# On groups with quasidiagonal C*-algebras

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## Abstract

We examine the question of quasidiagonality for C*-algebras of discrete amenable groups from a variety of angles. We give a quantitative version of Rosenberg's theorem via paradoxical decompositions and a characterization of quasidiagonality for group C*-algebras in terms of embeddability of the groups. We consider several notable examples of groups, such as topological full groups associated with Cantor minimal systems and Abels' celebrated example of a finitely presented solvable group that is not residually finite, and show that they have quasidiagonal C*-algebras. Finally, we study strong quasidiagonality for group C*-algebras, exhibiting classes of amenable groups with and without strongly quasidiagonal C*-algebras.
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## 1. Introduction

In [23] Lance provided a C*-algebraic characterization of amenability for discrete groups by proving that a discrete group $\Gamma$ is amenable if and only if its reduced C*-algebra, $C_r^*(\Gamma)$ is nuclear. Later Rosenberg showed [18] that if $C_r^*(\Gamma)$ is quasidiagonal (see Definition 1.4), then $\Gamma$ is amenable, a result which has absolutely no analog for general C*-algebras (see [12]). The

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converse to Rosenberg's theorem remains open, namely: if $\Gamma$ is a discrete, amenable group, is $C_r^*(\Gamma)$ quasidiagonal [31]?

The question of quasidiagonality for amenable groups is tantalizing for a number of reasons. First, quasidiagonality displays certain "topological" properties, such as homotopy invariance [30]. On the other hand, one might describe amenability as a "measure theoretic" property, as one can detect amenability of $\Gamma$ in the von Neumann algebras it generates. Hence an affirmative answer would provide a nice topological characterization of amenability to complement its measure theoretic description. Second, this question is a critical test case for a number of other open questions. Indeed, it is not known if every separable, nuclear and stably finite C*-algebra is quasidiagonal (a question with important implications for the classification program) and, much more generally, if every stably finite C*-algebra is an MF algebra. Thus an answer to the above question concerning groups will either provide a chain of counterexamples or some evidence to the validity of the more general conjectures.

There are some known converses to Rosenberg's theorem. Recall that a group $\Gamma$ is *maximally almost periodic* (MAP) if it embeds into a compact group. Because the C*-algebra of an amenable MAP group is residually finite dimensional [3], it follows that the C*-algebra of an amenable group that is the union of residually finite groups must be quasidiagonal. We generalize this result in Section 2.6.

Our main results are the following. First, if $\Gamma$ is not amenable, then the modulus of quasidiagonality of $C_r^*(\Gamma)$ is controlled by the number of pieces in a paradoxical decomposition of $\Gamma$ (Theorem 2.4). Second, if $\Gamma$ is amenable, then $C^*(\Gamma)$ is quasidiagonal if and only if $\Gamma$ embeds in the unitary group of $\prod_{n=1}^{\infty} M_n(\mathbb{C}) / \sum_{n=1}^{\infty} M_n(\mathbb{C})$ (Theorem 2.8). We expand this class of groups beyond the class of LEF groups of [29]. Third, if $\Gamma$ and $\Lambda$ are amenable groups such that $\Gamma$ is non-torsion and $\Lambda$ has a finite dimensional representation other than the trivial one, then $C^*(\Lambda \wr \Gamma)$ has a non-finite quotient and therefore cannot be strongly quasidiagonal (Theorem 3.4).

## 1.1. Organization of the paper

In Section 2.1 we revisit Rosenberg's previously mentioned result. His result implies that the modulus of quasidiagonality [28] does not vanish for some finite subset of a non-amenable group. The modulus of quasidiagonality measures how badly a C*-algebra violates quasidiagonality. We estimate this number and a closely related one using paradoxical decompositions, and give some calculations for free groups.

In Section 2.6 we consider an approximate version of MAP for groups that characterizes quasidiagonality for discrete amenable groups. We call groups with this property MF due to their connection with Blackadar and Kirchberg's MF algebras [5]. We then show that the groups that are locally embeddable into finite groups in the sense of Vershik and Gordon [29] (so-called LEF groups) are MF groups. Kerr had already proved that the C*-algebra of an amenable LEF group is quasidiagonal [21].

In Section 2.14, we use our characterization of quasidiagonality for amenable groups to give examples of solvable groups that are not LEF but have quasidiagonal C*-algebras. These groups are well-known examples due to Abels of finitely presented solvable groups that are not residually finite.

Finally, in Section 3 we discuss groups and strong quasidiagonality (see Definition 1.4). Theorem 3.4 provides examples of group C*-algebras that are not strongly quasidiagonal, such as

the C*-algebra of the lamplighter group. Section 3.8 exhibits some classes of nilpotent groups that have strongly quasidiagonal C*-algebras.

## 1.2. Some consequences

Let $X$ be the Cantor set and $T$ a minimal homeomorphism of $X$. The *topological full group* $[\![T]\!]$ is the group of all homeomorphisms of $X$ that are locally equal to an integer power of $T$. These groups are of interest for several reasons. For example, they are complete invariants for flip conjugacy [16] and studying their properties as abstract groups led to the first examples of infinite, simple, amenable groups that are finitely generated [17,20,26]. It follows from the results of Section 2.6 that the C*-algebra of $[\![T]\!]$ must be quasidiagonal, since $[\![T]\!]$ is LEF by [17] and amenable by [20]. Since the previously mentioned examples of infinite, simple, amenable and finitely generated groups arise as commutator subgroups of topological full groups associated to certain Cantor minimal systems, their C*-algebras are quasidiagonal as well.

On the other hand, an example of Abels provides an amenable group that is not LEF. We observe that if a group is not LEF, then it cannot be a union of residually finite groups and one cannot obtain quasidiagonality based on the result of Bekka mentioned above. However, we will see in Section 2.14 that the $C^*$-algebra of Abels' example has a quasidiagonal C*-algebra.

## 1.3. Definitions and notation

For completeness we record the definition of quasidiagonality. We refer the reader to the survey article [8] for more information on quasidiagonality.

**Definition 1.4.** Let $H$ be a separable Hilbert space. A (separable) set $\Omega \subset B(H)$ is *quasidiagonal* if there is an increasing sequence of (self-adjoint) projections $(P_n) \subset \mathcal{K}(H)$ with $P_n \to 1_H$ strongly and such that $\|[P_n, T]\| \to 0$ for every $T \in \Omega$. (We write $[S, T]$ for the commutator $ST - TS$.)

A separable $C^*$-algebra $A$ is *quasidiagonal* if it has a faithful representation as a set of quasidiagonal operators. We say $A$ is *strongly quasidiagonal* if $\sigma(A)$ is a quasidiagonal set of operators for every representation $\sigma$ of $A$.

**Theorem 1.5** *(Voiculescu [30]). A separable $C^*$-algebra is quasidiagonal if and only if there exists a sequence of contractive completely positive maps $\phi_n : A \to M_{k_n}(\mathbb{C})$ such that $\|\phi_n(a)\| \to \|a\|$ and $\|\phi(ab) - \phi(a)\phi(b)\| \to 0$ for every $a, b \in A$.*

In this paper we only consider discrete countable groups. The left regular representation of a group $\Gamma$ on $B(\ell^2\Gamma)$ maps $s \in \Gamma$ to the operator $\lambda_s \in B(\ell^2\Gamma)$ which is left-translation by $s$. For $t \in \Gamma$ we write $\delta_t \in \ell^2\Gamma$ for the characteristic function of the set $\{t\}$, so that $\lambda_s\delta_t = \delta_{st}$. The reduced C*-algebra of $\Gamma$ is the sub-C*-algebra $C_r^*(\Gamma)$ of $B(\ell^2\Gamma)$ generated by $\lambda(\Gamma)$. We will usually use $e$ for the neutral element of a group $\Gamma$ and $Z(\Gamma)$ for its center. We also write $Z(A)$ for the center of a C*-algebra $A$.

## 2. Quasidiagonality and groups

### 2.1. A quantitative version of Rosenberg's theorem

In [18] Rosenberg proved that if a group $\Gamma$ is not amenable, then $C_r^*(\Gamma)$ is not quasidiagonal. First we reformulate his result.

**Definition 2.2.** Let $\mathcal{P}$ be the set of non-zero finite-rank projections on $\ell^2\Gamma$. Given a finite subset $F \subset \Gamma$, set

$$C_F := \inf_{P \in \mathcal{P}} \sup_{x \in F} \big\| [\lambda_x, P] \big\|.$$

It is clear that if $C_r^*(\Gamma)$ is quasidiagonal, then $C_F = 0$ for every finite subset of $\Gamma$. Furthermore, if $\Gamma$ is amenable, then $\lambda$ has an approximately fixed vector, so $C_F = 0$ for all finite subsets as well. Rosenberg [18] has proved that if $\Gamma$ is not amenable, then there is a finite subset $F \subseteq \Gamma$ such that $C_F > 0$. In this section we give a quantitative version of this statement by estimating (and in some cases calculating) $C_F$ using paradoxical decompositions of $\Gamma$.

We point out a very similar concept due to Pimsner, Popa and Voiculescu [28]. Recall that the *modulus of quasidiagonality* of a set $\Omega \subset B(\ell^2\Gamma)$ is

$$\mathrm{qd}(\Omega) := \liminf_{P \in \mathcal{P}} \sup_{T \in \Omega} \big\| [T, P] \big\|,$$

where the order on projections is given by $P \leqslant Q$ if $PQ = P$. Clearly $C_F \leqslant \mathrm{qd}(\lambda(F))$.

Recall that a group $\Gamma$ is not amenable if and only if it admits a *paradoxical decomposition*: that is, there exist pairwise disjoint subsets $X_1, \ldots, X_n, Y_1, \ldots, Y_m \subseteq \Gamma$ and $g_1, \ldots, g_n, h_1, \ldots, h_m \in \Gamma$ with $g_1 = h_1 = e$ such that

$$\Gamma = \bigsqcup_{i=1}^{n} g_i X_i = \bigsqcup_{j=1}^{m} h_j Y_j = \left( \bigsqcup_{i=1}^{n} X_i \right) \sqcup \left( \bigsqcup_{j=1}^{m} Y_j \right). \tag{1}$$

In this case we say that the paradoxical decomposition has $n + m$ *pieces*. A non-amenable group always has a paradoxical decomposition with at least four pieces. It is well known that a group contains a copy of the free group on two generators if and only if one can find a paradoxical decomposition with exactly 4 pieces. (We refer the reader to [32] for more information on paradoxical decompositions.)

We will require an elementary lemma.

**Lemma 2.3.** *Let $H$ be a Hilbert space and let* $\mathrm{Tr}$ *denote the usual trace on $B(H)$. Let $X = X^* \in B(H)$ be finite rank with $\mathrm{Tr}(X) = 0$. Then for any $Q \in B(H)$ with $0 \leqslant Q \leqslant 1$ we have*

$$\big| \mathrm{Tr}(QX) \big| \leqslant \frac{1}{2} \, \mathrm{rank}(X) \| X \|.$$

**Proof.** If $Y$ is a finite-rank operator, then $\mathrm{Tr}(Y) \leqslant \mathrm{rank}(Y) \| Y \|$. Indeed, if $E$ is a projection onto the range of $Y$, then $\mathrm{Tr}(Y) = \mathrm{Tr}(EY) \leqslant \mathrm{Tr}(E) \| Y \| = \mathrm{rank}(Y) \| Y \|$.

Write $X = X_+ - X_-$ with $X_\pm \geqslant 0$ and $X_+ X_- = 0$. Then $\mathrm{Tr}(X_\pm) \leqslant \mathrm{rank}(X_\pm)\|X_\pm\| \leqslant \mathrm{rank}(X_\pm)\|X\|$. Since $\mathrm{rank}(X) = \mathrm{rank}(X_+) + \mathrm{rank}(X_-)$ and $\mathrm{Tr}(X_+) = \mathrm{Tr}(X_-)$ we obtain

$$\mathrm{Tr}(X_\pm) \leqslant \frac{1}{2}\,\mathrm{rank}(X)\|X\|.$$

Now, since

$$\mathrm{Tr}(QX) = \mathrm{Tr}\big(Q^{1/2}X_+Q^{1/2}\big) - \mathrm{Tr}\big(Q^{1/2}X_-Q^{1/2}\big)$$

and $\mathrm{Tr}(Q^{1/2}X_\pm Q^{1/2}) \leqslant \|Q\|\,\mathrm{Tr}(X_\pm) \leqslant \mathrm{Tr}(X_\pm)$, it follows that

$$-\frac{1}{2}\,\mathrm{rank}(X)\|X\| \leqslant -\mathrm{Tr}(X_-) \leqslant \mathrm{Tr}(QX) \leqslant \mathrm{Tr}(X_+) \leqslant \frac{1}{2}\,\mathrm{rank}(X)\|X\|. \qquad \square$$

**Theorem 2.4.** *Suppose $\Gamma$ is a non-amenable group with a paradoxical decomposition as in (1). If $F = \{g_1, \ldots, g_n, h_1, \ldots, h_m\}$, then*

$$C_F \geqslant \frac{1}{n+m-2}.$$

*In particular, if $\Gamma$ contains $\mathbb{F}_2$, then $C_F \geqslant 1/2$ by choosing a minimal decomposition with four pieces.*

*Since $\mathrm{qd}(\lambda(F)) \geqslant C_F$ we have the same statement for the modulus of quasidiagonality of $\lambda(F)$ instead of $C_F$.*

**Proof.** For each subset $A \subseteq \Gamma$, let $P_A$ be the projection onto $\overline{\mathrm{span}}\{\delta_a\colon a \in A\}$. Let $\mathrm{Tr}$ denote the usual semi-finite trace on $B(\ell^2\Gamma)$ and let $P \in B(\ell^2\Gamma)$ be a finite-rank projection of rank $k \geqslant 1$. Suppose that $\|[\lambda_x, P]\| \leqslant \varepsilon$ for all $x \in F$. We prove $\varepsilon \geqslant \frac{1}{n+m-2}$.

Let $1 \leqslant i \leqslant n$. By Lemma 2.3,

$$\big|\mathrm{Tr}\big(P_{g_i X_i}(P - \lambda_{g_i} P \lambda_{g_i^{-1}})\big)\big| \leqslant k\varepsilon. \tag{2}$$

Because $\lambda_{g_i} P_{X_i} \lambda_{g_i^{-1}} = P_{g_i X_i}$, we have

$$\mathrm{Tr}(P_{g_i X_i} P) = \mathrm{Tr}(P_{X_i} P) + \mathrm{Tr}\big(P_{g_i X_i}(P - \lambda_{g_i} P \lambda_{g_i^{-1}})\big).$$

From this and the estimate (2) it follows that for each $2 \leqslant i \leqslant n$

$$\mathrm{Tr}(P_{g_i X_i} P) \leqslant \mathrm{Tr}(P_{X_i} P) + k\varepsilon. \tag{3}$$

Let $X = \bigcup X_i$ and $Y = \bigcup Y_j$. By (3) and the fact that $g_1 = e$ we obtain

$$k = \mathrm{Tr}(P) = \mathrm{Tr}(P_{X_1} P) + \sum_{i=2}^{n} \mathrm{Tr}(P_{g_i X_i} P)$$

$$\leqslant \mathrm{Tr}(P_{X_1} P) + \sum_{i=2}^{n} \mathrm{Tr}(P_{X_i} P) + (n-1)k\varepsilon$$

$$= \mathrm{Tr}(P_X P) + (n-1)k\varepsilon. \tag{4}$$

From a similar calculation involving the $h_i$'s and $Y_i$'s we see that

$$k \leqslant \mathrm{Tr}(P_Y P) + (m-1)k\varepsilon. \tag{5}$$

Finally, add up (4) and (5) to obtain the conclusion.  □

Now we calculate $C_F$ when $\Gamma = \mathbb{F}_2$. Let us fix some notation first. Let $a, b$ be generators of $\mathbb{F}_2$ and for each word $w \in \mathbb{F}_2$ let $|w|$ denote the word length of $w$ with respect to the generating set $\{a, b, a^{-1}, b^{-1}\}$. For each $n \geqslant 0$ let $S_n$ denote the sphere of radius $n$, that is,

$$S_n = \{w \in \mathbb{F}_2 : |w| = n\}.$$

Note that $S_0 = \{e\}$. For each $x \in \{a, b, a^{-1}, b^{-1}\}$, let $S_n^x$ denote those elements of $S_n$ whose first letter is $x$. It is easy to see that

$$|S_n| = 4 \cdot 3^{n-1} \quad \text{and} \quad |S_n^x| = 3^{n-1} \quad \text{for } n \geqslant 1. \tag{6}$$

It is well known that (see [32, Theorem 4.2]) there is a paradoxical decomposition of $\mathbb{F}_2$ as

$$\mathbb{F}_2 = X_1 \sqcup a X_2 = Y_1 \sqcup b Y_2 = X_1 \sqcup X_2 \sqcup Y_1 \sqcup Y_2. \tag{7}$$

**Theorem 2.5.** *For any $\varepsilon > 0$ and any $n \geqslant 1$ there is a projection $P \in B(\ell^2 \mathbb{F}_2)$ of rank $n$ such that $\|[\lambda_a, P]\| < 1/2 + \varepsilon$ and $\|[\lambda_b, P]\| < 1/2 + \varepsilon$. In particular,*

$$C_{\{a,b\}} = 1/2.$$

**Proof.** By Voiculescu's Weyl–von Neumann type theorem, $\lambda : \mathbb{F}_2 \to B(\ell^2 \mathbb{F}_2)$ is approximately unitarily equivalent to $\lambda \otimes 1_n : \mathbb{F}_2 \to B(\ell^2 \mathbb{F}_2 \otimes \mathbb{C}^n)$. On the other hand $\|[\lambda_x \otimes 1_n, P \otimes 1_n]\| = \|[\lambda_x, P]\|$ for $P \in B(\ell^2 \mathbb{F}_2)$. It follows that it suffices to prove the first part of the statement for $n = 1$.

Let $P \in B(\ell^2 \mathbb{F}_2)$ be any projection and $U \in B(\ell^2 \mathbb{F}_2)$ a unitary. Since in a C*-algebra $\|x^* x\| = \|x\|^2$, using the identity $[U, P] = (1 - P)UP - PU(1 - P)$ we see that $\|[U, P]\| = \max\{\|PU(1 - P)\|, \|(1 - P)UP\|\}$. Moreover, if $P$ is rank 1 and $\xi$ is a norm one vector in its range, then

$$\|(1 - P)UP\|^2 = \|(1 - P)UP\xi\|^2 = \|(1 - P)U\xi\|^2$$

$$= \|U\xi\|^2 - \|PU\xi\|^2 = 1 - |\langle U(\xi), \xi \rangle|^2.$$

From the above observations it now suffices to find, for each $\varepsilon > 0$, a norm 1 vector $\xi \in \ell^2 \mathbb{F}_2$ such that

$$\left| \langle \lambda_x(\xi), \xi \rangle \right| > \frac{\sqrt{3}}{2} - \varepsilon \quad \text{for } x \in \{a, a^{-1}, b, b^{-1}\}. \tag{8}$$

Let $n > \sqrt{3}/2\varepsilon$. Define $\alpha_i = (|S_i|n)^{-1/2}$ and

$$\xi = \sum_{i=1}^{n} \alpha_i \left( \sum_{x \in S_i} \delta_x \right).$$

It is clear that $\|\xi\| = 1$. We then have

$$
\begin{aligned}
\langle \lambda_a(\xi), \xi \rangle &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle \sum_{x \in S_i} \alpha_i \delta_{ax}, \sum_{y \in S_j} \alpha_j \delta_y \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle \sum_{x \in (S_{i-1} \setminus S_{i-1}^a) \cup S_{i+1}^a} \alpha_i \delta_x, \sum_{y \in S_j} \alpha_j \delta_y \right\rangle \\
&= \sum_{i=1}^{n} \left\langle \sum_{x \in (S_{i-1} \setminus S_{i-1}^a) \cup S_{i+1}^a} \alpha_i \delta_x, \sum_{y \in S_{i-1}} \alpha_{i-1} \delta_y + \sum_{z \in S_{i+1}} \alpha_{i+1} \delta_z \right\rangle \\
&= \sum_{i=1}^{n} \left( \alpha_i \alpha_{i-1} |S_{i-1} \setminus S_{i-1}^a| + \alpha_i \alpha_{i+1} |S_{i+1}^a| \right) \\
&\geqslant \sum_{i=2}^{n} \left( \frac{1}{4n\sqrt{3^{i-1}}\sqrt{3^{i-2}}} (3)(3^{i-2}) + \frac{1}{4n\sqrt{3^i}\sqrt{3^{i-1}}} 3^i \right) \quad \text{(by (6))} \\
&= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2n} > \frac{\sqrt{3}}{2} - \varepsilon.
\end{aligned}
$$

The corresponding inequality for $\lambda_b$ follows by symmetry. Since $|\langle \lambda_x(\xi), \xi \rangle| = |\langle \lambda_{x^{-1}}(\xi), \xi \rangle|$, this proves (8). We complete the proof by applying Theorem 2.4 to the paradoxical decomposition of $\mathbb{F}_2$ given in (7). $\quad \square$

### 2.6. A characterization of quasidiagonality for amenable groups

For each increasing sequence $\vec{n} = (n_k)$ of positive integers, we consider the C*-algebra

$$Q_{\vec{n}} = \prod_k M_{n_k}(\mathbb{C}) / \sum_k M_{n_k}(\mathbb{C}).$$

Here the C*-algebra $\prod_k M_{n_k}(\mathbb{C})$ consists of all sequences $(a_k)$ of matrices $a_k \in M_{n_k}(\mathbb{C})$ such that $\sup_k \|a_k\| < \infty$ and $\sum_k M_{n_k}(\mathbb{C})$ is the two-sided closed ideal consisting of those sequences with the property that $\lim_{k \to \infty} \|a_k\| = 0$.

Recall that a separable C*-algebra is MF if it embeds as a sub-C*-algebra of $Q_{\vec{n}}$ for some $\vec{n}$, see [5]. In analogy with the class of MF algebras, we make the following definition.

**Definition 2.7.** A countable group $\Gamma$ is *MF* if it embeds in the unitary group of $Q_{\vec{n}}$ for some $\vec{n}$.

It is readily seen that $\Gamma$ is MF if and only if it embeds in $U(Q_{\vec{n}})$ where $\vec{n} = (1, 2, 3, \ldots)$.

Recall that a group $\Gamma$ is called *maximally almost periodic* (abbreviated MAP) if it embeds in a compact group. Equivalently, $\Gamma$ embeds in

$$ U\left( \prod_{k=1}^{\infty} M_k(\mathbb{C}) \right) = \prod_{k=1}^{\infty} U(k). $$

A discrete residually finite group is MAP.

Bekka [3] proved that if $\Gamma$ is a discrete countable amenable group, then $\Gamma$ is maximally almost periodic if and only if $C^*(\Gamma)$ is residually finite dimensional. That is,

$$ \Gamma \hookrightarrow U\left( \prod_{k=1}^{\infty} M_k(\mathbb{C}) \right) \quad \Leftrightarrow \quad C^*(\Gamma) \hookrightarrow \prod_{k=1}^{\infty} M_k(\mathbb{C}). $$

The following theorem says that a discrete countable amenable group $\Gamma$ embeds in $U(Q_{\vec{n}})$ for some sequence $\vec{n}$ if and only if $C^*(\Gamma)$ embeds in $Q_{\vec{m}}$ for some $\vec{m}$.

**Theorem 2.8.** *Let $\Gamma$ be a discrete countable amenable group. Then $\Gamma$ is MF if and only if the C*-algebra $C^*(\Gamma)$ is quasidiagonal.*

For the proof we will need the following result from [14].

**Proposition 2.9.** *(See Proposition 2.1 of [14].) Let $\Gamma$ be a discrete amenable group. Suppose there exist a sequence $(B_k)_{k=1}^{\infty}$ of unital C*-algebras and a sequence $(\omega_k)_{k=1}^{\infty}$ of group homomorphisms $\omega_k : \Gamma \to U(B_k)$ that separate the points of $\Gamma$, and that each $\omega_k$ appears infinitely many times in the sequence. Then $C^*(\Gamma)$ embeds unitally into the product C*-algebra $\prod_{n=1}^{\infty} C_n$, where $C_n = \bigotimes_{k=1}^{n} M_2(\mathbb{C}) \otimes B_k \otimes B_k$ (minimal tensor product).*

**Proof of Theorem 2.8.** By the Choi–Effros lifting theorem [11] and the local characterization of quasidiagonality given by Voiculescu, Theorem 1.5, it follows that a separable and nuclear C*-algebra $A$ is quasidiagonal if and only if $A$ is an MF algebra. Suppose that $\Gamma$ is MF. Then there is an injective homomorphism $\theta : \Gamma \to U(Q_{\vec{n}})$ for some $\vec{n}$. Let $B$ be the sub-C*-algebra of $Q_{\vec{n}}$ generated by $\theta(\Gamma)$. Since $\Gamma$ is amenable, $B$ is nuclear and hence quasidiagonal. By Proposition 2.9 $C^*(\Gamma)$ embeds the product $\prod_n C_n$, where $C_n = M_{2^n}(\mathbb{C}) \otimes B^{\otimes 2n}$. Since $B$ is quasidiagonal, so is each $C_n$. It follows that $C^*(\Gamma)$ is quasidiagonal.

Conversely, if $C^*(\Gamma)$ is quasidiagonal, then $C^*(\Gamma) \subset Q_{\vec{n}}$ for some $\vec{n}$ and hence $\Gamma$ is MF.   $\square$

**Definition 2.10.** A group $\Gamma$ is *locally embeddable into the class of finite groups* (or simply LEF) if for any finite subset $F \subset \Gamma$ there is a finite group $H$ and a map $\phi : \Gamma \to H$ that is both injective and multiplicative when restricted to $F$.

**Remark 2.11.** Vershik and Gordon introduced LEF groups in [29]. Theorems 1 and 2 of [29] illustrate the relationship between LEF groups and residually finite groups. Specifically, if every finitely generated subgroup of a group $\Gamma$ is residually finite, then $\Gamma$ is LEF. On the other hand, every finitely presented LEF group is residually finite. There are finitely presented solvable non-LEF groups, see [1].

We will show that an LEF group is MF. The following lemma will be used.

**Lemma 2.12.** *Let $\Gamma$ be a discrete countable group. Then $\Gamma$ is LEF if and only if there is a sequence of finite groups $(H_k)_{k=1}^{\infty}$ and an injective homomorphism $\Phi : \Gamma \to \prod_k H_k / \sum_k H_k$.*

The proof is straightforward.

**Proposition 2.13.** *Let $\Gamma$ be a countable discrete group. If $\Gamma$ is LEF, then $\Gamma$ is MF.*

**Proof.** Let $H_k$ and $\Phi$ be as in Lemma 2.12 and let $\phi = (\phi_k)_k : \Gamma \to \prod_k H_k$ be a set-theoretic lifting of $\Phi$. Let $\lambda_k : H_k \to B(\ell^2(H_k))$ be the left regular representation of $H_k$. If $s, t \in F_k$, $s \neq t$, then $\phi_k(s) \neq \phi_k(t)$ and hence $\|\lambda_k(\phi_k(s)) - \lambda_k(\phi_k(t))\| \geqslant \sqrt{2}$. Let $n_k = |H_k|$ and set $\vec{n} = (n_k)$ as above. Consider the maps $\pi_k = \lambda_k \circ \phi_k : \Gamma \to U(n_k)$. Then the sequence of maps $(\pi_k)$ induces a group homomorphism $\pi : \Gamma \to U(Q_{\vec{n}})$ which is injective since $\|\pi(s) - \pi(t)\| \geqslant \sqrt{2}$ whenever $s, t$ are distinct elements of $\Gamma$. $\square$

*2.14. An MF group that is not LEF*

Let $p$ be a prime number. We recall the following group from [1]:

$$\Gamma = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & p^k & x_{23} & x_{24} \\ 0 & 0 & p^n & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{ij} \in \mathbb{Z}\left[\frac{1}{p}\right], \ k, n \in \mathbb{Z} \right\}.$$

One sees that

$$Z(\Gamma) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{Z}\left[\frac{1}{p}\right] \right\}.$$

We define

$$N = \left\{ \begin{pmatrix} 1 & 0 & 0 & k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}.$$

Abels showed in [1] that $\Gamma$ and $\Gamma/N$ are a finitely presented groups. Moreover, as observed by Abels, ideas similar to those in [19, p. 349], show that $\Gamma/N$ does not have the Hopf property. It is well known that any finitely generated residually finite group has the Hopf property (see [25]);

hence $\Gamma/N$ is not residually finite. In particular $\Gamma/N$ is not LEF (see Remark 2.11). As observed in [1], the group $\Gamma/N$ is a solvable—hence amenable—group. Finally, we observe that $G$ is residually finite. This follows from Mal'cev's theorem [25] (stating that a finitely generated subgroup of $GL(N, F)$ is residually finite for any field $F$).

The proof of the following lemma basically consists of writing down the definitions of induced representations, which we do for the convenience of the reader.

**Lemma 2.15.** *Let $F$ be a finite group and $H \leqslant Z(F)$. Let $\gamma : H \to L(E)$ be a finite dimensional unitary representation of $H$, and $\mathrm{Ind}_H^F \gamma$ be the induced unitary representation of $F$. Then*

(i) $\| \mathrm{Ind}_H^F \gamma (g) - 1 \| \geqslant \sqrt{2}$ *for all $g \in F \setminus H$ and*
(ii) $\mathrm{Ind}_H^F \gamma |_H$ *is unitary equivalent to $[F : H]\gamma$.*

**Proof.** Recall that (see e.g. [4, Appendix E]) $\mathrm{Ind}_H^F \gamma$ is defined on the Hilbert space

$$\mathcal{A} = \left\{ \xi : F \to E \colon \xi(xh) = \gamma\left(h^{-1}\right)\xi(x) \text{ for all } x \in F, \ h \in H \right\},$$

with inner product defined by

$$\langle \xi, \eta \rangle = \sum_{xH \in F/H} \left\langle \xi(x), \eta(x) \right\rangle.$$

(Note that if $xH = yH$, then $\langle \xi(x), \eta(x) \rangle = \langle \xi(y), \eta(y) \rangle$ so the above inner product is well defined.) One then defines the induced representation on the finite dimensional Hilbert space $\mathcal{A}$ by the equations

$$\mathrm{Ind}_H^F \chi(g)\xi(x) = \xi\left(g^{-1}x\right) \quad \text{for all } g, x \in F.$$

Suppose now that $g \notin H$. Define $\eta \in \mathcal{A}$ by

$$\eta(x) = \begin{cases} \gamma(x^{-1})\xi_0 & \text{if } x \in H, \\ 0 & \text{if } x \notin H, \end{cases}$$

where $\xi_0 \in E$ is a unit vector. Then $\|\eta\| = 1$ and

$$\left\langle \mathrm{Ind}_H^F \gamma(g)\eta, \eta \right\rangle = \left\langle \eta\left(g^{-1}\right), \eta(e) \right\rangle = 0.$$

This proves (i).

Let $h \in H$. Then

$$\left( \mathrm{Ind}_H^F \gamma(h)\xi \right)(x) = \xi\left(h^{-1}x\right) = \xi\left(xh^{-1}\right) = \gamma(h)\xi(x).$$

This proves (ii). $\quad \square$

**Remark 2.16.** For a group $G$ we denote by $\lambda_G$ its left regular representation. Let $\Gamma$ be a discrete countable residually finite group. It follows that there is a decreasing sequence of finite index normal subgroups $(L_n)_{n \geqslant 1}$ of $\Gamma$ such that $\bigcap_{n=1}^{\infty} L_n = \{e\}$. We denote by $\pi_n$ the corresponding

surjective homomorphisms $\pi_n : \Gamma \to \Gamma_n := \Gamma/L_n$. It is known (and not hard to verify) that $\lambda_\Gamma$ is weakly contained in the direct sum of the representations $\lambda_{\Gamma_n} \circ \pi_n$. If in addition $\Gamma$ is amenable, then it follows that the set of irreducible subrepresentations of all of $\lambda_{\Gamma_n} \circ \pi_n$ is dense in the primitive spectrum of $\Gamma$.

**Theorem 2.17.** *Let $\Gamma$ be a countable discrete residually finite group and let $N$ be a central subgroup of $\Gamma$. Then $\Gamma/N$ is MF and hence if in addition $\Gamma$ is amenable, then $C^*(\Gamma/N)$ is quasidiagonal.*

**Proof.** We will construct a sequence of finite dimensional unitary representations $\sigma_n$ of $\Gamma$, such that

$$\lim_{n\to\infty} \left\| \sigma_n(x) - 1 \right\| = 0 \quad \text{if and only if} \quad x \in N. \tag{9}$$

In particular, this will prove that $\Gamma/N$ is MF. Indeed, writing $\sigma_n : \Gamma \to U(k(n))$, one sees that the map of $G/N$ to $U(\prod_n M_{k(n)} / \sum_n M_{k(n)})$ given by $x \mapsto (\sigma_n(x))$ is an embedding.

Let $L_n$, and $\pi_n : \Gamma \to \Gamma_n := \Gamma/L_n$ be as in Remark 2.16. Let $Z$ be the center of $\Gamma$ and set $Z_n = \pi_n(Z) \cong Z/Z \cap L_n$. The restriction of $\pi_n$ to $Z$ is denoted again by $\pi_n : Z \to Z_n$. Let $\hat{\pi}_n : \hat{Z}_n \to \hat{Z}$ be the dual map of the restriction of $\pi_n$ to $Z$. It follows that the union of $\hat{\pi}_n(\hat{Z}_n)$ is dense in $\hat{Z}$ as noted in Remark 2.16 applied to $Z$ and its finite index subgroups $Z \cap L_n$. Let $(\omega_i)_{i\geqslant 1}$ be a dense sequence in the Pontriagin dual $(Z/N)\hat{\ }$. We will regard $\omega_i$ as characters on $Z$ that are trivial on $N$. Set $\eta_n = \omega_1 \oplus \cdots \oplus \omega_n : Z \to U(n)$. Let us observe that

$$\lim_{n\to\infty} \left\| \eta_n(x) - 1 \right\| = 0, \quad \text{if and only if} \quad x \in N. \tag{10}$$

Write $Z$ as an increasing union of finite subsets $F_n$. Since the union of $\hat{\pi}_n(\hat{Z}_n)$ is dense in $\hat{Z}$, we can replace $\Gamma_n$ by $\Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_{m(n)}$, $\pi_n$ by $\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_{m(n)}$ and $Z_n$ by $Z_1 \oplus Z_2 \oplus \cdots \oplus Z_{m(n)}$ so that in the new setup $\pi_n(Z) \subset Z_n$, $Z_n$ is a central subgroup of $\Gamma_n$ and the following properties will hold.

(i) For each $n \geqslant 1$ there is a unitary representation $\gamma_n : Z_n \to U(n)$ such that

$$\left\| \eta_n(x) - \gamma_n \circ \pi_n(x) \right\| < 1/n, \quad \text{for all } x \in F_n. \tag{11}$$

(ii) For any $x \in \Gamma \setminus Z$ there is $m$ such that $\pi_n(x) \notin Z_n$ for all $n \geqslant m$.

Concerning (ii) let us note that if $x \in \Gamma$ and $\pi_n(x) \in Z_n$ for all $n \geqslant m$ then $\pi_n(xgx^{-1}g^{-1}) = 1$ for all $g \in \Gamma$ and $n \geqslant m$ and that implies that $x \in Z$ since the sequence $(\pi_n)_{n\geqslant m}$ separates the points of $\Gamma$.

Define the finite dimensional unitary representation of $\Gamma$

$$\sigma_n = \left( \operatorname{Ind}_{Z_n}^{\Gamma_n} \gamma_n \right) \circ \pi_n.$$

Let $x \in Z$. By Lemma 2.15(2) we have

$$\left\| \sigma_n(x) - 1 \right\| = \left\| \gamma_n \circ \pi_n(x) - 1 \right\|$$

and hence in conjunction with (11)

$$\lim_{n\to\infty}\big\|\sigma_n(x)-1\big\|=0 \quad\Leftrightarrow\quad \lim_{n\to\infty}\big\|\gamma_n\circ\pi_n(x)-1\big\|=0 \quad\Leftrightarrow\quad \lim_{n\to\infty}\big\|\eta_n(x)-1\big\|=0.$$

By (11), it follows that $\|\sigma_n(x)-1\|\to 0$ if and only if $x\in N$.

Now let $x\in\Gamma\setminus Z(\Gamma)$. By (ii) there is an $m$ large enough so $\pi_n(x)\notin Z_n$ for all $n\geqslant m$. By Lemma 2.15(1), we have $\|\sigma_n(x)-1\|\geqslant\sqrt{2}$ for all $n\geqslant m$.

This proves that (9) holds and therefore that $\Gamma/N$ is MF.   $\square$

**Corollary 2.18.** *Let $\Gamma$ and $N$ be as in 2.14. Then $\Gamma/N$ is MF but not LEF. Since $\Gamma/N$ is amenable it also follows that $C^*(\Gamma/N)$ is quasidiagonal.*

**Proof.** This follows from Theorem 2.17. We have already noted that $\Gamma/N$ cannot be LEF since it is finitely presented but not residually finite.   $\square$

## 3. Strong quasidiagonality and groups

We exhibit some classes of amenable groups that have non-strongly quasidiagonal C*-algebras. See Theorem 3.4. All of these groups arise as wreath products. We do not know if these C*-algebras are quasidiagonal, except for a certain subclass. See Proposition 3.6.

Let us establish some notation related to crossed product C*-algebras. (We refer the reader to [9, Section 4.1] for details.) Let $A$ be a unital C*-algebra, $\Gamma$ a discrete countable group, and $\alpha:\Gamma\to\operatorname{Aut}(A)$ a homomorphism. A $*$-representation $(\pi,H)$ of $A$ induces $*$-representation $\pi\times\lambda$ of $A\rtimes_\alpha\Gamma$ on $B(H\otimes\ell^2\Gamma)$ defined by

$$(\pi\times\lambda)(a)(\xi\otimes\delta_t)=\pi\big(\alpha_{t^{-1}}(a)\big)(\xi)\otimes\delta_t,$$
$$(\pi\times\lambda)(s)(\xi\otimes\delta_t)=\xi\otimes\delta_{st} \tag{12}$$

for $a\in A$, $s,t\in\Gamma$, $\xi\in H$, and where $\{\delta_s\}_{s\in\Gamma}$ is the usual orthonormal basis of $\ell^2\Gamma$.

We denote by $A^{\otimes\Gamma}$ the $\Gamma$-fold maximal tensor product of $A$ with itself. This infinite tensor product is defined as an inductive limit indexed by finite subsets of $\Gamma$. The Bernoulli action $\beta$ of $\Gamma$ on $A^{\otimes\Gamma}$ may be described formally by

$$\beta_s(a_{t_1}\otimes\cdots\otimes a_{t_n})=a_{st_1}\otimes\cdots\otimes a_{st_n}.$$

The proof of the next proposition was inspired by [18, Theorem 25].

**Proposition 3.1.** *Let $A$ be a unital C*-algebra which is generated by two proper two-sided closed ideals. Let $\Gamma$ be a discrete countable non-torsion group. Then $A^{\otimes\Gamma}\rtimes_\beta\Gamma$ has a non-finite quotient. In particular, it is not strongly quasidiagonal.*

**Proof.** Write $A=I_1+I_2$ where $I_i$ are proper two-sided closed ideals of $A$ and write $1_A=y+x$ where $y\in I_1$ and $x\in I_2$. Let $\pi_i$ be a unital representations of $A$ with kernel $I_i$, $i=1,2$. Then $\pi_1(x)=1$ and $\pi_2(x)=0$.

Let $u \in \Gamma$ be an element of infinite order. It generates a subgroup $\mathbb{Z} \leqslant \Gamma$. Choose a subset $F \subseteq \Gamma$ of left coset representatives, that is,

$$\Gamma = \bigsqcup_{s \in F} g\mathbb{Z}.$$

Set

$$\Gamma_1 = \{su^n : n < 0 \text{ and } s \in F\}, \qquad \Gamma_2 = \{su^n : n \geqslant 0 \text{ and } s \in F\}$$

and observe that $u\Gamma_1^{-1}$ is a proper subset of $\Gamma_1^{-1}$. Define the representation $(\pi, H)$ of $A^{\otimes \Gamma}$ by

$$\pi := \left( \bigotimes_{s \in \Gamma_1} \pi_1 \right) \otimes \left( \bigotimes_{s \in \Gamma_2} \pi_2 \right).$$

For $t \in \Gamma$, let $x_t \in A^{\otimes \Gamma}$ be the elementary tensor with $x$ in the "$t$"-position and 1 elsewhere, in particular $\beta_g(x_t) = x_{gt}$. It follows from the properties of $\pi_1, \pi_2$ and $x$ that

$$\pi(x_t) = 1_H \quad \text{if } t \in \Gamma_1, \quad \text{and} \quad \pi(x_t) = 0 \quad \text{if } t \in \Gamma_2. \tag{13}$$

For a set $S \subset \Gamma$ we denote by $\chi_S$ the characteristic function of $S$ as well as the corresponding multiplication operator by $\chi_S$ on $\ell^2 \Gamma$. Using (12) and (13) one verifies immediately that

$$(\pi \times \lambda)(x_e) = 1_H \otimes \chi_{\Gamma_1^{-1}}.$$

Define the partial isometry

$$V := (\pi \times \lambda)(u) \cdot (\pi \times \lambda)(x_e) = 1_H \otimes \lambda(u)\chi_{\Gamma_1^{-1}}.$$

Then $V^*V = 1_H \otimes \chi_{\Gamma_1^{-1}}$ and $VV^* = 1_H \otimes \chi_{u\Gamma_1^{-1}}$. It follows that $V^*V - VV^* = 1 \otimes \chi_{\Gamma_1^{-1} \setminus u\Gamma_1^{-1}} > 0$. $\quad\square$

**Corollary 3.2.** *Let $A$ be a unital C\*-algebra which admits a quotient with nontrivial center. Let $\Gamma$ be a discrete countable non-torsion group. Then $A^{\otimes \Gamma} \rtimes \Gamma$ has a non-finite quotient.*

**Proof.** If $B$ is a quotient of $A$, then $B^{\otimes \Gamma} \rtimes \Gamma$ is a quotient of $A^{\otimes \Gamma} \rtimes \Gamma$. Thus we may assume that the center $Z(A)$ is nontrivial. Write $Z(A)$ as the sum of two maximal ideals $Z(A) = J_1 + J_2$. We conclude the proof by applying Proposition 3.1 for the ideals $I_1 = J_1 A$ and $I_2 = J_2 A$. $\quad\square$

**Remark 3.3.** The condition on $A$ in Proposition 3.1 is equivalent to the requirement that the primitive spectrum of $A$ contains two non-empty disjoint closed subsets. It is not hard to see that the primitive spectrum of a separable $A$ contains two distinct closed points if and only if $A$ has two distinct maximal ideals. Moreover, in this case $A$ admits a quotient with nontrivial center.

Although we state the next result in greater generality, perhaps the most interesting case is when the groups are amenable.

**Theorem 3.4.** *Let $\Gamma$ be a discrete countable non-torsion group and let $\Lambda$ be any discrete countable group such that either*

 (i)  *$\Lambda$ admits a finite dimensional representation other than the trivial representation, or*
(ii)  *$\Lambda$ has a finite conjugacy class other than $\{e\}$.*

*Then, $C^*(\Lambda \wr \Gamma)$ has a non-finite quotient.*

**Proof.** We first notice that $C^*(\Lambda \wr \Gamma) \cong C^*(\Lambda)^{\otimes \Gamma} \rtimes_\beta \Gamma$. We first assume (i). By assumption, there are two inequivalent finite dimensional irreducible representations $\pi_1$ and $\pi_2$ of $C^*(\Lambda)$. Setting $I_i = \ker(\pi_i)$, $i = 1, 2$, we see that $I_1$ and $I_2$ are distinct maximal ideals that satisfy the hypothesis of Proposition 3.1.

Now assume (ii). It is well known that if $\Lambda$ has a finite conjugacy class, then $C^*(\Lambda)$ has a non-trivial center (simply add up the elements of the conjugacy class to produce a central element). The conclusion follows from Corollary 3.2.  □

We observe that if $\Gamma$ is as in Theorem 3.4 and $\Lambda$ is not amenable, then the same conclusion follows. Indeed, in this case the C*-algebra $C_r^*(\Lambda)$ cannot have a character. Thus $\ker(\lambda)$ is not contained in the kernel $I_1 \triangleleft C^*(\Lambda)$ of the trivial representation, but in some other maximal ideal $I_2$ of $C^*(\Lambda)$. Hence $I_1$ and $I_2$ are distinct maximal ideals of $C^*(\Lambda)$ and we can apply Proposition 3.1.

As a special case of Theorem 3.4 we obtain the simplest example we know of an amenable group with a non-strongly quasidiagonal C*-algebra.

**Corollary 3.5.** *The C*-algebra of the group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is not strongly quasidiagonal.*

Since $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is residually finite (one may find a separating family of homomorphisms $\pi_n : \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}/n\mathbb{Z}$), its C*-algebra is quasidiagonal (it is actually residually finite dimensional and in particular inner quasidiagonal, see [6] for relevant definitions). More generally, we have the following:

**Proposition 3.6.** *Let $\Lambda$ be an amenable group that is a union of residually finite groups. Then the C*-algebra of the group $\Lambda \wr \mathbb{Z}^k$ is quasidiagonal.*

**Proof.** Write $\Lambda$ as an increasing union of residually finite groups $\Lambda_i$. Then $\Lambda \wr \mathbb{Z}^k$ is the union of $\Lambda_i \wr \mathbb{Z}^k$. Therefore, we may assume that $\Lambda$ itself is residually finite. By [13], $C^*(\Lambda)$ embeds in the UHF algebra of type $2^\infty$, denoted here by $D$. Then

$$C^*(\Lambda \wr \mathbb{Z}^k) \cong \left( \bigotimes_{\mathbb{Z}^k} C^*(\Lambda) \right) \rtimes \mathbb{Z}^k \subset \left( \bigotimes_{\mathbb{Z}^k} D \right) \rtimes \mathbb{Z}^k$$

and $(\bigotimes_{\mathbb{Z}^k} D) \rtimes \mathbb{Z}^k \cong D \rtimes \mathbb{Z}^k$ embeds in a simple unital AF algebra by a result of N. Brown [7].  □

If $C^*(\Lambda)$ has two distinct maximal ideals we do not need to assume (1) or (2) in Theorem 3.4 to obtain its conclusion. This raises an interesting question.

**Question 3.7.** *Are there any nontrivial groups $\Lambda$ such that $C^*(\Lambda)$ has a unique maximal ideal? (Such a group would have to be amenable.)*

### 3.8. Groups with strongly quasidiagonal C*-algebras

Now we exhibit some classes of (amenable) groups $\Gamma$ with strongly quasidiagonal C*-algebras. These will arise as extensions

$$1 \to \Delta \to \Gamma \to \Lambda \to 1$$

where $\Delta$ is a central subgroup of $\Gamma$, with some additional hypotheses on $\Lambda$ and $\Delta$. For example, we have the following proposition.

**Proposition 3.9.** *Suppose $\Gamma$ has a central subgroup $\Delta$ such that both $\Delta$ and $\Gamma/\Delta$ are finitely generated abelian groups. Then $C^*(\Gamma)$ is strongly quasidiagonal.*

**Proof.** Theorem 2.2 of [10] shows that such groups have finite decomposition rank. A C*-algebra with finite decomposition rank must be strongly quasidiagonal, as proved in [22, Theorem 5.3]. $\square$

The use of decomposition rank only serves to simplify the exposition, although it is perhaps of interest in itself. Proving strong quasidiagonality in all the cases obtained here could be done using analogous permanence properties of strong quasidiagonality.

**Lemma 3.10.** *Let $A$ be a separable continuous field C*-algebra over a locally compact and metrizable space $X$. Write $A_x$ for the fiber at $x \in X$. If $A$ is nuclear, then the set*

$$X_{\mathrm{QD}} := \{x \in X \colon A_x \text{ is quasidiagonal}\}$$

*is closed.*

**Proof.** Let $y \in \overline{X_{\mathrm{QD}}}$. Fix a finite subset $\mathcal{F}$ of $A_y$ and $\varepsilon > 0$. For $x \in X$ write $\pi_x \colon A \to A_x$ for the quotient map. Because $A_y$ is nuclear, the Choi–Effros lifting theorem affords us a contractive completely positive lift $\psi \colon A_y \to A$ of $\pi_y$. Using the fact that $x \mapsto \|\pi_x(\tilde{a})\|$ is continuous for every $\tilde{a} \in A$ we see that the sets

$$U = \bigcap_{a \in \mathcal{F}} \{x \in X \colon \|\pi_x(\psi(a))\| > \|a\| - \varepsilon\}$$

and

$$V = \bigcap_{(a,b) \in \mathcal{F} \times \mathcal{F}} \{x \in X \colon \|\pi_x(\psi(ab) - \psi(a)\psi(b))\| < \varepsilon\}$$

are finite intersections of open sets containing $y \in \overline{X_{\mathrm{QD}}}$. Then there exists $z \in X_{\mathrm{QD}} \cap U \cap V$.

Let $\phi = \pi_z \circ \psi \colon A_y \to A_z$. This is a completely positive contraction satisfying $\|\phi(a)\| > \|a\| - \varepsilon$ and $\|\phi(ab) - \phi(a)\phi(b)\| < \varepsilon$ for all $a, b \in \mathcal{F}$. It follows from Theorem 1.5 that $A_y$ is quasidiagonal. $\square$

The integer Heisenberg group $\mathbb{H}_3$ has a central subgroup isomorphic to $\mathbb{Z}$ such that the corresponding quotient of $\mathbb{H}_3$ is $\mathbb{Z}^2$. Therefore $\mathbb{H}_3$ satisfies the conditions of Proposition 3.9. One could also consider the case when $\Gamma$ has a central subgroup $\Delta$ such that $\Gamma/\Delta \cong \mathbb{H}_3$.

**Proposition 3.11.** *Suppose $\Gamma$ has a finitely generated central subgroup $\Delta$ such that $\Gamma/\Delta$ is a finitely generated, torsion-free, two-step nilpotent group with rank one center. Then $C^*(\Gamma)$ is strongly quasidiagonal.*

**Proof.** As noted in Section 2 of [24], the analysis of [2, Corollary 3.4] shows that every discrete, finitely generated, torsion-free, two-step nilpotent group with rank one center is isomorphic to a "generalized discrete Heisenberg group" $H(d_1, \ldots, d_n)$. If $n$ is a positive integer and $d_1, \ldots, d_n$ are positive integers with $d_1 | \ldots | d_n$, then $H(d_1, \ldots, d_n)$ is defined as the set $\mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}^n$ with multiplication

$$(r, s, t) \cdot (r', s', t') = \left(r + r' + \sum d_i t_i s_i', s + s', t + t'\right).$$

Write $\Gamma/\Delta \cong H(d_1, \ldots, d_n)$.

*Case $n > 1$.* It follows from Theorem 3.4 of [24] that every twisted group algebra $C^*(\Gamma/\Delta, \sigma)$ of $\Gamma/\Delta$ is isomorphic to the section algebra of a continuous field of C*-algebras over a one-dimensional space with each fiber stably isomorphic to a noncommutative torus of dimension at most $2n$. Every noncommutative torus of dimension at most $2n$ has decomposition rank at most $4n + 1$ (by [10, Lemma 4.4]) and decomposition rank is invariant under stable isomorphism, so Lemma 4.1 of [10] implies that $C^*(\Gamma/\Delta, \sigma)$ has finite decomposition rank. Now, by Theorem 1.2 of [27] we have that $C^*(\Gamma)$ is itself a continuous field C*-algebra over the finite dimensional space $\hat{\Delta}$ with every fiber isomorphic to some twisted group C*-algebra of $\Gamma/\Delta$. By Lemma 4.1 of [10] we obtain that $C^*(\Gamma)$ has finite decomposition rank and is therefore strongly quasidiagonal.

*Case $n = 1$.* Write $H$ for $H(d_1)$. There is an isomorphism $H^2(H, \mathbb{T}) \cong \mathbb{T}^2$ such that whenever a multiplier $\sigma$ corresponds to $(\lambda, \mu) \in \mathbb{T}^2$ with both of $\lambda$ and $\mu$ torsion elements, then the twisted group C*-algebra $C^*(H, \sigma)$ is stably isomorphic to a noncommutative torus [24, Theorem 3.9]. When at least one of $\lambda$ or $\mu$ is non-torsion, we have that $C^*(H, \sigma)$ is simple and has a unique trace [24, Theorem 3.7]. Now, there is a continuous field of C*-algebras over $H^2(H, \mathbb{T})$ where the fiber over (the class of) a given multiplier $\sigma$ is $C^*(H, \sigma)$ [27, Corollary 1.3]. Since the fibers are quasidiagonal over a dense set of points, every fiber must be quasidiagonal (by Lemma 3.10). In fact, every fiber must be strongly quasidiagonal, owing either to simplicity or to having finite decomposition rank.

By Theorem 1.2 of [27] we have that $C^*(\Gamma)$ is the section algebra of a continuous field of C*-algebras with strongly quasidiagonal fibers (the fibers are of the form $C^*(H, \sigma)$). Therefore, every primitive quotient of $C^*(\Gamma)$ is quasidiagonal, since it is a primitive quotient of some fiber of the field [15, Theorem 10.4.3]. It follows from Proposition 5 of [18] that $C^*(\Gamma)$ is strongly quasidiagonal. □

**Conjecture.** *If $\Gamma$ is a finitely generated countable discrete nilpotent group, then $C^*(\Gamma)$ is strongly quasidiagonal.*

A group is *supramenable* if it contains no paradoxical subsets. (A subset is paradoxical if it admits a paradoxical decomposition as in 2.1.) For solvable groups, this is the same as saying it

has a nilpotent subgroup of finite index. Does every countable discrete supramenable group have a (strongly) quasidiagonal C*-algebra?

## 4. Note added in proof

The third author has announced a proof of the above conjecture (in an even stronger form).

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