_j \mathbb{Z}/p_j^{r_j}$) together with additivity of $\otimes$ and Tor, and the middle vertical map is therefore an isomorphism by the five lemma.

[157] In this section, we use $K_i(D)/nK_i(D)$ rather that the isomorphic group $K_i(D) \otimes \mathbb{Z}/n$ to more easily describe maps between such groups.

Lemma A.5 (Bödigheimer)**.** *For any C^*-algebra D there are (unnatural) homomorphisms*

$$\operatorname{Tor}(K_{1-i}(D),\mathbb{Z}/n) \xrightarrow{s_{i,D}^{(n)}} K_i(D;\mathbb{Z}/n) \xrightarrow{\iota_{i,D}^{(n)}} K_i(D)/nK_i(D) \tag{A.26}$$

for $n \geq 2$ such that

$$\overline{\nu}_{i,D}^{(n)} \circ s_{i,D}^{(n)} = \mathrm{id}_{\operatorname{Tor}(K_{1-i}(D),\mathbb{Z}/n)}, \qquad \iota_{i,D}^{(n)} \circ \overline{\mu}_{i,D}^{(n)} = \mathrm{id}_{K_i(D)/nK_i(D)}, \tag{A.27}$$

and the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Tor}(K_{1-i}(D),\mathbb{Z}/n) \arrow[r, "s_{i,D}^{(n)}"] \arrow[d] & K_i(D;\mathbb{Z}/n) \arrow[r, "\iota_{i,D}^{(n)}"] \arrow[d, "\kappa_{i,D}^{(nm,n)}"] & K_i(D)/nK_i(D) \arrow[d, "\times m"] \\
\operatorname{Tor}(K_{1-i}(D),\mathbb{Z}/nm) \arrow[r, "s_{i,D}^{(nm)}"] \arrow[d, "\times m"] & K_i(D;\mathbb{Z}/nm) \arrow[r, "\iota_{i,D}^{(nm)}"] \arrow[d, "\kappa_{i,D}^{(n,nm)}"] & K_i(D)/nmK_i(D) \arrow[d] \\
\operatorname{Tor}(K_{1-i}(D),\mathbb{Z}/n) \arrow[r, "s_{i,D}^{(n)}"] & K_i(D;\mathbb{Z}/n) \arrow[r, "\iota_{i,D}^{(n)}"] & K_i(D)/nK_i(D).
\end{tikzcd} \tag{A.28}$$

Proof. We start with the maps $s_{i,D}^{(n)}$. Recall that for $n, m \geq 2$, the maps

$$\begin{aligned} &\kappa_{*,D}^{(nm,n)} \colon K_*(D;\mathbb{Z}/n) \longrightarrow K_*(D;\mathbb{Z}/nm) \text{ and} \\ &\kappa_{*,D}^{(n,nm)} \colon K_*(D;\mathbb{Z}/nm) \longrightarrow K_*(D;\mathbb{Z}/n) \end{aligned} \tag{A.29}$$

are induced by the canonical inclusions $\mathbb{I}_n \to \mathbb{I}_{nm}$ and $\mathbb{I}_{nm} \to \mathbb{I}_n \otimes M_m$, respectively. It follows that the compositions $\kappa_{*,D}^{(nm,n)} \circ \kappa_{*,D}^{(n,nm)}$ and $\kappa_{*,D}^{(n,nm)} \circ \kappa_{*,D}^{(nm,n)}$ are given by multiplication by m since the compositions of the corresponding *-homomorphisms, in both directions, are the diagonal embeddings into the $m \times m$ matrices. Further, note that $nx = 0$ for all $x \in K_*(D;\mathbb{Z}/n)$. It follows that for a prime p, the diagram (A.30)

$$\begin{tikzcd}
\vdots \arrow[d, shift left] & \vdots \arrow[d, shift left] & \vdots \arrow[d, shift left] \\
K_i(D)/p^kK_i(D) \arrow[u, shift left] \arrow[r, tail, "\bar{\nu}_{i,D}^{(p^k)}"] \arrow[d, shift left] & K_i(D;\mathbb{Z}/p^k) \arrow[u, shift left] \arrow[r, two heads, "\bar{\mu}_{i,D}^{(p^k)}"] \arrow[d, shift left, "\kappa_{i,D}^{(p^{k+1},p^k)}"] & \operatorname{Tor}(K_{1-i}(D),\mathbb{Z}/p^k) \arrow[u, shift left] \arrow[d, shift left, "\times p"] \\
K_i(D)/p^{k+1}K_i(D) \arrow[u, shift left, "\times p"] \arrow[r, tail, "\bar{\nu}_{i,D}^{(p^{k+1})}"] \arrow[d, shift left] & K_i(D;\mathbb{Z}/p^{k+1}) \arrow[u, shift left, "\kappa_{i,D}^{(p^k,p^{k+1})}"] \arrow[r, two heads, "\bar{\mu}_{i,D}^{(p^{k+1})}"] \arrow[d, shift left] & \operatorname{Tor}(K_{1-i}(D),\mathbb{Z}/p^{k+1}) \arrow[u, shift left] \arrow[d, shift left] \\
\vdots \arrow[u, shift left] & \vdots \arrow[u, shift left] & \vdots \arrow[u, shift left]
\end{tikzcd}$$

satisfies the hypotheses of the (purely algebraic) lemma in [20, Section 2] for $i = 0, 1$,[158] where the vertical maps on the left and right are induced by the canonical maps between $\mathbb{Z}/p^k$ and $\mathbb{Z}/p^{k+1}$. This lemma implies that the maps $s_{i,D}^{(n)}$ exist satisfying (A.27) and such that (A.28) commutes when n and m above are powers of the same prime p; we therefore fix such $s_{i,D}^{(p^r)}$ for all primes p and $r \geq 1$.

We now use the identifications in the right-hand square of (A.25) to define maps $s_{i,D}^{(n)}$ for general n. For $n \geq 2$ arbitrary, let $n = p_1^{r_1} \cdots p_k^{r_k}$ be its prime factorization. Then we define $s_{i,D}^{(n)} \colon \mathrm{Tor}(K_{1-i}(D); \mathbb{Z}/n) \to K_i(D; \mathbb{Z}/n)$ such that the following commutes

$$\begin{tikzcd}
\bigoplus_j \mathrm{Tor}(K_{1-i}(D), \mathbb{Z}/p_j^{r_j}) \arrow[r, "\bigoplus_j s_{i,D}^{(p_j^{r_j})}"] \arrow[d, "\cong"] & \bigoplus_j K_i(D; \mathbb{Z}/p_j^{r_j}) \arrow[d, "\cong"', "\bigoplus_j \kappa_{i,D}^{(n,p_j^{r_j})}"] \\
\mathrm{Tor}(K_{1-i}(D), \mathbb{Z}/n) \arrow[r, "s_{i,D}^{(n)}"] & K_i(D; \mathbb{Z}/n).
\end{tikzcd} \tag{A.31}$$

Since the top map, $\bigoplus_j s_{i,D}^{(p_j^{r_j})}$, is a right splitting for the top row of (A.25), it follows that the $s_{i,D}^{(n)}$ is a right splitting for the bottom row, i.e., $\overline{\nu}_{i,D}^{(n)} \circ s_{i,D}^{(n)} = \mathrm{id}_{\mathrm{Tor}(K_{1-i}(D),\mathbb{Z}/n)}$.

To check commutativity of the left side of (A.28), we consider $m, n \geq 2$. Let $n = p_1^{r_1} \cdots p_k^{r_k}$ and $m = p_1^{t_1} \cdots p_k^{t_k}$ be their prime factorizations, allowing some exponents to possibly be zero. For convenience, we take $K_i(D; \mathbb{Z}/1)$ to be 0 by convention; then (A.25) and the definition of $s_{i,D}^{(n)}$ from (A.31) are still correct with some r_j equal to 0.

Since $\kappa_{i,D}^{(nm,p_j^{r_j+t_j})} \circ \kappa_{i,D}^{(p_j^{r_j+t_j},p_j^{r_j})} = \kappa_{i,D}^{(nm,p_j^{r_j})} = \kappa_{i,D}^{(nm,n)} \circ \kappa_{i,D}^{(n,p_j^{r_j})}$ (by the definition of these maps), the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_j K_i(D; \mathbb{Z}/p_j^{r_j}) \arrow[r, "\bigoplus_j \kappa_{i,D}^{(p_j^{r_j+t_j},p_j^{r_j})}"] \arrow[d, "\cong"', "\bigoplus_j \kappa_{i,D}^{(n,p_j^{r_j})}"] & \bigoplus_j K_i(D; \mathbb{Z}/p_j^{r_j+t_j}) \arrow[d, "\cong"', "\bigoplus_j \kappa_{i,D}^{(nm,p_j^{r_j+t_j})}"] \\
K_i(D; \mathbb{Z}/n) \arrow[r, "\kappa_{i,D}^{(nm,n)}"] & K_i(D; \mathbb{Z}/nm).
\end{tikzcd} \tag{A.32}$$

From this and (A.31), one can deduce commutativity of the top-left square of (A.28) for general n and m from Bödigheimer's commutativity of this square for the case that n and m are powers of the same prime.

Fix $j \in \{1, \ldots, k\}$. Then we have

$$\kappa_{i,D}^{(n,nm)} \circ \kappa_{i,D}^{(nm,p_j^{r_j+t_j})} = \frac{m}{p_j^{t_j}} \kappa_{i,D}^{(n,p_j^{r_j})} \circ \kappa_{i,D}^{(p_j^{r_j},p_j^{r_j+t_j})} \tag{A.33}$$

[158]In [20], for an abelian group G and natural number $n \geq 2$, the group $\mathrm{Tor}(G, \mathbb{Z}/n)$ is denoted $G[n]$.

since the left side comes from the inclusion $\mathbb{I}_{p_j^{r_j+t_j}} \to \mathbb{I}_n \otimes M_m$ given by $f \mapsto f \otimes 1_{M_{mn/p_j^{r_j+t_j}}}$, while the right side (ignoring the factor $m/p_j^{t_j}$) comes from the inclusion $\mathbb{I}_{p_j^{r_j+t_j}} \to \mathbb{I}_n \otimes M_{p_j^{t_j}}$ given by $f \mapsto f \otimes 1_{M_{n/p_j^{r_j}}}$.

Therefore, for an element x of the direct summand $\mathrm{Tor}(K_i(D), \mathbb{Z}/p_j^{r_j+t_j})$ of $\mathrm{Tor}(K_i(D), \mathbb{Z}/nm)$, we have

$$\begin{aligned}
\kappa_{i,D}^{(n,nm)}\big(s_{i,D}^{(nm)}(x)\big) &\overset{(A.31)}{=} \kappa_{i,D}^{(n,nm)}\Big(\kappa_{i,D}^{(nm,p_j^{r_j+t_j})}\big(s_{i,D}^{(p_j^{r_j+t_j})}(x)\big)\Big) \\
&\overset{(A.33)}{=} \frac{m}{p_j^{t_j}}\Big(\kappa_{i,D}^{(n,p_j^{r_j})}\Big(\kappa_{i,D}^{(p_j^{r_j},p_j^{r_j+t_j})}\big(s_{i,D}^{(p_j^{r_j+t_j})}(x)\big)\Big)\Big) \\
&= \frac{m}{p_j^{t_j}}\kappa_{i,D}^{(n,p_j^{r_j})}\Big(s_{i,D}^{(p_j^{r_j})}(p_j^{t_j}x)\Big) \\
&= \kappa_{i,D}^{(n,p_j^{r_j})}\Big(s_{i,D}^{(p_j^{r_j})}(mx)\Big) \\
&\overset{(A.31)}{=} s_{i,D}^{(n)}(mx),
\end{aligned} \tag{A.34}$$

where the third equation comes from Bödigheimer's commutativity of the bottom-left square of (A.28) in the case that n and m are powers of p_j. Since

$$\mathrm{Tor}(K_i(D), \mathbb{Z}/nm) = \bigoplus_j \mathrm{Tor}(K_i(D), \mathbb{Z}/p_j^{r_j+t_j}) \tag{A.35}$$

and the above shows commutativity of the bottom-left square of (A.28) on each of these summands, it follows that the bottom-left square of (A.28) commutes.

The splitting lemma provides the maps $\iota_{i,D}^{(n)}$. A version that suffices for our purposes says that given a commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_1 & \longrightarrow & H_1 & \longrightarrow & L_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_2 & \longrightarrow & H_2 & \longrightarrow & L_2 & \longrightarrow & 0
\end{array} \tag{A.36}$$

with exact rows, if there is a left splitting for each row that commutes with the downward maps, then the corresponding right splittings also commute with the downward maps.[159] $\square$

Proof of Proposition 2.15. Fix homomorphisms $\iota_{i,A}^{(n)}$ and $s_{i,B}^{(n)}$ as in Lemma A.5. Given $\alpha_*\colon K_*(A) \to K_*(B)$ we define group homomorphisms $\alpha_i^{(n)}\colon K_i(A;\mathbb{Z}/n) \to K_i(B;\mathbb{Z}/n)$ by

$$\alpha_i^{(n)} := \overline{\mu}_{i,B}^{(n)} \circ (\alpha_i \otimes \mathrm{id}_{\mathbb{Z}/n}) \circ \iota_{i,A}^{(n)} + s_{i,B}^{(n)} \circ \mathrm{Tor}(\alpha_{1-i}, \mathbb{Z}/n) \circ \overline{\nu}_{i,A}^{(n)}. \tag{A.37}$$

It follows from the construction that the collection of maps α_i and $\alpha_i^{(n)}$ for $i = 0, 1$ and $n \geq 2$ intertwines the natural transformations μ_i and ν_i, and by Lemma A.5, they also intertwine the κ_i. $\square$

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[159] If an exact sequence $0 \to G \xrightarrow{i} H \xrightarrow{p} L \to 0$ is an exact sequence of abelian groups and $q\colon H \to G$ is a splitting of i, then a splitting $j\colon L \to H$ of q is given explicitly as follows: the map $\mathrm{id}_H - i \circ q\colon H \to H$ vanishes on $\mathrm{im}\, i = \ker p$, and hence there is a map j with $j \circ p = \mathrm{id}_H - i \circ q$. Then $p \circ j \circ p = p$, and as p is surjective, $p \circ j = \mathrm{id}_L$. Using this explicit formula for the splitting, the claim follows.

Appendix B. KK-theory outside the σ-unital setting

In this appendix we give some details regarding the inductive limit description of KK-theory in the case when the second variable is not σ-unital, extending some standard results from the σ-unital case to the general framework.

B.1. **Cuntz's picture of KK-theory.** We will make use of Cuntz's picture of KK-theory from [56]. This (like its precursor in [54]) works smoothly for non-σ-unital algebras in the second variable and allows for a very clean description of $KK(A,\theta)$ for a *-homomorphism θ.

Definition B.1 (Cuntz, [56])**.** Let A and I be C^*-algebras with A separable. Let qA be the kernel of the map $\mathrm{id}_A * \mathrm{id}_A\colon A * A \to A$ (where $*$ denotes the full free product). Define

$$KK_c(A,I) \coloneqq [qA, I\otimes\mathcal{K}], \tag{B.1}$$

(i.e. homotopy classes of *-homomorphisms from qA to $I\otimes\mathcal{K}$). Write $[\rho]_{KK_c(A,I)} \in KK_c(A,I)$ for the homotopy class of $\rho\colon qA \to I\otimes\mathcal{K}$.

Given another C^*-algebra J and a *-homomorphism $\theta\colon I\to J$, the induced map $KK_c(A,\theta)\colon KK_c(A,I)\to KK_c(A,J)$ is given by

$$KK_c(A,\theta)([\rho]_{KK_c(A,I)}) \coloneqq [(\theta\otimes\mathrm{id}_{\mathcal{K}})\circ\rho]_{KK_c(A,J)}. \tag{B.2}$$

The above definition of $KK_c(A,\theta)$ makes $KK_c(A,\,\cdot\,)$ a covariant functor, and indeed $KK_c(\,\cdot\,,\,\cdot\,)$ is a bifunctor (though we do not need an explicit description of functoriality in the first variable). Cuntz used Higson's characterization of KK-theory for separable C^*-algebras ([133]) to show that $KK_c(\,\cdot\,,\,\cdot\,)$ gives another realization of KK-theory in the separable setting. This was extended to the case when the second variable is σ-unital in [142], where the isomorphism between the Cuntz picture and Cuntz–Thomsen picture is described explicitly (the proof reveals that the isomorphism is natural) as in the following proposition. To set this up, note that given a Cuntz pair $(\phi,\psi)\colon A \rightrightarrows E \rhd I\otimes\mathcal{K}$, we have $(\phi*\psi)(qA)\subseteq I\otimes\mathcal{K}$ (see [142, Lemma 5.14]); we write

$$[\phi,\psi]_{KK_c(A,I)} \coloneqq \big[(\phi*\psi)|_{qA}^{I\otimes\mathcal{K}}\big]_{KK_c(A,I)}. \tag{B.3}$$

Proposition B.2 (cf. [142, Theorem 5.2.4])**.** *Let A and I be C^*-algebras with A separable and I σ-unital. Then $[\phi,\psi]_{KK(A,I)}\mapsto[\phi,\psi]_{KK_c(A,I)}$ defines an isomorphism $KK(A,I)\cong KK_c(A,I)$ that is natural in both variables.*

The clean description of $KK_c(A,\theta)$ above facilitates the following computation.[160]

Proposition B.3. *Let A be a separable C^*-algebra and let $\theta\colon I_1\to I_2$ be a *-homomorphism between C^*-algebras. Suppose that E_i is a C^*-algebra containing I_i as an ideal for $i=1,2$ and that θ extends to a *-homomorphism $\bar\theta\colon E_1\to E_2$. Given a Cuntz pair $(\phi,\psi)\colon A\rightrightarrows E_1\rhd I_1$, we have*

$$KK_c(A,\theta)([\phi,\psi]_{KK_c(A,I_1)}) = [\bar\theta\circ\phi,\bar\theta\circ\psi]_{KK_c(A,I_2)}. \tag{B.4}$$

In particular, if I_1 and I_2 are σ-unital, then

$$KK(A,\theta)([\phi,\psi]_{KK(A,I_1)}) = [\bar\theta\circ\phi,\bar\theta\circ\psi]_{KK(A,I_2)}. \tag{B.5}$$

[160] This calculation was performed in Kasparov's original Fredholm module picture, in the case that $I_2=E_1=E_2$ separable, in [253, Proposition 2.1]. Comparing the computations demonstrates the power of Cuntz's picture.

Proof. To shorten notation, we may assume (by stabilizing everything) that I_1, I_2 are stable. One may verify (using [142, Lemma 5.1.2]) that $((\bar\theta \circ \phi) * (\bar\theta \circ \psi))|_{qA} = \theta \circ (\phi * \psi)|_{qA}$. We use this for the third equality below:

$$\begin{aligned} &KK_c(A,\theta)\big([\phi,\psi]_{KK_c(A,I_1)}\big) \\ &\quad\overset{\text{(B.3)}}{=} KK_c(A,\theta)\big(\big[(\phi * \psi)|_{qA}^{I_1}\big]_{KK_c(A,I_1)}\big) \\ &\quad\overset{\text{(B.2)}}{=} \big[\theta \circ \big((\phi * \psi)|_{qA}^{I_1}\big)\big]_{KK_c(A,I_2)} \\ &\quad= \big[\big((\bar\theta \circ \phi) * (\bar\theta \circ \psi)\big)|_{qA}^{I_2}\big]_{KK_c(A,I_2)} \\ &\quad= [\bar\theta \circ \phi, \bar\theta \circ \psi]_{KK_c(A,I_2)}. \end{aligned} \tag{B.6}$$

The last line of the proposition follows from Proposition B.2 □

B.2. **The inductive limit picture of KK-theory.** Recall that in Definition 5.3 for C^*-algebras A and I with A separable, we defined

$$KK(A,I) \coloneqq \varinjlim_{I_0 \text{ sep.}} KK(A,I_0), \tag{B.7}$$

with the limit taken over all separable C^*-subalgebras of I, ordered by inclusion. We now justify the interpretation of the class of an (A,I)-Cuntz pair in this picture.

Proposition B.4. *Let A and I be C^*-algebras with A separable. Given a Cuntz pair $(\phi,\psi)\colon A \rightrightarrows E \rhd I$, the class $[\phi,\psi]_{KK(A,I)}$ in Definition 5.3 is well-defined. That is,*

(i) there exist separable C^*-subalgebras $E_0 \subseteq E$ and $I_0 \subseteq I$ such that $I_0 \lhd E_0$, $\phi(A) \cup \psi(A) \subseteq E_0$, and $(\phi - \psi)(A) \subseteq I_0$, and

(ii) the class of $[\phi|^{E_0}, \psi|^{E_0}]_{KK(A,I_0)}$ in $\varinjlim_{I_0 \text{ sep.}} KK(A,I_0)$ does not depend on the choice of I_0 and E_0 in (i).

Proof. (i): Let E_0 denote the C^*-algebra generated by $\phi(A) \cup \psi(A)$ and let I_0 be the ideal generated by $(\phi - \psi)(A)$ in E_0.

(ii): Suppose that $I_1 \subseteq I$ and $E_1 \subseteq E$ are separable C^*-subalgebras that also satisfy the conditions in (i). Let E_2 be the C^*-algebra generated by $E_0 \cup E_1$ and let I_2 be the ideal generated by $I_0 \cup I_1$ in E_2. These are separable C^*-subalgebras of E and I, respectively.

Using Proposition B.3 twice (with θ equal to the inclusions of I_0 and I_1 into I_2 in turn), we have

$$\begin{aligned} KK(A,\iota_{I_0 \subseteq I_2})\big([\phi|^{E_0}, \psi|^{E_0}]_{KK(A,I_0)}\big) &= [\phi|^{E_2}, \psi|^{E_2}]_{KK(A,I_2)} \\ &= KK(A,\iota_{I_1 \subseteq I_2})\big([\phi|^{E_1}, \psi|^{E_1}]_{KK(A,I_1)}\big). \end{aligned} \tag{B.8}$$

Therefore, $[\phi|^{E_0}, \psi|^{E_0}]_{KK(A,I_0)}$ and $[\phi|^{E_1}, \psi|^{E_1}]_{KK(A,I_1)}$ represent the same class in $\varinjlim_{I_0 \text{ sep.}} KK(A,I_0)$. □

Proposition B.5. $KK(\,\cdot\,,\,\cdot\,)$ *(as in Definition 5.3) is a bifunctor.*

Proof. Functoriality in the first variable is entirely straightforward.

For the second variable, given C^*-algebras A, I, and J such that A is separable, and a *-homomorphism $\theta\colon I \to J$, we must define a group homomorphism $KK(A,\theta)\colon KK(A,I) \to KK(A,J)$. First, fix a separable C^*-subalgebra $I_0 \subseteq I$;

then there exists a separable C^*-subalgebra $J_0 \subseteq J$ such that $\theta(I_0) \subseteq J_0$. Moreover, by functoriality of $KK(A, \cdot\,)$ in the separable case, we see that the map

$$KK(A, I_0) \to \varinjlim_{J_0 \text{ sep.}} KK(A, J_0) \tag{B.9}$$

given by $KK(A, \theta|_{I_0}^{J_0})$ does not depend on the choice of J_0. We may call this map $KK(A, \theta|_{I_0}) \colon KK(A, I_0) \to KK(A, J)$.

The system of maps $(KK(A, \theta|_{I_0}))_{I_0 \subseteq I \text{ sep.}}$ is compatible with the connecting maps defining $KK(A, I)$ (by functoriality for separable codomains) and therefore they induce a map $KK(A, \theta) \colon KK(A, I) \to KK(A, J)$. It is a straightforward check that this definition makes $KK(A, \cdot\,)$ a covariant functor, using that it is a covariant functor in the separable case. It is likewise straightforward to check that the functoriality in the first and second variable commute, so that $KK(\,\cdot\,, \cdot\,)$ is a bifunctor. □

We end this subsection by noting that Cuntz's picture of KK-theory agrees with the inductive limit definition in the non-separable case. (The left-hand side of (B.10) below is, by definition, the inductive limit over separable subalgebras of I.) This justifies that the inductive limit picture agrees with the usual picture in the case of σ-unital but non-separable algebras I.

Proposition B.6. *Let A and I be C^*-algebras with A separable. Then*

$$KK(A, I) \cong KK_c(A, I), \tag{B.10}$$

and the isomorphism is natural in both variables.

In particular, if I is σ-unital, then the definitions of $KK(A, I)$ given in Definitions 5.2 and 5.3 agree.

Proof. Using Proposition B.2, there is a natural identification

$$KK(A, I) \cong \varinjlim_{I_0 \text{ sep.}} KK_c(A, I_0). \tag{B.11}$$

We will therefore prove the existence of a natural isomorphism between the right-hand side and $KK_c(A, I)$.

Given separable C^*-subalgebras $I_0 \subseteq I_1 \subseteq I$, we have

$$KK_c(A, \iota_{I_1 \subseteq I}) \circ KK_c(A, \iota_{I_0 \subseteq I_1}) = KK_c(A, \iota_{I_0 \subseteq I}), \tag{B.12}$$

since $KK_c(A, \cdot\,)$ is designed to be functorial without restriction on the codomain. Therefore, we may define

$$\Theta \coloneqq \varinjlim_{I_0 \text{ sep.}} KK_c(A, \iota_{I_0 \subseteq I}) \colon \varinjlim_{I_0 \text{ sep.}} KK_c(A, I_0) \to KK_c(A, I), \tag{B.13}$$

a homomorphism that is natural in both variables.

For surjectivity, consider $\rho \colon qA \to I \otimes \mathcal{K}$. Let I_1 be a separable C^*-subalgebra of I such that $\rho(qA) \subseteq I_1 \otimes \mathcal{K}$. Then ρ corestricts to $I_1 \otimes \mathcal{K}$, thus giving an element $\kappa \coloneqq \big[\rho|^{I_1 \otimes \mathcal{K}}\big]_{KK_c(A, I_1)} \in \varinjlim_{I_0 \text{ sep.}} KK_c(A, I_0)$. We have

$$\Theta(\kappa) = KK_c(A, \iota_{I_1 \subseteq I})\big(\big[\rho|^{I_1 \otimes \mathcal{K}}\big]_{KK_c(A, I_1)}\big) = [\rho]_{KK_c(A, I)} \tag{B.14}$$

by (B.2).

For injectivity, suppose that I_1 is a separable C^*-subalgebra of I and $\rho\colon qA \to I_1 \otimes \mathcal{K}$ is a *-homomorphism such that

$$\Theta([\rho]_{KK_c(A,I_1)}) = 0. \tag{B.15}$$

That is to say, $\iota_{I_1\subseteq I} \circ \rho$ is homotopic to the zero *-homomorphism within the space of *-homomorphisms $qA \to I \otimes \mathcal{K}$. Let $(\rho_t\colon qA \to I \otimes \mathcal{K})_{t\in[0,1]}$ be such a homotopy, so that $\rho_0 = \iota_{I_1\subseteq I} \circ \rho$ and $\rho_1 = 0$. Then let $I_2 \subseteq I$ be a separable C^*-subalgebra containing I_1 and $\rho_t(A) \subseteq I_2 \otimes \mathcal{K}$ for $t \in [0,1] \cap \mathbb{Q}$. By density, it follows that $(\rho_t)_{t\in[0,1]}$ corestricts to a homotopy of *-homomorphisms $qA \to I_2 \otimes \mathcal{K}$, from $\iota_{I_1\subseteq I_2} \circ \rho$ to 0. Thus

$$KK_c(A, \iota_{I_1\subseteq I_2})\big([\rho]_{KK_c(A,I_1)}\big) = [\iota_{I_1\subseteq I_2} \circ \rho]_{KK_c(A,I_2)} = 0. \tag{B.16}$$

This shows that $[\rho]_{KK_c(A,I_1)}$ represents 0 in $\varinjlim_{I_0 \text{ sep.}} KK(A, I_0)$, as required.

The last statement follows using Proposition B.2. $\square$

Remark B.7. If A and I are separable C^*-algebras and J is a σ-unital and non-separable C^*-algebra, then the Kasparov product $KK(A,I) \times KK(I,J) \to KK(A,J)$ agrees with taking the limit of Kasparov products over separable subalgebras J_0 of J. To see this, let $\kappa \in KK(A,I)$ and $\kappa' \in KK(I,J)$. According to Definition 5.3, we may find a separable subalgebra J_0 of J such that κ' comes from $\kappa_0' \in KK(A, J_0)$ — that is, $\kappa' = KK(A, \iota_{J_0\subseteq J})(\kappa_0') = [\iota_{J_0\subseteq J}]_{KK(J_0,J)} \circ \kappa_0'$. Then

$$\begin{aligned} \kappa' \circ \kappa &= ([\iota_{J_0\subseteq J}]_{KK(J_0,J)} \circ \kappa_0') \circ \kappa \\ &= [\iota_{J_0\subseteq J}]_{KK(J_0,J)} \circ (\kappa_0' \circ \kappa) \\ &= KK(A, \iota_{J_0\subseteq J})(\kappa_0' \circ \kappa). \end{aligned} \tag{B.17}$$

The above argument only uses that the Kasparov product is associative and that $KK(A, \iota_{J_0\subseteq J})(\,\cdot\,) = [\iota_{J_0\subseteq J}]_{KK(J_0,J)} \circ (\,\cdot\,)$, so it holds for the Kasparov product as defined in any picture.

B.3. **KL-theory outside the σ-unital setting.** Next we turn to KL, aiming to prove Proposition 5.6 that $KL(A,I)$ can also be described as a limit over separable C^*-subalgebras.

Lemma B.8. *Let A and I be C^*-algebras with A separable. Then*

$$Z_{KK(A,I)} = \varinjlim_{I_0 \text{ sep.}} Z_{KK(A,I_0)}, \tag{B.18}$$

inside $\varinjlim_{I_0 \text{ sep.}} KK(A, I_0)$.

Proof. Suppose $I_0 \subseteq I$ is separable and $\kappa_0 \in Z_{KK(A,I_0)}$. Then there exists an element $\bar\kappa_0 \in KK(A, C(\overline{\mathbb{N}}, I_0))$ such that $KK(A, \mathrm{ev}_n)(\bar\kappa_0) = 0$ for $n \in \mathbb{N}$ and $KK(A, \mathrm{ev}_\infty)(\bar\kappa_0) = \kappa_0$. We see that $KK(A, \iota_{I_0\subseteq I})(\kappa_0) \in Z_{KK(A,I)}$ by pushing this forward to $C(\overline{\mathbb{N}}, I)$.

On the other hand, suppose that $\kappa \in Z_{KK(A,I)}$. Then there exists an element $\bar\kappa \in KK(A, C(\overline{\mathbb{N}}, I))$ such that $KK(A, \mathrm{ev}_n)(\bar\kappa) = 0$ for all $n \in \mathbb{N}$ and $KK(A, \mathrm{ev}_\infty)(\bar\kappa) = KK(A, \iota_{I_0\subseteq I})(\kappa)$.

Using the definition of $KK(A, C(\overline{\mathbb{N}}, I))$ as a limit, there exists a separable subalgebra $J_0 \subseteq C(\overline{\mathbb{N}}, I)$ and $\bar\kappa_0 \in KK(A, J_0)$ such that $KK(A, \iota_{J_0\subseteq C(\overline{\mathbb{N}},I)})(\bar\kappa_0) = \bar\kappa$.

By increasing J_0, we can arrange that J_0 has the form $C(\overline{\mathbb{N}}, I_0)$ for some separable $I_0 \subseteq I$, and that $KK(A, \mathrm{ev}_n)(\bar{\kappa}_0) = 0$ in $KK(A, I_0)$ for all $n \in \mathbb{N}$. Setting $\kappa_0 \coloneqq KK(A, \mathrm{ev}_\infty)(\bar{\kappa}_0) \in Z_{KK(A,I_0)}$, we then have $\kappa = KK(A, \iota_{I_0 \subseteq I})(\kappa_0)$, as required. $\square$

Proof of Proposition 5.6. Functoriality of KL descends from functoriality of KK (Proposition B.5). Given separable C^*-subalgebras $I_0 \subseteq I_1 \subseteq I_2 \subseteq I$, functoriality applied to the inclusions $I_0 \subseteq I_1 \subseteq I_2$ tells us that the limit $\varinjlim_{I_0 \text{ sep.}} KL(A, I_0)$ exist, and functoriality applied to $I_0 \subseteq I_1 \subseteq I$ provides a map

$$\Theta \coloneqq \varinjlim_{I_0 \text{ sep.}} KL(A, \iota_{I_0 \subseteq I})\colon \varinjlim_{I_0 \text{ sep.}} KL(A, I_0) \to KL(A, I). \tag{B.19}$$

By naturality of the map from KK to KL, we get a commuting diagram

$$\begin{tikzcd} \varinjlim_{I_0 \text{ sep.}} KK(A, I_0) \arrow[r, equal] \arrow[d, two heads] & KK(A, I) \arrow[d, two heads] \\ \varinjlim_{I_0 \text{ sep.}} KL(A, I_0) \arrow[r, "\Theta"] & KL(A, I). \end{tikzcd} \tag{B.20}$$

In particular, this shows that Θ is surjective.

For injectivity, let $I_0 \subseteq I$ and $\lambda \in KL(A, I_0)$ satisfy $KL(A, \iota_{I_0 \subseteq I})(\lambda) = 0$. Let $\kappa \in KK(A, I_0)$ be a lift of λ. Then $KK(A, \iota_{I_0 \subseteq I})(\kappa) \in Z_{KK(A,I)}$, so by Lemma B.8, there exists $I_1 \subseteq I$ separable such that $I_0 \subseteq I_1$ and $KK(A, \iota_{I_0 \subseteq I_1})(\kappa) \in Z_{KK(A,I_1)}$. Therefore,

$$KL(A, \iota_{I_0 \subseteq I_1})(\lambda) = 0, \tag{B.21}$$

as required. $\square$

Now we recall and prove Propositions 5.4 and 5.7.

Proposition B.9. *Let A and I be C^*-algebras with A separable, let E be a C^*-algebra containing I as an ideal, and let $(\phi, \psi)\colon A \rightrightarrows E \rhd I$ be an (A, I)-Cuntz pair.*

(i) Let $\iota_I^{(2)}\colon I \to M_2(I)$ be the top-left corner inclusion. Then the induced maps $KK(A, \iota_I^{(2)})\colon KK(A, I) \to KK(A, M_2(I))$ and $KL(A, \iota_I^{(2)})\colon KL(A, I) \to KL(A, M_2(I))$ are isomorphisms. Moreover,

$$\begin{aligned} KK(A, \iota_I^{(2)})([\phi, \psi]_{KK(A,I)}) &= [\iota_E^{(2)} \circ \phi, \iota_E^{(2)} \circ \psi]_{KK(A,M_2(I))}, \quad \textit{and} \\ KL(A, \iota_I^{(2)})([\phi, \psi]_{KL(A,I)}) &= [\iota_E^{(2)} \circ \phi, \iota_E^{(2)} \circ \psi]_{KL(A,M_2(I))}. \end{aligned} \tag{B.22}$$

(ii) For any unitary $u \in I^\dagger$,

$$\begin{aligned} [\phi, \psi]_{KK(A,I)} &= [\operatorname{Ad} u \circ \phi, \psi]_{KK(A,I)}, \ \textit{and} \\ [\phi, \psi]_{KL(A,I)} &= [\operatorname{Ad} u \circ \phi, \psi]_{KL(A,I)}. \end{aligned} \tag{B.23}$$

(iii) $[\phi, \psi]_{KK(A,I)} + [\psi, \phi]_{KK(A,I)} = 0$ and $[\phi, \psi]_{KL(A,I)} + [\psi, \phi]_{KL(A,I)} = 0$.

(iv) If $J \lhd F$ and $\theta\colon I \to J$ is a *-homomorphism that extends to $\bar{\theta}\colon E \to F$, then $KK(A, \theta)([\phi, \psi]_{KK(A,I)}) = [\bar{\theta} \circ \phi, \bar{\theta} \circ \psi]_{KK(A,J)}$ and similarly $KL(A, \theta)([\phi, \psi]_{KL(A,I)}) = [\bar{\theta} \circ \phi, \bar{\theta} \circ \psi]_{KL(A,J)}$.

(v) Let A be nuclear, and suppose

$$\mathsf{e}\colon 0 \longrightarrow I \xrightarrow{j_{\mathsf{e}}} E \xrightarrow{q_{\mathsf{e}}} D \longrightarrow 0 \tag{B.24}$$

is an extension of C^*-algebras. Then the sequence

$$KK(A,I) \xrightarrow{KK(A,j_{\mathsf{e}})} KK(A,E) \xrightarrow{KK(A,q_{\mathsf{e}})} KK(A,D) \tag{B.25}$$

is exact. The same holds when A is KK-equivalent to a nuclear C^*-algebra.

Proof. (i): In the case when I is separable, the KK isomorphism result follows from [10, Example 17.8.2(c)], for example. The formula for $KK(A,\iota_I^{(2)})$ follows from Proposition B.3. For general I, the result follows from the definition of $KK(A,\iota_I^{(2)})$ as a limit of maps $KK(A,\iota_{I_0}^{(2)})$ (where $I_0 \subseteq I$ is separable), in the same spirit as the proof of Proposition B.5.

(ii): First suppose that I is separable. We may assume that the scalar part of u is $1_{I^\dagger}$. Then $u \oplus u^*$ is homotopic to $1_{I^\dagger} \oplus 1_{I^\dagger}$ via the unitary path

$$[0,\pi/2] \ni t \mapsto \begin{pmatrix} \cos^2(t)(u-1_{I^\dagger}) + 1_{I^\dagger} & -\cos(t)\sin(t)(u-1_{I^\dagger}) \\ \cos(t)\sin(t)(u^*-1_{I^\dagger}) & \cos^2(t)(u^*-1_{I^\dagger}) + 1_{I^\dagger} \end{pmatrix} \tag{B.26}$$

which is constantly $1_{I^\dagger} \oplus 1_{I^\dagger}$ modulo $M_2(I)$, since $u - 1_{I^\dagger} \in I$. Conjugating with this unitary path in the first entry induces a homotopy of $(A, M_2(I))$-Cuntz pairs from $((\operatorname{Ad} u \circ \phi) \oplus 0, \psi \oplus 0) = (\operatorname{Ad}(u \oplus u^*) \circ (\phi \oplus 0), \psi \oplus 0)$ to $(\phi \oplus 0, \psi \oplus 0)$. Thus (using the identification in (i)), $[\phi,\psi]_{KK(A,I)} = [\operatorname{Ad} u \circ \phi, \psi]_{KK(A,I)}$.

For non-separable I, the result follows from the separable case and the definition $KK(A,I)$ as a direct limit (see Definition 5.3). The KL version follows by taking quotients.

(iii): When I is separable, the KK formula can be found in [142, Proposition 4.1.5], for example. For non-separable I, the result follows from the separable case, and the KL version follows by taking quotients.

(iv): When I and J are separable, this is the last part of Proposition B.3. For the non-separable case, set $E_0 \coloneqq C^*(\phi(A) \cup \psi(A)) \subseteq E$ and $F_0 \coloneqq C^*(\theta(I_0)) \subseteq F$. Then let $I_0 \lhd E_0$ and $J_0 \lhd F_0$ be the ideals generated by $(\phi-\psi)(A)$ and $\theta(I_0)$, respectively. Then Proposition B.3 gives

$$KK(A, \theta|_{I_0}^{J_0})\big([\phi|^{E_0}, \psi|^{E_0}]_{KK(A,I_0)}\big) = \big[(\bar\theta \circ \phi)|^{F_0}, (\bar\theta \circ \psi)|^{F_0}\big]_{KK(A,J_0)}. \tag{B.27}$$

By the definition of $KK(A,\theta)$ in the proof of Proposition B.5, it follows that $KK(A,\theta)([\phi,\psi]_{KK(A,I)}) = [\bar\theta \circ \phi, \bar\theta \circ \psi]_{KK(A,J)}$. The computation for KL follows by taking quotients.

(v): This is standard when E is separable, being a fragment of the six-term exact sequence (see [10, Theorem 19.5.7 and Example 19.5.2(a)], or [10, Exercise 20.10.2(f)] for the case that A is KK-equivalent to a nuclear C^*-algebra, for example). For the non-separable case, as $\operatorname{im} KK(A, j_{\mathsf{e}}) \subseteq \ker KK(A, q_{\mathsf{e}})$ by functoriality, the only thing to check is that $\ker KK(A,q_{\mathsf{e}}) \subseteq \operatorname{im} KK(A, j_{\mathsf{e}})$. Fix $\kappa \in \ker KK(A, q_{\mathsf{e}})$. As in the proof of Proposition B.5, there exist separable subalgebras E_0 and D_0 of E and D, respectively, such that $q_{\mathsf{e}}(E_0) \subseteq D_0$, κ comes from some element $\kappa_0 \in KK(A, E_0)$, and $KK(A, q_{\mathsf{e}}|_{E_0}^{D_0})(\kappa_0) = 0$ in $KK(A, D_0)$. Setting $I_0 \coloneqq I \cap E_0$, we have an extension of separable C^*-algebras

$$\mathsf{e}_0\colon 0 \to I_0 \xrightarrow{j_{\mathsf{e}_0}} E_0 \xrightarrow{q_{\mathsf{e}_0}} D_0 \to 0. \tag{B.28}$$

Thus by the separable case, there exists $\kappa_0' \in KK(A, I_0)$ such that $KK(A, j_{\mathsf{e}_0})(\kappa_0') = \kappa_0$. Letting $\kappa' \in KK(A, I)$ be the image of κ_0', it follows that $KK(A, j_{\mathsf{e}})(\kappa') = \kappa$. □

B.4. **The universal multicoefficient theorem.** Now we turn to the natural map

$$\Gamma^{(A,I)} \colon KK(A, I) \to \mathrm{Hom}(K_*(A), K_*(I)). \tag{B.29}$$

In the case that I is σ-unital, the natural identification of $KK(\mathbb{C}, S^i I)$ with $K_i(I)$ for $i = 0, 1$ ([10, Proposition 17.5.5]) combined with the Kasparov product provides this map. For $\phi\colon A \to I$, since $KK(\mathbb{C}, S^i\phi)$ identifies with $K_i(\phi)$ and since taking the Kasparov product with $[\phi]_{KK(A,I)}$ corresponds to applying $KK(\,\cdot\,, \phi)$ ([10, Example 18.4.2(a)]), we have

$$\Gamma_i^{(A,I)}([\phi]_{KK(A,I)}) = K_i(\phi). \tag{B.30}$$

For I non-separable, since $K_*(A)$ is countable, we have a natural isomorphism

$$\mathrm{Hom}(K_*(A), K_*(I)) \cong \varinjlim_{I_0 \text{ sep.}} \mathrm{Hom}(K_*(A), K_*(I_0)), \tag{B.31}$$

(see [253, Proposition 1.10], for example), and upon making this identification we may define

$$\Gamma^{(A,I)} \coloneqq \varinjlim_{I_0 \text{ sep.}} \Gamma^{(A,I_0)}. \tag{B.32}$$

When I is both non-separable and σ-unital, a priori this gives two competing definitions for $\Gamma^{(A,I)}$. By Remark B.7, they agree.

More generally, identifying $KK(\mathbb{I}_n, S^i I)$ naturally with $K_i(I; \mathbb{Z}/n)$ as per the picture in [66] (using continuity of $K_i(\,\cdot\,; \mathbb{Z}/n)$ for non-separable I), we likewise have[161]

$$\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I)) \cong \varinjlim_{I_0 \text{ sep.}} \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I_0)). \tag{B.33}$$

This gives a homomorphism

$$\Gamma_\Lambda^{(A,I)} \colon KK(A, I) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I)). \tag{B.34}$$

Just as with $\Gamma^{(A,I)}$, the two possible definitions of $\Gamma_\Lambda^{(A,I)}$ for I σ-unital and non-separable agree. When A satisfies the UCT, Dadarlat and Loring's universal multicoefficient theorem says that $\Gamma_\Lambda^{(A,I)}$ descends to an isomorphism when taking the KL quotient on the left. While their statement only allows separable codomains, the result extends to the non-separable case.

Proof of Theorem 8.5. First let us justify that $\tilde{\Gamma}_\Lambda^{(A,I)}$ is well-defined, without assuming that A satisfies the UCT. When I is separable, this is implicit in the paragraph

[161]Injectivity of the map $\varinjlim_{I_0 \text{ sep.}} \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I_0)) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I))$ follows directly from injectivity of the corresponding map (B.31). For surjectivity, given an element $\underline{\alpha}$ of $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I))$, first by surjectivity of the corresponding map (B.31), we can lift $\underline{\alpha}$ to a graded group homomorphism $\underline{K}(A) \to \underline{K}(I_0)$ for some separable $I_0 \subseteq I$. Then using injectivity of the map (B.31), we can enlarge I_0 to ensure compatibility with each of the countably many Bockstein operations.

following [61, Eq. (13)].[162] For I non-separable, since $Z_{KK(A,I)} = \varinjlim_{I_0 \text{ sep.}} Z_{KK(A,I_0)}$ (Lemma B.8), it follows from the separable case that $\Gamma_\Lambda^{(A,I)}$ vanishes on $Z_{KK(A,I)}$.

Now, suppose that A satisfies the UCT. If I is separable, then $\tilde{\Gamma}_\Lambda^{(A,I)}$ is an isomorphism by [61, Theorem 4.1] (see also [66]). For the non-separable case, we have

$$\tilde{\Gamma}_\Lambda^{(A,I)} = \varinjlim_{I_0 \text{ sep.}} \tilde{\Gamma}_\Lambda^{(A,I_0)}. \tag{B.35}$$

Since $\tilde{\Gamma}_\Lambda^{(A,I_0)} \colon KL(A,I_0) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I_0))$ is an isomorphism for each I_0, it follows that $\tilde{\Gamma}_\Lambda^{(A,I)}$ is as well. □

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[162] The idea is that for $\kappa \in KK(A, C(\overline{\mathbb{N}}, I))$, the sequence

$$\left(\Gamma_\Lambda^{(A,I)}(KK(A,\mathrm{ev}_n)(\kappa))_i^{(m)} \colon K_i(A;\mathbb{Z}/m) \to K_i(I;\mathbb{Z}/m)\right)_{n=1}^\infty$$

of homomorphisms is eventually equal to $\Gamma_\Lambda^{(A,I)}(KK(A,\mathrm{ev}_\infty)(\kappa))_i^{(m)}$. Hence any element of $Z_{KK(A,I)}$ is mapped to zero by $\Gamma_\Lambda^{(A,I)}$.

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José R. Carrión, Department of Mathematics, Texas Christian University, Fort Worth, Texas 76109, United States
Email address: `j.carrion@tcu.edu`

James Gabe, Department of Mathematics and Computer Science, University of Southern Denmark, 5230 Odense, Denmark
Email address: `gabe@imada.sdu.dk`

Christopher Schafhauser, Department of Mathematics, University of Nebraska-Lincoln, Lincoln, Nebraska 68588, United States
Email address: `cschafhauser2@unl.edu`

Aaron Tikuisis, Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa, ON, K1N 6N5, Canada
Email address: `aaron.tikuisis@uottawa.ca`

Stuart White, Mathematical Institute, University of Oxford, Oxford, OX2 6GG, United Kingdom
Email address: `stuart.white@maths.ox.ac.uk`