TRACIALLY COMPLETE C^*-ALGEBRAS

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Abstract. We introduce a new class of operator algebras – tracially complete C^*-algebras – as a vehicle for transferring ideas and results between C^*-algebras and their tracial von Neumann algebra completions. We obtain structure and classification results for amenable tracially complete C^*-algebras satisfying an appropriate version of Murray and von Neumann's property Γ for II_1 factors. In a precise sense, these results fit between Connes' celebrated theorems for injective II_1 factors and the unital classification theorem for separable simple nuclear C^*-algebras. The theory also underpins arguments for the known parts of the Toms–Winter conjecture.

Overview of results

There are two major classes of self-adjoint operator algebras: C^*-algebras and von Neumann algebras. Despite significant differences in their theory and their stages of development, techniques for transferring structure between these two classes are important on both sides. Examples of this are found in the deep connections between nuclearity of a C^*-algebra A and injectivity of its enveloping von Neumann algebra A^{**} ([29, 24, 25, 34]), which are obtained through Connes' celebrated work on injectivity and hyperfiniteness of von Neumann algebras. Other prominent examples of structural transfer between operator algebras include [13, 56, 75, 76, 85, 87].

The focus of this paper is on operator algebras with a trace, by which we will always mean a tracial state. One of Murray and von Neumann's very early results is that a finite factor has a unique tracial state ([81, 82]). In contrast, a simple unital stably finite C^*-algebra can have many tracial states. Indeed, by results of Blackadar ([6]) and Goodearl ([54]), any Choquet simplex may arise as the trace simplex $T(A)$ of a simple unital approximately finite dimensional (AF) C^*-algebra A. While a finite von Neumann algebra $\mathcal{M}$ always has sufficiently many traces, in that for any non-zero positive element $x \in \mathcal{M}$, there is a trace τ with $\tau(x) \neq 0$, the existence of traces is much more subtle in the C^*-setting. A deep theorem of Haagerup ([58]), together with [8, 9] is needed to obtain the corresponding result for a unital

This work was supported by: long term structural funding – a Methusalem grant of the Flemish Government (Castillejos); UNAM-PAPIIT IA103124 (Castillejos); Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 427320536 – SFB 1442 (Evington); Germany's Excellence Strategy EXC 2044 390685587 Mathematics Münster: Dynamics–Geometry–Structure (Evington); ERC Advanced Grant 834267 - AMAREC (Evington); Engineering and Physical Sciences Research Council [Grant Refs: EP/R025061/1, EP/R025061/2, and EP/X026647/1] (Evington, White); Australian Research Council grant DP180100595 (Gabe); NSF grant DMS-2000129 (Schafhauser); NSERC Discovery Grant (Tikuisis).

stably finite exact C^*-algebras, and without exactness, it is open whether traces necessarily exist.

Associated to every trace τ on a operator algebra A are a 2-seminorm $\|\cdot\|_{2,\tau}$, a GNS representation $\pi_\tau\colon A \to \mathcal{B}(\mathcal{H}_\tau)$, and a finite von Neumann algebra $\pi_\tau(A)''$. These objects are closely related. Indeed, the unit ball of $\pi_\tau(A)''$ is the $\|\cdot\|_{2,\tau}$-completion of the unit ball of A. In [86], motivated by developments on the Toms–Winter conjecture ([68, 110, 97]), Ozawa considers the C^*-algebra $\overline{A}^{\mathrm{u}}$ obtained by completing the unit ball of A with respect to the uniform 2-seminorm

$$\|a\|_{2,T(A)} \coloneqq \sup_{\tau\in T(A)} \|a\|_{2,\tau}.$$

When A has a unique trace τ, the uniform tracial completion $\overline{A}^{\mathrm{u}}$ coincides with the von Neumann algebra $\pi_\tau(A)''$.

When $T(A)$ has infinitely many extreme points, the uniform tracial completion $\overline{A}^{\mathrm{u}}$ is no longer a von Neumann algebra. Ozawa's theory of W^*-bundles (see Section 3.6 below) axiomatises the structure of $\overline{A}^{\mathrm{u}}$ in the special case that the extreme points of $T(A)$ are weak*-closed. The techniques developed in this paper allow us to generalise Ozawa's theory to arbitrary trace simplices. In Theorem A, the condition of tensorially absorbing the Jiang–Su algebra (from [60]) is one of the two major hypotheses in the unital classification theorem for separable simple nuclear C^*-algebras ([52, 53, 39, 107, 16]; see [16, Section 1.1] for a discussion of this property and examples). In particular the theorem applies to all C^*-algebras covered by this classification.

Theorem A. *Let A and B be unital separable nuclear C^*-algebras that absorb the Jiang–Su algebra tensorially. Then $\overline{A}^{\mathrm{u}} \cong \overline{B}^{\mathrm{u}}$ if and only if A and B have affinely homeomorphic spaces of tracial states.*[1]

Note that, by contrast, if A is a separable, nuclear C^*-algebra with no finite dimensional quotients, then the finite part of the bidual A^{**}_{fin} *only* remembers the number of extreme traces in $T(A)$ by results that go back to Connes and Elliott ([29, 37]). Indeed, A^{**}_{fin} is type II_1 as A has no finite dimensional representations, and the minimal direct summands of A^{**}_{fin} are in bijection with the extremal traces of A, so the result follows from [37, Theorem 4.3].

Over the last decade, major advances in the Elliott classification programme for simple nuclear C^*-algebras with traces[2] have culminated in two major themes.

Classification: The finite case of the *unital classification theorem* ([23, Corollary D], using [39, 52, 53, 107]; an abstract proof is given in [16]), which correspond to the uniqueness of the injective II_1 factor (combining Connes' equivalence of injectivity and hyperfiniteness from [29] with Murray and von Neumann's uniqueness of the hyperfinite II_1 factor).

[1]The isomorphism is to be interpreted as $(\overline{A}^{\mathrm{u}}, T(A)) \cong (\overline{B}^{\mathrm{u}}, T(B))$ in the category of tracially complete C^*-algebras as defined in Section 3.1. In this case, this amounts to a uniform 2-norm continuous C^*-isomorphism with a uniform 2-norm continuous inverse.

[2]Purely infinite C^*-algebras, and type III factors, are out of the scope of this paper.

Structure: Structural theorems ([117, 118, 106, 75, 76, 98, 10, 23, 19]), which combine to prove most of the *Toms–Winter conjecture* ([119, Conjecture 9.3], cf. [112, 41]) for C^*-algebras. These results mirror aspects of Connes' equivalence of injectivity and hyperfiniteness for von Neumann II_1 factors ([29]).

With the benefit of hindsight, the advances above have been heavily driven by a subtle interplay between nuclear C^*-algebras A and their uniform tracial completions $\overline{A}^{\mathrm{u}}$. This goes back to Matui and Sato's breakthrough work on the Toms–Winter regularity conjecture ([75, 76]). In a nutshell, these 'von Neumann techniques' allow for direct access to Connes' work in C^*-arguments. This is true both conceptually and technically.

In this paper, we provide an abstract framework for studying uniform tracial completions: *tracially complete C^*-algebras*, consisting of a unital C^*-algebra together with a distinguished set of tracial states such that the operator norm unit ball is complete in the resulting uniform 2-norm. The motivating example is the pair $\big(\overline{A}^{\mathrm{u}}, T(A)\big)$, consisting of the uniform tracial completion of a C^*-algebra A together with the set of traces arising from traces on A. We regard tracially complete C^*-algebras as a bridge between tracial von Neumann algebras and tracial C^*-algebras, allowing one to solve problems about tracial C^*-algebras in three stages.

(i) Apply von Neumann algebra theory in each tracial GNS representation.
(ii) Obtain results for uniform tracial completions via local-to-global arguments.
(iii) Pull results back from the tracial completion to the C^*-level.

The main objective of the paper is to obtain structure and classification theorems for tracially complete C^*-algebras – i.e. results that fit into step (ii) of the programme above. These theorems fit between Connes' Theorem (together with Murray and von Neumann's uniqueness of the hyperfinite II_1 factor) and the structure and classification theorems in the setting of simple nuclear C^*-algebras. Our work is organised into the following topics.

Amenability: We will show that the two natural definitions of amenability, one in terms of amenability of all tracial von Neumann algebra completions, and the other defined by a uniform 2-norm version of the completely positive approximation property, agree.

Factoriality: We identify a subclass of algebras – the (type II_1) *factorial tracially complete C^* algebras* – in which each GNS representation coming from an extreme point of the distinguished traces is a factor. Uniform tracial completions of C^*-algebras are always factorial, and Ozawa's W^*-bundles are examples of type II_1 factorial tracially complete C^*-algebras precisely when its fibres are factors.

Regularity: We introduce the McDuff property and property Γ for tracially complete C^*-algebras. This is simultaneously motivated by the following implications due to Connes for a II_1 factor ([29]):

$$\text{injective} \Rightarrow \begin{array}{c}\text{injective with}\\ \text{property } \Gamma\end{array} \Rightarrow \begin{array}{c}\text{injective}\\ \text{and McDuff}\end{array} \Rightarrow \text{hyperfinite},$$

as well as by work on regularity properties (Jiang–Su stability, uniform property Γ) for C^*-algebras ([23, 22, 94, 75]).

Hyperfiniteness: We consider inductive limits of finite dimensional tracially complete C^*-algebras, in the spirit of Murray and von Neumann's hyperfinite von Neumann algebras and Bratteli's AF C^*-algebras.

Concrete models: For each metrisable Choquet simplex X, we construct a hyperfinite II_1 tracially complete C^*-algebra $(\mathcal{R}_X, X)$ in the same spirit as Blackadar's and Goodearl's constructions of a simple unital AF C^*-algebra with trace simplex affinely homeomorphic to X. Unlike the case of AF C^*-algebras, the resulting tracially complete C^*-algebra is independent of all choices made in the construction. When X is an n-dimensional simplex, $(\mathcal{R}_X, X)$ is isomorphic to the direct sum of $n+1$ copies of the hyperfinite II_1 factor with its entire trace simplex. When X is a metrisable Bauer simplex (i.e. the extreme boundary $\partial_e X$ is compact), $(\mathcal{R}_X, X)$ can be identified with the trivial W^*-bundle over $\partial_e X$ whose fibres are the hyperfinite II_1 factor $\mathcal{R}$.

With this setup, our structure and classification theorems for amenable tracially complete C^*-algebras are as follows.

Theorem B (Structure theorem)**.** *For an amenable type* II_1 *factorial tracially complete* C^**-algebra, property* Γ*, the McDuff property, and hyperfiniteness are equivalent.*

Theorem C (Classification theorem)**.** *Any amenable factorial tracially complete* C^**-algebra which is separable (in the uniform 2-norm) and has property* Γ *is isomorphic to the hyperfinite model* $(\mathcal{R}_X, X)$ *with the corresponding Choquet simplex* X *of traces.*

The equivalence of property Γ and the McDuff property in Theorem B was established in [22] for the uniform tracial completions of separable nuclear C^*-algebras with no finite dimensional representations.

As with the classification programme for C^*-algebras and Murray and von Neumann's uniqueness of the hyperfinite II_1 factor before that, our classification theorem is powered by classification results for *-homomorphisms satisfying a suitable notion of amenability together with an Elliott-style intertwining argument. In the setting of tracially complete C^*-algebras amenability is given in terms of the completely positive approximation property in the uniform 2-norm, or equivalently using uniformly amenable traces – we call such maps *tracially nuclear*.[3]

Theorem D (Classification of tracially nuclear *-homomorphisms)**.** *Consider tracially complete* C^**-algebras* $(\mathcal{M}, X)$ *and* (N, Y) *with* $(\mathcal{M}, X)$ *being* $\|\cdot\|_{2,X}$*-separable and* $(\mathcal{N}, Y)$ *being factorial with property* Γ*. Then a map* $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ *is tracially nuclear if and only if the induced map* $\phi^*\colon Y \to X$ *takes values in the uniformly amenable traces on* $\mathcal{M}$*. Any continuous affine* $\gamma\colon Y \to X$ *taking values in the uniformly amenable traces*

[3]We state Theorem D for tracially complete C^*-algebras satisfying the regularity hypothesis of property Γ. In the main body, we will work with a more general hypothesis – complemented partitions of unity (discussed in Section 1.4 below) – so that that the classification of tracially nuclear *-homomorphisms generalises the classification of weakly nuclear *-homomorphisms from C^*-algebras into finite von Neumann algebras.

on $\mathcal{M}$ is induced by a tracially nuclear map $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$, which is unique up to approximate unitary equivalence in the uniform 2-norm given by $(\mathcal{N}, Y)$.

An important special case of the previous theorem is given by taking $(\mathcal{M}, X) \coloneqq \big(\mathbb{C}^2, T(\mathbb{C}^2)\big)$, giving a classification of projections. In this case we can do better: unitary equivalence and Murray–von Neumann subequivalence of projections in factorial tracially complete C^*-algebras with property Γ are determined by the designated set of traces (see Theorem 7.18).[4] We highlight the following special case of this in the language of von Neumann algebras, showing that Murray and von Neumann's foundational classification of projections can be performed continuously in II_1 factors with property Γ. It is obtained by applying our classification of projections to the trivial W^*-bundle over X with fibre $\mathcal{M}$.

Theorem E. *Let K be a compact Hausdorff space and let $\mathcal{M}$ be a II_1 factor with property Γ. Suppose that $p, q\colon K \to \mathcal{M}$ are projection-valued functions which are continuous with respect to $\|\cdot\|_2$.*

(i) *There is a $\|\cdot\|_2$-continuous $v\colon K \to \mathcal{M}$ with $v(x)^*v(x) = p(x)$ and $v(x)v(x)^* \leq q(x)$ for all $x \in K$ if and only if $\tau(p(x)) \leq \tau(q(x))$ for all $x \in K$.*

(ii) *There is a $\|\cdot\|_2$-continuous $u\colon K \to \mathcal{M}$ taking values in the unitary group of $\mathcal{M}$ with $u(x)p(x)u(x)^* = q(x)$ for all $x \in K$ if and only if $\tau(p(x)) = \tau(q(x))$ for all $x \in K$.*

The existence theorem corresponding to Theorem E is immediate. Given a continuous $f\colon K \to [0,1]$, fix a copy of $L^\infty[0,1] \subseteq \mathcal{M}$. Then the projection-valued function $p(x) = \chi_{[0,f(x)]}$ is $\|\cdot\|_2$-continuous and has $\tau(p(x)) = f(x)$ for all $x \in K$.

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[4] This also holds with complemented partitions in place of property Γ, but we emphasis that we do not know Theorem E for general II_1 factors $\mathcal{M}$ unless X is totally disconnected. It is open whether Theorem E holds when $X = [0,1]$ and $\mathcal{M}$ is a free group factor.

1. Introduction

1.1. **Tracially complete C^*-algebras.** Abstractly, we define a tracially complete C^*-algebra to be a pair $(\mathcal{M}, X)$ where $\mathcal{M}$ is a C^*-algebra and X is a (weak*) compact and convex subset of $T(\mathcal{M})$ such that the unit ball of $\mathcal{M}$ is complete in the uniform 2-norm $\|\cdot\|_{2,X}$ induced by X. This definition is designed to encompass both tracial von Neumann algebras and Ozawa's uniform tracial completions. While in the overview above we only considered the uniform tracial completion of a C^*-algebra A with respect to all its traces, Ozawa's work considers uniform tracial completions with respect to any closed convex subset $X \subset T(A)$. We write $\overline{A}^X$ for the uniform tracial completion with respect to X (and, in particular, use the notation $\overline{A}^{T(A)}$ rather than $\overline{A}^{\mathrm{u}}$ in the rest of the paper). Every trace $\tau \in X$ will extend uniquely to a $\|\cdot\|_{2,X}$-continuous trace on $\overline{A}^X$; these extensions are the designated set of traces on $\overline{A}^X$ in the definition of a tracially complete

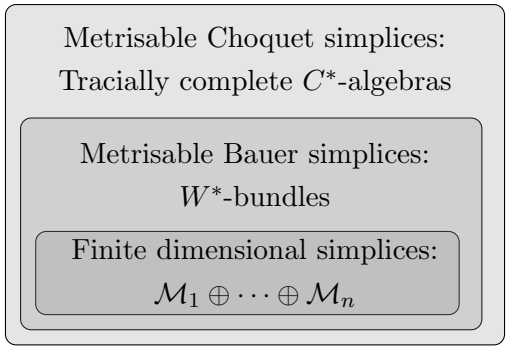
C^*-algebra. With a minor abuse of notation, the extension of $\tau \in X$ to $\overline{A}^X$ is still denoted τ, and the set of such extensions is still denoted X.

Given a tracial von Neumann algebra $(\mathcal{M}, \tau)$,[5] it is typically necessary in classification results to consider all (normal) traces on $\mathcal{M}$ instead of only τ itself. For example, the specified trace τ will classify projections in $\mathcal{M}$ only when it is the unique trace on $\mathcal{M}$ – i.e. when $\mathcal{M}$ is a factor. By analogy, we would like to work with a subclass of tracially complete C^*-algebras $(\mathcal{M}, X)$, where the traces in X provide enough information for classification. In the case when X is a face in $T(\mathcal{M})$, the set X is precisely the set of $\|\cdot\|_{2,X}$-continuous traces on $\mathcal{M}$ (and conversely) – see Proposition 3.15. We call such tracially complete C^*-algebras *factorial*. The name is further justified by the fact that a W^*-bundle is factorial as a tracially complete C^*-algebra if and only if each fibre is a factor (Proposition 3.6) and, more generally, a tracially complete C^*-algebra is factorial if and only if every extreme point of X gives rise to a factor representation of $\mathcal{M}$ (Proposition 3.14).

The examples of greatest interest – uniform tracial completions of a C^*-algebras – are automatically factorial. Further, as well as being necessary to classify projections by the distinguished traces, we have found factoriality to be an important technical condition when applying local-to-global arguments. Accordingly, we have come to regard factorial tracially complete C^*-algebras as the fundamental objects in the class of tracially complete C^*-algebras.

When working with factorial tracially complete C^*-algebras $(\mathcal{M}, X)$ (such as uniform tracial completions of C^*-algebras), difficulties increase with the complexity of X. The space X is always always a Choquet simplex as it is a closed face in the Choquet simplex $T(\mathcal{M})$. When X is finite dimensional, or more generally a Bauer simplex, a factorial tracially complete C^*-algebra over X has extra structure and is easier to analyse. This is illustrated schematically as follows.

(1.1)



Firstly, when X is a finite dimensional simplex, $\mathcal{M}$ is just a finite direct sum $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$ of factors. In this case, Theorems B and C are Connes' Theorem[6] ([29]) and the regularity hypothesis (property Γ) is automatic from amenability.

[5] A tracial von Neumann algebra $(\mathcal{M}, \tau)$ is von Neumann algebra $\mathcal{M}$ equipped with a faithful normal trace τ.

[6] Theorem D also follows directly from Connes' Theorem (see Proposition 9.2), though this is most often stated in the literature when A is nuclear.

The next threshold of complexity occurs when X is a *Bauer simplex*: that is, when the extreme boundary $\partial_e X$ is compact. Ozawa paid particular attention to this situation in [86], introducing the abstract notion of (the section algebra of) a *continuous W^*-bundle* over a compact Hausdorff space K (see Section 3.6 for the precise definition). The most basic examples of W^*-bundles are trivial bundles. The trivial W^*-bundle over K with fibre a tracial von Neumann algebra $(\mathcal{M}, \tau)$ is given by

$$C_\sigma(K, \mathcal{M}) := \{f : K \to \mathcal{M} : f \text{ is } \|\cdot\|\text{-bounded and } \|\cdot\|_{2,\tau}\text{-continuous}\}, \tag{1.2}$$

together with the conditional expectation $E\colon C_\sigma(K, \mathcal{M}) \to C(K)$ given by composing with τ.

Ozawa showed that if A is a C^*-algebra whose tracial state space $T(A)$ is a Bauer simplex, then the uniform tracial completion $\overline{A}^{T(A)}$ naturally has the structure of a continuous W^*-bundle over $\partial_e T(A)$ whose fibre at $\tau \in \partial_e T(A)$ is the von Neumann factor $\pi_\tau(A)''$. Moreover, he proved a 'trivialisation theorem' ([86, Theorem 15]), characterising when W^*-bundles whose fibres are the hyperfinite II_1 factor are trivial – this amounts to asking when Connes' theorem can be established in a continuous fashion over K. Theorem A follows from Ozawa's trivialisation theorem whenever the C^*-algebras involved have Bauer trace simplices. Using Ozawa's methods, we show in Section 3.6 that factorial tracially complete C^*-algebras $(\mathcal{M}, X)$ whose traces X form a Bauer simplex have a natural structure of a W^*-bundle over $\partial_e X$, whose fibre at $\tau \in \partial_e X$ is the factor $\pi_\tau(\mathcal{M})$.[7] In this way, Theorems B and C also follow from Ozawa's trivialisation theorem in the case of factorial tracially complete C^*-algebras with a Bauer simplex of traces.

The trace simplex of a C^*-algebras can be very far from Bauer and can have highly complex affine structure. In fact, any metrisable Choquet simplex can arise as the trace space of a separable C^*-algebras, including, for example, the Poulsen simplex characterised as the unique (non-trivial) metrisable Choquet simplex where the extreme points form a dense subset.

We think of factorial tracially complete C^*-algebras $(\mathcal{M}, X)$ as providing a formalism for (the section algebra of) a bundle over a Choquet simplex X, with 'fibres' coming from the GNS representations of $\mathcal{M}$ at points $\tau \in X$, which takes into account the affine relations between the fibres (see the discussion in Section 3). Outside the Bauer simplex setting, the subtle interaction between the affine structure of X and the operator algebraic structure of $\mathcal{M}$ is quite challenging and leads to very different behaviour compared to W^*-bundles. To give one example, suppose that $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra; if X is a Bauer simplex then the centre of $\mathcal{M}$ can be identified with the continuous affine functions on X, whereas in the general case, the centre of $\mathcal{M}$ can be trivial.

1.2. **The trace problem.** We take a brief interlude to discuss why we keep track of the designated collection of traces in our definition of a tracially complete C^*-algebra. Every II_1 factor $\mathcal{M}$ has a unique trace τ, so in

[7] In this case, by Ozawa's work $\pi_\tau(\mathcal{M}) = \pi_\tau(\mathcal{M})''$.

particular all traces on $\mathcal{M}$ are normal. We do not know if the analogous statement holds for factorial tracially complete C^*-algebras.

When $(\mathcal{M}, X)$ is a tracially complete C^*-algebra, all the traces in X are evidently $\|\cdot\|_{2,X}$-continuous – the appropriate notion of continuity in this setting. Moreover all $\|\cdot\|_{2,X}$-continuous traces on $\mathcal{M}$ belong to the closed face in $T(\mathcal{M})$ generated by X (see Proposition 3.15). Accordingly, when $(\mathcal{M}, X)$ is factorial, X consists precisely of the $\|\cdot\|_{2,X}$ continuous traces on $\mathcal{M}$. Are there any other traces? We regard this as a foundational problem in the theory of tracially complete C^*-algebras.

Question 1.1 (Trace Problem)**.** Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra. Are all traces on $\mathcal{M}$ automatically $\|\cdot\|_{2,X}$-continuous? Equivalently, is the inclusion $X \subseteq T(\mathcal{M})$ an equality?

When X is a finite dimensional simplex, $\mathcal{M}$ is a finite direct sum of factors, and hence the trace problem has a positive solution since all traces on $\mathcal{M}$ are normal. In this paper we resolve the trace problem for ultrapowers (and reduced products) of tracially complete C^*-algebras with property Γ (Theorem 7.5); property Γ is discussed in Section 1.4 and Section 5.3. This is in the spirit of various 'no silly trace' results asserting that, under appropriate regularity conditions, traces on ultrapowers are generated by the limit traces (such as [86, Theorem 8], [84, Theorem 1.2], [10, Theorem 3.22]; see [4, Theorem A] for a very general C^*-algebra result).

The trace problem is particularly pertinent when we take tracial completions. Given a C^*-algebra A, Ozawa's uniform tracial completion $\overline{A}^{T(A)}$ gives rise to a factorial tracially complete C^*-algebra $\big(\overline{A}^{T(A)}, T(A)\big)$, but is $\overline{A}^{T(A)}$ uniformly tracially complete with respect to *all* its traces? If there are additional traces on $\overline{A}^{T(A)}$ which are not $\|\cdot\|_{2,T(A)}$-continuous, then there seems to be no reason why this should be the case. A positive answer to the trace problem would ensure that the uniform tracial completion process stabilises.[8]

1.3. **Local-to-global: amenability.** Our main approach to understanding the structure of tracially complete C^*-algebras is to pass from local properties at each trace, i.e. properties of the fibres $\pi_\tau(\mathcal{M})''$, to global properties that hold uniformly over all traces. Identifying the appropriate class of *amenable* tracially complete C^*-algebras through completely positive approximations gives a first example of this idea.

Recall that nuclearity of a C^*-algebra A is characterised through the completely positive approximation property ([64]): there is a net $(F_i, \phi_i, \psi_i)_i$ consisting of finite dimensional C^*-algebras F_i and completely positive and contractive (c.p.c.) maps

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A \tag{1.3}$$

[8]During the long gestation period of this paper, the trace problem was resolved positively by the third-named author for tracially complete C^*-algebras with property Γ. In particular, this means that for a $\mathcal{Z}$-stable C^*-algebra A, Ozawa's $\overline{A}^{T(A)}$ has a complete unit ball in its uniform 2-norm.

such that $\|\phi_i(\psi_i(a)) - a\| \to 0$ for all $a \in A$. The directly analogous concept for a von Neumann algebra $\mathcal{M}$ is *semidiscreteness*, which asks for c.p.c. maps

$$\mathcal{M} \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} \mathcal{M} \tag{1.4}$$

which approximate the identity on $\mathcal{M}$ in the point-weak*-topology (rather than the point-norm topology). When $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra, (i.e. τ is a specified faithful normal trace on $\mathcal{M}$), a standard Hahn–Banach argument shows that this is equivalent to the completely positive approximation property in the $\|\cdot\|_{2,\tau}$-norm. A large body of deep work, including Connes' theorem, gives a plethora of other conditions equivalent to the completely positive approximation property for C^*-algebras and to semidiscreteness for a von Neumann algebras.

If we look for analogous conditions on a tracially complete C^*-algebra $(\mathcal{M}, X)$, one option is to work locally and ask for all the von Neumann fibres $\pi_\tau(\mathcal{M})''$ for $\tau \in X$ to be semidiscrete. Our local-to-global result for amenability allows us to go from such a local condition to a single system of completely positive approximations which works uniformly over all traces.

Theorem 1.2. *Let $(\mathcal{M}, X)$ be a tracially complete C^*-algebra. The following are equivalent:*

(i) *$(\mathcal{M}, X)$ is amenable, in the sense that the completely positive approximation property holds in the point-$\|\cdot\|_{2,X}$-norm;*
(ii) *for all $\tau \in X$, $\pi_\tau(\mathcal{M})''$ is semidiscrete, in the sense that the completely positive approximation property holds for $\pi_\tau(X)''$ in the point-weak*-topology;*
(iii) *every $\tau \in X$ is uniformly amenable in the sense of* [12, Definition 3.2.1].

In particular, via Connes' theorem the uniform tracial completion of a nuclear C^*-algebra is amenable as a tracially complete C^*-algebra.[9]

Theorem 1.2 generalises to *-homomorphisms between C^*-algebras and tracially complete C^*-algebras (Theorem 4.9, which proves the first statement in Theorem D) characterising the completely positive approximation property (with respect to the uniform 2-norm) in terms of pointwise amenability conditions. We call such maps *tracially nuclear*, and this is the appropriate amenability condition on *-homomorphisms for classification; see the discussion in Section 1.5.

Theorem 1.2, and its generalisation to morphisms in Theorem 4.9, are both proved by means of a Hahn–Banach trick which we learnt from [69, 48]. The key point – used by Ozawa in [86] – is that the weak topology on the space $\mathrm{Aff}(X)$ of continuous affine functions on a Choquet simplex is given by pointwise convergence. Since the set of c.p.c. maps which factorise through finite dimensional C^*-algebras is closed under convex combinations, it is then

[9] A naive argument for this fails in the same way that a direct proof that the bidual of a nuclear C^*-algebra is semidiscrete as a von Neumann algebra fails – an extension of the approximations witnessing nuclearity of A will not a priori approximate the identity map on the tracial completion in the point-$\|\cdot\|_{2,X}$-norm.

possible to take convex combinations of a finite set of local approximations (obtained through compactness) to build a global approximation.

1.4. **Local-to-global: CPoU and property Γ.** The use of the Hahn–Banach Theorem with $\mathrm{Aff}(X)$ described in the previous section provides a general local-to-global tool in the setting of tracially complete C^*-algebras for conditions witnessed by elements of a convex set (see Lemma 4.7). However, the Hahn–Banach strategy is unlikely to apply to conditions which are not affine in nature. To discuss this, let us consider the problem (which is open in general) of whether a factorial II_1 tracially complete C^*-algebra $(\mathcal{M}, X)$ admits approximate projections which are approximately of trace $1/2$; i.e. given $\epsilon > 0$, does there exist a positive contraction $p \in \mathcal{M}$ with $\|p - p^2\|_{2,X} < \epsilon$ and $|\tau(p) - 1/2| < \epsilon$ for all $\tau \in X$?

As we think of tracially complete C^*-algebras as a kind of affine bundle, it is natural to approach local-to-global problems via partition of unity techniques. For a W^*-bundle $\mathcal{M}$ with II_1 factor fibres over a compact Hausdorff space K, the algebra $C(K)$ embeds centrally into $\mathcal{M}$, and so it is straightforward to combine elements in $\mathcal{M}$ with a partition of unity in $C(K)$ – however such a direct approach does not preserve algebraic conditions such as (in our sample problem) being a projection. To see where it fails, for each fibre $\tau \in K$ fix a positive contraction $p_\tau \in \mathcal{M}$ such that $\pi_\tau(p_\tau)$ is a projection of trace $1/2$ in the fibre $\pi_\tau(\mathcal{M})$ (which is automatically a II_1 factor). Fixing $\epsilon > 0$, compactness gives an open cover $U_1, \dots, U_n$ of K and positive contractions $p_1, \dots, p_n \in \mathcal{M}$ such that

$$\|p_i^2 - p_i\|_{2,\tau} < \epsilon \text{ and } \tau(p_i) \approx_\epsilon 1/2, \quad \tau \in U_i. \tag{1.5}$$

Taking a partition of unity $f_1, \dots, f_n \in C(K)$ subordinate to $U_1, \dots, U_n$, one can form $p \coloneqq \sum_{i=1}^n f_i p_i$. While this will have $\tau(p) \approx_\epsilon 1/2$ for all $\tau \in K$, there is no reason for it to be an approximate projection in uniform 2-norm. Partitions of unity from $C(K)$ do not interact well with multiplication due to the lack of orthogonality of the f_i.

The solution is to ask that the f_i are approximate projections (giving rise to approximate orthogonality). However, unless K is zero dimensional, this requires that they come from $\mathcal{M}$ instead of $C(K)$, and we must weaken centrality to approximate centrality. If such f_i can be found, then $p \coloneqq \sum_{i=1}^n f_i p_i$ will give the required approximate projection in $\mathcal{M}$.[10] In general, such partitions of unity need not exist (see Example 6.6), but when they do, they can be used for local-to-global results. Ozawa's trivialisation theorem is an example par excellence: the passage from the existence of approximately central approximate projections of trace $1/2$ in a W^*-bundle with hyperfinite II_1 factor fibres to global triviality of the bundle ([86, Theorem 15(ii)⇒(iii)⇒(i)]) is underpinned by such partition of unity arguments. Subsequently, this strategy was made explicit and systematically used in [10] for local-to-global transfer in trivial W^*-bundles whose fibre is a McDuff II_1 factors.

[10]The reason this works is that $\tau(f_i p_i) \approx \tau(f_i)\tau(p_i)$ uniformly over $\tau \in K$. For subtle reasons, it is essential for this approximation that K is compact, i.e. that as a tracially complete C^*-algebra, the underlying simplex is Bauer.

Beyond the W^*-bundle setting, one needs even more control on the approximately central projections f_i forming a partition of unity. It is necessary to be able to uniformly control $\tau(f_i a)$ for certain a, whereas in the W^*-bundle setting, this control comes for free from knowing that $\tau(f_i)$ vanishes outside of U_i (see Footnote 10). A solution was identified in [23] in the form of *complemented partitions of unity* (CPoU) for a C^*-algebra. The technical definition is designed to give the required control with respect to a given family of positive elements, which should be thought of as complementary to the tracial support of the f_i. See the second part of the introduction to [23] for a further discussion of this and the challenges that must be overcome outside the Bauer simplex setting.

In [23], the concept of complemented partitions of unity was set out using the uniform tracial ultraproduct of a C^*-algebra but, as we describe in Section 6, this definition naturally lives at the level of tracially complete C^*-algebras. When complemented partitions of unity can be found, they give rise to a very general local-to-global transfer process. We give a number of examples in Section 7 (some of which are based on applications of CPoU for uniform tracial closures of nuclear C^*-algebras in [22, 21]).

As with the example described above of approximate projections, the local-to-global transfer process produces approximate properties in factorial tracially complete C^*-algebras with CPoU, i.e. properties holding up to a small error in uniform 2-norm. In order to cleanly encode such approximate conditions, we develop the theory of reduced products (including ultraproducts) of tracially complete C^*-algebras in Section 5. The effect of the local-to-global argument is that the ultrapower of factorial tracially complete C^*-algebras with CPoU enjoys several of the fundamental properties of a finite von Neumann algebra: real rank zero, stable rank one, all unitaries are exponentials, and Murray–von Neumann comparison of projections is determined by traces. Our solution to the trace problem for such reduced products (Theorem 7.5) is obtained as a consequence of these results, avoiding the use of sums of commutators found in precursor results such as [10, Section 3.4] (see Remark 7.6).

Since CPoU arguments inevitably produce approximate conclusions, there is a more work to be done to achieve the exact classification of projections needed for Theorem E. This is achieved by means of an intertwining argument, based on explicit estimates showing that projections of the same trace which are close in 2-norm are approximately conjugate by a unitary close to the unit. This is in the spirit of similar perturbation results for finite von Neumann algebras and leads to the following result (proved as Theorems 7.18 and 7.19).

Theorem 1.3. *Let $(\mathcal{M}, X)$ be a factorial type II_1 tracially complete C^*-algebra with CPoU.*

(i) *If $p, q \in \mathcal{M}$ are projections and $\tau(p) = \tau(q)$ for all $\tau \in X$, then p and q are unitarily equivalent.*
(ii) *For any continuous affine $f\colon X \to [0,1]$ there is a projection $p \in \mathcal{M}$ with $\tau(p) = f(\tau)$ for all $\tau \in X$.*

The fundamental challenge is to determine when complemented partitions of unity can be found, i.e. when a factorial tracially complete C^*-algebra has

CPoU. In [23], a subset of the authors and Winter introduced the concept of *uniform property* Γ for a C^*-algebra as a uniform 2-norm version of Murray and von Neumann's property Γ for II_1 factors. This too is most naturally a property of tracially complete C^*-algebras, and we say that $(\mathcal{M}, X)$ has property Γ when there exist uniform 2-norm approximately central approximate projections, which approximately divide the trace of elements of $\mathcal{M}$ in half (Definition 5.19 and Proposition 5.23). The main technical result of [23] is that unital nuclear C^*-algebras with uniform property Γ have complemented partitions of unity. We strengthen the main result of [23] by removing the condition of nuclearity, allowing CPoU to be obtained from property Γ in general. As set out in Section 1.6, this forms a major ingredient in the forthcoming general framework for the classification of *-homomorphisms ([17]).

Theorem 1.4. *Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra with property Γ. Then $(\mathcal{M}, X)$ has CPoU. In particular, unital C^*-algebras with uniform property Γ (e.g. unital $\mathcal{Z}$-stable C^*-algebras) have CPoU.*

For a factorial tracially complete C^*-algebra $(\mathcal{M}, X)$, all of whose fibres $\pi_\tau(\mathcal{M})''$ have property Γ as tracial von Neumann algebras, obtaining property Γ for $(\mathcal{M}, X)$ is itself a local-to-global transfer problem. Theorem 1.4, together with the local-to-global technology, shows that transferring property Γ from fibres to a global result is to some extent a universal local-to-global problem.

The strategy to prove Theorem 1.4 follows the overall framework used in [23], which, in the language of this paper, proves Theorem 1.4 for the uniform tracial completion of a separable nuclear C^*-algebra. The argument splits into two steps:

- Obtain a weak form of CPoU in which all the approximate projections making up the partition of unity are replaced by contractions (Theorem 6.15) and are not orthogonal.
- Use property Γ to convert the weak form of CPoU to CPoU by means of orthogonalisation, projectionisation, and a maximality argument (see the discussion in the last section of the introduction to [23] for an outline).

The second step works generally as was foreshadowed in [23, Lemma 3.7]. However, in [23], nuclearity was instrumental in performing the first step ([23, Lemma 3.6]) through a refined form of the completely positive approximation property from [14]. In particular, Connes' theorem on the equivalence of injectivity and hyperfiniteness underpins these approximations. Our proof of Theorem 1.4 establishes this weak form of CPoU in general (Theorem 6.15), a result which is already of independent interest (see [102, 120]). We do this by means of a Hahn–Banach-driven local-to-global transfer of the form described in Section 1.3. The point is that all finite von Neumann algebras (viewed as tracially complete C^*-algebras with respect to all their traces) satisfy CPoU. Taking suitable convex combinations of the elements witnessing CPoU in finitely many fibres gives rise to the required weak form of CPoU, and we do not need to rely on anything as deep as Connes' work. Specialising to the uniform tracial completion of a nuclear

$\mathcal{Z}$-stable C^*-algebra, this approach gives a much simpler overall argument for CPoU as compared with [23].

1.5. **Structure and classification.** Our structure and classification results (Theorems A–D) are proved by a combination of local-to-global transfer in the same spirit as [23, 22, 21] and Elliott-style intertwining arguments.

For a separable nuclear C^*-algebra A, a local-to-global argument was given in [22, Theorem 4.6] to pass from uniform property Γ to the uniform McDuff property via CPoU. The key point is that as A is nuclear, all tracial von Neumann algebras $\pi_\tau(A)''$ associated to traces on A are McDuff, and this can then be transformed into a global statement via CPoU. This proves the equivalence of property Γ and the McDuff property in Theorem B for the uniform tracial closures $\big(\overline{A}^{T(A)}, T(A)\big)$ of such C^*-algebras. An identical argument can be used to obtain this equivalence for amenable factorial II_1 tracially complete C^*-algebras, the only difference being the use of Theorem 1.4 in place of the main result of [23]. For the additional equivalence of hyperfiniteness in Theorem B, we go through classification. Our local characterisations of amenability ensure that hyperfinite tracially complete C^*-algebras are amenable (Theorem 8.2), so once we show hyperfinite factorial tracially complete C^*-algebras have CPoU (Theorem 8.3), the rest of the structure theorem (Theorem B) will follow from the classification theorem.

As factorial finite dimensional tracially complete C^*-algebras have CPoU, and CPoU is preserved under inductive limits, it is easy to obtain CPoU for limits of factorial finite dimensional C^*-algebras. It is similarly straightforward to show CPoU holds for a tracially complete C^*-algebra that is locally approximated by embedded factorial finite dimensional tracially complete subalgebras. The problem comes when a factorial tracially complete C^*-algebra $(\mathcal{M}, X)$ has approximations by finite dimensional C^*-subalgebras in which not all traces extend to elements of X – and so they don't embed as tracially complete subalgebras. This is reminiscent of Murray and von Neumann's work ([83]) on the uniqueness of hyperfinite II_1 factors, where they have to be concerned with approximating finite dimensional subalgebras which are not factors. Our solution, found in Section 8, is the same as Murray and von Neumann's: reduce to the case that the building blocks can always be taken to be factorial. This proceeds by showing that for separable tracially complete C^*-algebras, local and inductive limit definitions of hyperfiniteness agree, which allows us to realise the given algebra as a tracial completion of an AF C^*-algebra.

Turning to classification, the primary objective is the classification of tracially nuclear maps in Theorem D (proved as Theorem 9.12(ii)). As is standard for classification results for maps, this consists of two components: existence (of tracially nuclear *-homomorphisms with specified behaviour at the level of traces) and uniqueness (of such maps up to approximate unitary equivalence). The uniqueness aspect of Theorem D (Theorem 9.3) is a direct application of the corresponding uniqueness result for weakly nuclear *-homomorphisms into finite von Neumann algebras by traces (a folklore consequence of Connes' theorem), and a CPoU-powered local-to-global argument. This works in essentially the same fashion as the uniqueness result

found in [21, Theorem 2.2], and the only difference is the increased generality of the statement.

As with the corresponding theorems in the C^*-classification programme, the existence aspect of Theorem D is obtained in two stages. At the first pass, one only aims for an 'approximate' result – for a C^*-algebra A, and factorial tracially complete C^*-algebra $(\mathcal{M}, X)$ we want uniform 2-norm approximately multiplicative maps $A \to \mathcal{M}$ which approximately implement a given continuous affine map $\gamma\colon X \to T(A)$. For this to hold generally, it will be necessary for the traces $\gamma(X) \subseteq T(A)$ to satisfy suitable approximation properties – in particular, we require the range of γ to consist of *hyperlinear* traces on A (i.e. those factoring through the tracial ultrapower $\mathcal{R}^\omega$ of $\mathcal{R}$).[11] The approximate existence result uses another CPoU-powered local-to-global argument to patch together approximate morphisms into the fibres $\pi_\tau(\mathcal{M})''$, $\tau \in X$, which arise from composing approximate embeddings into $\mathcal{R}$ with a unital embedding $\mathcal{R} \to \pi_\tau(\mathcal{M})''$. Under the stronger hypothesis that the range of $\gamma\colon X \to T(A)$ consists of uniformly amenable traces, the approximate morphisms $A \to \mathcal{M}$ approximately realising γ can be further arranged to be tracially nuclear.

While it would be possible (though technically somewhat awkward) to prove Theorem 9.8 in a similar fashion to that of [21, Theorem 2.6], we instead take advantage of our classification of projections (Theorems 7.18 and 7.19), and hence maps from finite dimensional algebras,[12] to give a different and arguably more conceptual approach to these results. The point is that as $(\mathcal{M}, X)$ is factorial, X is a Choquet simplex, and so we can approximate γ by affine maps $X \to Z \to T(A)$ factoring through finite dimensional simplices Z using a result of Lazar and Lindenstrauss from the early 1970s ([70]). Then one uses hyperlinearity to produce approximately multiplicative maps from A into finite dimensional algebras which approximately realise the maps $Z \to T(A)$, and use the the classification of projections to embed these finite dimensional algebras into $\mathcal{M}$ compatibly with the map $X \to Z$.

At the second pass one aims for exact existence results by means of a one-sided Elliott intertwining argument. This is by now a standard technique, and there are no additional difficulties caused by working with tracially complete C^*-algebras and uniform 2-norms.[13] As ever, it is important that the uniqueness theorem is strong enough to cover approximately multiplicative maps. This gives rise to the exact existence result in Theorem 9.12, and completes the proof of Theorem D.

Classification results for uniformly tracially complete C^*-algebras are then obtained from Theorem D using a two-sided Elliott intertwining argument. This classifies the family of amenable tracially complete C^*-algebras $(\mathcal{M}, X)$

[11] When $(\mathcal{M}, X)$ is the hyperfinite II_1 factor with its unique trace $\tau_{\mathcal{R}}$, this approximate existence result is equivalent to hyperlinearity of the relevant trace on A.

[12] The existence result for maps out of finite-dimensional C^*-algebras is Lemma 9.5. Uniqueness does appear explicitly but is implicitly contained in the proof of Proposition 9.2 – in particular, see Footnote 57.

[13] The one subtle point is that the intertwining by reparameterisation technique for constructing genuine morphisms from approximate morphisms (Theorem 5.11) requires stability of unitaries in the uniform 2-norm. We prove that this follows from CPoU in Corollary 7.11.

satisfying both the domain and codomain hypotheses of Theorem D. The amenability hypothesis is needed so that the relevant identity maps are tracially nuclear (and so fall within the scope of Theorem D), as the intertwining argument will use uniqueness to compare the identity maps with the compositions of the maps obtained from the existence portion of Theorem D. This process (which is an instance of Elliott's abstract classification framework from [38]) yields Theorem C, which contains Theorem A as a special case. See [115, Section 6] for a general description of the passage from approximate existence and uniqueness, to the classification of operator algebras via one-sided intertwining and then a symmetrisation of hypotheses.

Just as we often prefer to describe the outcomes of the local-to-global transfer procedure at the level of ultraproducts or sequence algebras in order to suppress explicit error tolerances, we do the same for approximately multiplicative maps. So, in the main body our approximate existence result and the corresponding uniqueness theorem are given in terms of exact classification results into reduced products, as was done with the precursor results in [21].

1.6. **Classification and the Toms–Winter conjecture.** We end the introduction by discussing the role tracially complete C^*-algebras play in the structure and classification of simple stably finite amenable C^*-algebras. These are some instances of step (iii), 'pulling results back from the tracial completion to the C^*-level', of the scheme on page 3.

Given a simple unital C^*-algebra B, with $T(B) \neq \emptyset$, write $\mathcal{M} := \overline{B}^{T(B)}$, $B_\infty := \ell^\infty(B)/c_0(B)$ for the norm approximate sequence algebra of B, and $\mathcal{M}^\infty$ for the uniform 2-norm approximate sequence algebra of $\mathcal{M}$, i.e. $\ell^\infty(\mathcal{M})$ modulo the $\|\cdot\|_{2,T(B)}$-null sequences. By construction, there is a Kaplansky density type theorem – the unit ball of B is $\|\cdot\|_{2,T(B)}$-dense in the unit ball of $\mathcal{M}$ – and this ensures that the canonical inclusion $B \to \mathcal{M}$ gives a surjection $B_\infty \to \mathcal{M}^\infty$. The *trace kernel extension* is the short exact sequence induced by this surjection:[14]

$$0 \to J_B \to B_\infty \to \mathcal{M}^\infty \to 0. \tag{1.6}$$

The ideal J_B is known as the *trace-kernel ideal*, and it inherits regularity properties (such as separable $\mathcal{Z}$-stability and strict comparison) from corresponding properties of B.

The main objective in the abstract approach to the unital classification theorem for simple nuclear C^*-algebras in [16] is a classification of full unital *-homomorphisms $A \to B_\infty$ where A is a unital separable nuclear C^*-algebra satisfying the UCT, and B is a unital simple separable nuclear $\mathcal{Z}$-stable finite C^*-algebra ([16, Theorem 1.1]). Once this is in place, the classification of C^*-algebras follows from intertwining arguments as discussed in Section 1.5. As set out in [16, Section 1.3],[15] at a very high level, the strategy for the

[14] In some cases in the literature, an ultrapower version of the trace-kernel extension is used; accordingly, in this paper, we develop the theory in terms of reduced powers with respect to a free filter, simultaneously covering both cases.

[15] There the argument is given using the uniform tracial sequence algebra B^∞; this is canonically isomorphic to $\mathcal{M}^\infty$ by a Kaplansky density argument.

classification of full approximate *-homomorphisms falls into the three step plan from page 3.

(i) Classify maps from A into a finite von Neumann algebra by traces; this is a consequence of Connes' theorem.
(ii) Classify maps from A into $\mathcal{M}^\infty$ by traces; this is the classification of approximate *-homomorphisms from A into $\mathcal{M}$ described in Section 1.5, and the instance used here is the case covered in the combination of [23, 21] (as $\mathcal{M}$ is the uniform tracial completion of a nuclear $\mathcal{Z}$-stable C^*-algebra).
(iii) Classify lifts of a full map $\theta\colon A \to \mathcal{M}^\infty$ back to B_∞ in terms of K-theoretic data. This is the main task of [16], and a detailed outline of the strategy for this step is given there.

In particular, none of the results in this paper are necessary for the abstract proof of the unital classification paper, though we do contend that, in hindsight, the classification results for maps $A \to \mathcal{M}$ given here make the three-step plan above more transparent.

To take stably finite classification beyond the setting of nuclear codomains, our Theorem 1.4 will be crucial to step (ii). One objective is a stably finite version of Kirchberg's very general classification results for full morphisms $A \to B_\infty$, where A is a separable exact C^*-algebras satisfying the UCT and B is an $\mathcal{O}_\infty$-stable C^*-algebra (see [93, Theorem 8.3.3], [45, Theorems A and B], [63]). In the forthcoming work [17], a subset of us will extend the stably finite classification of morphisms to a level of generality corresponding to Kirchberg's framework. Sticking to the unital case, this will classify unital full nuclear *-homomorphisms $A \to B_\infty$ for domains A in analogy with Kirchberg's theorem, and where B is unital, finite, $\mathcal{Z}$-stable, and has comparison of positive elements by bounded traces.[16] Via intertwining, this entails a classification of unital *-homomorphisms from A to B which map traces on B to faithful traces on A. Outside the nuclear setting, one cannot obtain CPoU for B (or equivalently, for its uniform tracial completion) from [23].

Looking to the future, we hope that the breakthrough results in [47] on the classification of stronger outer actions of countable discrete amenable groups[17] on Kirchberg algebras will, over time, have powerful stably finite counterparts. Here, we expect that developing a suitable classification of group actions on classifiable tracially complete C^*-algebras (step (ii)) will help break up the overall task in to more manageable parts, particularly in the case where the underlying action on the trace space is complex. Even more generally one can imagine more general notions of quantum symmetries acting on uniform tracially complete C^*-algebras, as a bridge towards studying such actions on classifiable C^*-algebras whose tracial state space is large.

On the structure side, the remaining open part of the Toms–Winter conjecture is intimately linked with tracially complete C^*-algebras. By now,

[16]This will be discussed further in [17], but it forces all quasitraces on B to be traces.

[17]The results of [47] are much more general than this, encompassing amenable isometrically shift absorbing actions of locally compact groups

for a simple separable non-elementary nuclear C^*-algebra B, the conditions of $\mathcal{Z}$-stability and finite nuclear dimension are known to coincide ([118, 117, 106], and [23, 19], building on the line of work [10, 98, 76]). Back in 2004, Rørdam showed that if B is $\mathcal{Z}$-stable, then B has strict comparison ([94]) and the remaining piece of the Toms–Winter conjecture is the converse.[18]

The trace-kernel extension, as we view it today, has its origins in Matui and Sato's breakthrough work on the implication from strict comparison to $\mathcal{Z}$-stability ([75]). A key fact is that it induces a short exact sequence at the level of central sequence algebras (see [68, Theorem 3.3])

$$0 \to J_B \cap B' \to B_\infty \cap B' \to \mathcal{M}^\infty \cap \mathcal{M}' \to 0, \tag{1.7}$$

and tensorial absorption results for B and $\mathcal{M}$ can be written in terms of the central sequence algebras $B_\infty \cap B'$ and $\mathcal{M}^\infty \cap \mathcal{M}'$. Let us split the problem of whether strict comparison implies $\mathcal{Z}$-stability of B into the three steps.

(i) Injective II_1 von Neumann algebras are McDuff as a consequence of Connes' theorem.[19]

(ii) Attempt to lift the McDuff property back from von Neumann algebras to $\mathcal{M}$. This is an immediate consequence of the local-to-global argument, *provided* one has CPoU. For the uniform tracial completion of a nuclear C^*-algebra with no finite dimensional representations, CPoU is equivalent to property Γ.

(iii) Lift an embedding $M_n \to \mathcal{M}^\infty \cap \mathcal{M}'$ to an order zero map $\phi\colon M_n \to B_\infty \cap B'$. Such lifts always exist by projectivity of order zero maps from matrix algebras ([71, Theorem 4.9]), and in fact, any *-homomorphism from a separable C^*-algebra into $\mathcal{M}^\infty \cap \mathcal{M}'$ has an order zero lift by [68, Propositions 4.5 and 4.6]. Matui and Sato's notion of property (SI) – a kind of large-to-small comparison condition in $B_\infty \cap B'$, which they obtain from strict comparison and nuclearity of B – is designed to ensure that ϕ gives rise to a copy of $\mathcal{Z}$ in $B_\infty \cap B'$ and hence to $\mathcal{Z}$-stability of B.

Steps (i) and (iii) work generally; the challenge is at step (ii). This abstraction of Matui and Sato's strategy led to [22, Theorem 5.6], showing that for unital C^*-algebras as in the Toms–Winter conjecture, $\mathcal{Z}$-stability is equivalent to the combination of strict comparison and uniform property Γ (i.e. property Γ for the tracial completion).[20]

Thus the following question is fundamental; by [22, Theorem 5.6] a positive answer would resolve the unital case of the Toms–Winter conjecture.

[18] A weaker conjecture, of the form that pure simple separable non-elementary nuclear C^*-algebras are $\mathcal{Z}$-stable, is discussed by Winter in [116, Section 5.4]. Here, pureness is the combination of strict comparison with a (tracial) divisibility condition on the Cuntz semigroup and can be thought of as a combination of a weak uniqueness theorem and an existence theorem for positive elements in terms of their rank functions.

[19] For II_1 factors this was a step along the road to Connes' theorem (this follows from 7⇒2 of [29, Theorem 5.1], defining ϕ in 7 as the composition of conditional expectation and the trace), and it is folklore that it holds generally; a proof from hyperfiniteness to the McDuff property can be found as [22, Proposition 1.6]. See also [101], which obtains the equivariant McDuff property, extending results from Ocneanu beyond the factor setting.

[20] See [20, Theorem A] for a non-unital statement.

The analogous result in the von Neumann setting is the first component of Connes' theorem: injective II_1 factors have property Γ (which goes through the passage from failure of property Γ to a spectral gap condition; see also the two new proofs [74, 73]).

Question 1.5. Does every amenable type II_1 factorial tracially complete C^*-algebra satisfy property Γ?

When the designated set of traces X is reasonably small, Question 1.5 has a positive answer; indeed in the unique trace case this is due to Connes as mentioned above. We give a positive answer for $\partial_e X$ compact and zero dimensional as Proposition 5.28. More generally, it will be shown in forthcoming work by the third- and fifth-named authors that the same holds for $\partial_e X$ compact and finite-dimensional. It remains mysterious whether one should expect a positive answer in general, or whether strict comparison for a C^*-algebra A would imply property Γ for its uniform tracial completion – which if true would establish the Toms–Winter conjecture. If neither of these situations hold, then it is reasonable to ask whether amenable tracially complete C^*-algebras with a suitable tracial divisibility property satisfy property Γ. In other words, if a version of Winter's tracial divisibility holds for a amenable tracially complete C^*-algebra, must it have property Γ? A positive answer to this question would resolve the modified conjecture mentioned in Footnote 18.

We end with a comparison of the hypotheses in the classification of amenable algebras that we have discussed.

Type of algebra	Regularity
Finite amenable von Neumann algebra	Automatic
Amenable factorial II_1 tracially complete C^*-algebra	Open
Simple separable non-elementary nuclear C^*-algebra	Not automatic

With the hypotheses listed in the table, 'regularity' for von Neumann algebras simply means the McDuff property; for tracially complete algebras it means either McDuff or property Γ (equivalent in this setting). For C^*-algebras as in the table, regularity can be interpreted as $\mathcal{Z}$-stability, and non-$\mathcal{Z}$-stable examples are known to exist. The other key hypothesis to C^*-classification is the universal coefficient theorem, which is inherently topological, and therefore doesn't have von Neumann algebra or tracially complete counterparts; whether it holds automatically remains open.

Acknowledgements. We'd like to thank Ilijas Farah, Bradd Hart, Ilan Hirshberg, Gábor Szabó and, Andrea Vaccaro for discussions, and Valerie Morris, Mikkel Munkholm, and Pawel Sarkowicz for comments an drafts of the paper. Parts of the research in this paper was undertaken at: the American Institute of Mathematics as part of the SQuaRE *von Neumann techniques in the classification of C*-algebras* (2019-2023), the *Workshop on C*-algebras: structure and dynamics* (Sde Boker, 2022), and the Fields Institute as part of the *Thematic program on operator algebras and their applications* (2023).

2. Traces and Choquet simplices

Choquet theory will play an important role throughout this paper, and we begin this preliminary section by recalling some definitions and theorems about Choquet simplices in Section 2.1. The main examples of a Choquet simplices in this paper are the spaces $T(A)$ of tracial states on a unital C^*-algebra A with its weak* topology, along with the closed faces of $T(A)$. We will discuss traces in more depth in Section 2.2 and recall some results related to their GNS representations and induced von Neumann algebras for later use.

2.1. **Choquet theory.** We refer the reader to [3] and [55] for general references on Choquet theory. We collect some basic definitions here along with the results needed in the main body of the paper.

Let X be a compact convex set in a locally convex space and write $\mathrm{Aff}(X)$ for the Banach space of all continuous affine functions $X \to \mathbb{R}$ equipped with the supremum norm. Then $\mathrm{Aff}(X)$ is an Archimedean order unit space in the sense of [3, Section II.1] with the pointwise order and the order unit $1_{\mathrm{Aff}(X)}$, the constant function 1.[21] Conversely, if V is a Archimedean order unit space, then the state space of V,

$$S(V) \coloneqq \{f \in V^* : \|f\| = f(1_V) = 1\}, \tag{2.1}$$

is a weak*-compact convex subset of V^*. By a result of Kadison [61] (see also [55, Chapter 7]), the natural maps

$$X \to S(\mathrm{Aff}(X)) \qquad \text{and} \qquad V \to \mathrm{Aff}(S(V)) \tag{2.2}$$

are isomorphisms. This gives an anti-equivalence between the categories of compact convex sets and Archimedean order unit spaces. We refer to this result as *Kadison duality*. One important consequence is that the weak topology on $\mathrm{Aff}(X)$ is the topology of pointwise convergence. This is standard and used, for example, in [86] and [48].

Proposition 2.1. *Let X be a compact convex set.*

(i) *All states on $\mathrm{Aff}(X)$ are given by point evaluations.*
(ii) *If $(f_\lambda) \subseteq \mathrm{Aff}(X)$ is a net and $f \in \mathrm{Aff}(X)$, then $f_\lambda \to f$ weakly if and only if $f_\lambda(\tau) \to f(\tau)$ for all $\tau \in X$.*

Proof. (i). This is a consequence of Kadison's duality.

(ii). Note that bounded linear functionals on $\mathrm{Aff}(X)$ extend to bounded linear functionals on $C(X)$ by the Hahn–Banach theorem. Since the dual of $C(X)$ is spanned by states on $C(X)$ and all states on $C(X)$ restrict to states on $\mathrm{Aff}(X)$, it follows that $S(\mathrm{Aff}(X))$ spans $\mathrm{Aff}(X)^*$. The result now follows from (i).

□

[21] Briefly, an *Archimedean order unit space* is a triple $V = (V, V_+, 1_V)$ where V is a real vector space, $V_+ \subseteq V$ is a spanning cone, and $1_V \in V_+$ is a distinguished element $1_V \in V_+$, called the *order unit*, such that

$$\|v\| \coloneqq \inf\{r > 0 : -r1_V \leq v \leq r1_V\}, \qquad v \in V,$$

is a complete norm on V. (Note: in the literature, completeness is not always assumed as part of the definition, but for us it is.)

For a compact convex set X, write $\mathrm{Prob}(X)$ for the set of Radon probability measures on X. Given $\mu \in \mathrm{Prob}(X)$, the *barycentre* of μ is the unique point $x \in X$ such that

$$\int_X f \, d\mu = f(x), \qquad f \in \mathrm{Aff}(X). \tag{2.3}$$

Let $\partial_e X \subseteq X$ denote the set of extreme points of X. We say a complex Radon measure μ is *supported on* $\partial_e X$ if for every Baire measurable set $E \subseteq X$ with $\partial_e X \subseteq E$, we have $|\mu|(X \setminus E) = 0$.[22] By the Choquet–Bishop–de Leeuw theorem (see [3, Theorem I.4.8 and Corollary 1.4.12]), for every $x \in X$, there exists $\mu \in \mathrm{Prob}(X)$ supported on $\partial_e X$ with barycentre x. A compact convex set X is called a *Choquet simplex* if this measure μ is unique for every $x \in X$.

In the finite dimensional setting, there is a unique Choquet simplex of every dimension. Specifically, if $n \geq 0$ is an integer, then the n-dimensional Choquet simplex is given as the convex hull of an orthonormal set of $n+1$ vectors in a Hilbert space.

A Choquet simplex X is called a *Bauer simplex* if $\partial_e X$ is compact. In this case, we view $\mathrm{Prob}(\partial_e X)$ as a compact convex set in the locally convex space $C(\partial_e X)^*$. Note that there is an affine homeomorphism

$$\mathrm{Prob}(\partial_e X) \to X \tag{2.4}$$

given by sending a measure to the barycentre of its canonical extension to X (given by declaring that $X \setminus \partial_e X$ has measure zero). Indeed, this map is bijective by the definition of a Choquet simplex, and it is easily seen to be a homeomorphism.

When X is a metrisable Choquet simplex, a result of Lazar and Lindenstrauss in [70, Corollary of Theorem 5.2] (see also [55, Theorem 11.6]) shows X can be written as a projective limit of finite dimensional Choquet simplices. As noted in [40, Lemma 2.8], this implies X satisfies the finite dimensional approximation property in Theorem 2.2 below. As we set out in Appendix A.2, this result holds generally, i.e. without a metrisability assumption on X. Naturally, for results which only concern the separable situation (C^*-algebras which are separable in norm and tracially complete C^*-algebras which are separable in their uniform 2-norm), the metrisable version of Theorem 2.2 will suffice.

Theorem 2.2. *If X is a Choquet simplex, then there are nets of finite dimensional Choquet simplices Z_λ and continuous affine maps*

$$X \xrightarrow{\beta_\lambda} Z_\lambda \xrightarrow{\alpha_\lambda} X \tag{2.5}$$

such that $\lim_\lambda \|f \circ \alpha_\lambda \circ \beta_\lambda - f\| = 0$ for all $f \in \mathrm{Aff}(X)$.

We end this subsection with some results about closed faces in a Choquet simplex. A *face* in a compact convex set X is a convex set $F \subseteq X$ with the following property: for all $x_1, x_2 \in X$, if a non-trivial convex combination of x_1 and x_2 is in F, then both x_1 and x_2 are in F.

[22] When X is metrisable, the set $\partial_e X$ is a G_δ-set, so $\mu \in \mathrm{Prob}(X)$ is supported on $\partial_e X$ if and only if $\mu(\partial_e X) = 1$. In general, $\partial_e X$ is not Baire measurable (and hence not Borel measurable) – see [5, Section VII] (and [5, Lemma 4.1] to identify $\partial_e X$ with $M(B)$).

The next two results can be viewed in analogy with the Hahn–Banach theorem. The first is an extension result, and the second is a separation result.

Theorem 2.3 ([3, Theorem II.5.19])**.** *Let F be a closed face of a Choquet simplex X. For every $f \in \mathrm{Aff}(F)$, there exists $\hat{f} \in \mathrm{Aff}(X)$ with $\hat{f}|_F = f$ and $\|\hat{f}\| = \|f\|$.*

The following is an easy consequence of [3, Corollary II.5.20]. The first part follows the proof of [3, Proposition II.5.16].

Theorem 2.4. *Let F be a closed face in a Choquet simplex X.*

(i) *If $S \subseteq X$ is an F_σ-set with $S \cap F = \emptyset$, then there is a continuous affine function $f\colon X \to [0,1]$ such that $f(x) = 0$ for all $x \in F$ and $f(x) > 0$ for all $x \in S$.*
(ii) *If $x \in X$ and $f(x) = 0$ for all $f \in \mathrm{Aff}(X)_+$ with $f|_F = 0$, then $x \in F$.*

Proof. By [3, Corollary II.5.20], closed faces in Choquet simplices are relatively exposed, which means that (i) holds when S is a point. From here, (ii) is immediate.

In (i), suppose first that S is closed (and hence compact as X is compact). For $s \in S$, let $f_s\colon X \to [0,1]$ be a continuous affine function such that $f_s(x) = 0$ for $x \in F$ and $f_s(s) > 0$. As S is compact and each f_s is continuous, there are $s_1, \dots, s_n \in S$ such that

$$\inf_{s \in S} \max_{1 \le i \le n} f_{s_i}(s) > 0. \tag{2.6}$$

Set $f \coloneqq \frac{1}{n}\sum_{i=1}^n f_{s_i}$. To see the general case, consider an F_σ-set S and write S as the union of closed sets $(S_n)_{n=1}^\infty$. For each $n \ge 1$, let $f_n\colon X \to [0,1]$ be a continuous affine function such that $f_n(x) = 0$ for all $x \in F$ and $f_n(x) > 0$ for all $x \in S_n$. Then set $f \coloneqq \sum_{n=1}^\infty 2^{-n} f_n$. □

The final result of this subsection concerns detecting closed faces. In general, if X is a Choquet simplex and $S \subseteq \partial_e X$, then the convex hull of S, written $\mathrm{co}(S)$, is a face in X. However, the closed convex hull of S, written $\overline{\mathrm{co}}(S)$, need not be a face (see [2, Theorem 1]). The following result of Roy gives a replacement.

Theorem 2.5 (cf. [95, Proposition 4.4])**.** *If X is a Choquet simplex and $F \subseteq X$ is a closed convex set, then F is a face in X if and only if $\partial_e F \subseteq \partial_e X$.*

Proof. The forward direction is clear. Conversely, if $\partial_e F \subseteq \partial_e X$, then $F_0 \coloneqq \mathrm{co}(\partial_e F)$ is a face in X, and by the Krein–Milman Theorem, F is the closure of F_0. Using again that $\partial_e F \subseteq \partial_e X$, we have that F is a face in X by [95, Proposition 4.4]. □

2.2. **Traces and the GNS construction.** By a *trace* on a C^*-algebra, we will always mean a tracial state. For a C^*-algebra A, let $T(A)$ denote the set of traces on A equipped with the weak* topology. Then $T(A)$ is convex. We will typically be interested in C^*-algebras where $T(A)$ is compact – this is the case for unital C^*-algebras, for example. The following result is folklore.

Theorem 2.6. *If A is a C^*-algebra, then every compact face in $T(A)$ is a Choquet simplex.*

Proof. When A is unital, $T(A)$ is a Choquet simplex by [96, Theorem 3.1.18]. By [55, Proposition 10.9], every closed face in a Choquet simplex is a Choquet simplex.

When A is non-unital with unitisation $A^\dagger$, traces on A can be extended to traces on $A^\dagger$, and $\tau \in T(A^\dagger)$ induces a trace on A if and only if $\|\tau|_A\| = 1$. In this way $T(A)$ can be viewed as a face in $T(A^\dagger)$, and the result follows from the unital case.

□

We turn to the structure of continuous affine functions $T(A) \to \mathbb{R}$, which will be used repeatedly in this paper. The result is well-known and is often attributed to Cuntz and Pedersen in [30], but the result does not explicitly appear in their paper. A proof of the first part of the result can be found in [68, Lemma 6.2] or [16, Proposition 2.1], for example.

Proposition 2.7 (cf. [30, Proposition 2.7])**.** *Let A be a unital C^*-algebra and let $f\colon T(A) \to \mathbb{R}$ be a continuous affine function. Then for any $\epsilon > 0$, there is a self-adjoint element $a \in A$ such that*

$$\|a\| \leq \|f\|_\infty + \epsilon \quad \text{and} \quad \tau(a) = f(\tau), \quad \tau \in T(A). \tag{2.7}$$

Moreover, if f is strictly positive, we may assume $a \geq 0$.

Proof. We only prove the last sentence. Assume that f is strictly positive. Since $T(A)$ is compact, there is a $\delta \in (0, 2\epsilon)$ such that $f(\tau) > \delta$ for all $\tau \in T(A)$. Apply the first part of the proof to $f - \frac{1}{2}(\|f\|_\infty + \delta)$, which has norm at most $\frac{1}{2}(\|f\|_\infty - \delta)$, to obtain a self-adjoint $b \in A$ such that

$$\|b\| \leq \frac{1}{2}\|f\|_\infty \qquad \text{and} \qquad \tau(b) = f(\tau) - \frac{1}{2}(\|f\|_\infty + \delta) \tag{2.8}$$

for all $\tau \in T(A)$. Then set $a \coloneqq b + \frac{1}{2}(\|f\|_\infty + \delta)1_A$. □

The essential starting point for all of our classification results is the classification of projections in finite von Neumann algebras by traces (parts (iii) and (iv) in the proposition below), which goes back to Murray and von Neumann. We review these results below; while these are well-known to experts, they are most often stated for factors – particularly the existence of projections realising arbitrary continuous affine functions – and in this paper we need to work with arbitrary finite von Neumann algebras.

Proposition 2.8. *Let $\mathcal{M}$ be a finite von Neumann algebra.*

(i) *Every trace on $\mathcal{M}$ factors uniquely through the centre-valued trace.*[23]
(ii) *The normal traces on $\mathcal{M}$ are dense in the traces on $\mathcal{M}$.*
(iii) *Projections $p, q \in \mathcal{M}$ are unitarily equivalent if and only if $\tau(p) = \tau(q)$ for all $\tau \in T(\mathcal{M})$; p is Murray–von Neumann subequivalent to q if and only if $\tau(p) \leq \tau(q)$ for all $\tau \in \mathcal{T}(\mathcal{M})$.*

[23] See [104, Theorem V.2.6] or [62, Theorem 8.2.8] for properties of the centre-valued trace.

(iv) *If $\mathcal{M}$ is type* II_1 *and $f\colon T(\mathcal{M}) \to [0,1]$ is a continuous affine function, then there is a projection $p \in \mathcal{M}$ such that $\tau(p) = f(\tau)$ for all $\tau \in T(\mathcal{M})$.*

Proof. (i). This is a consequence of Dixmier's approximation theorem (see [62, Proposition 8.3.10] or [7, Theorem III.2.5.7(iv)]).

(ii). As the centre-valued trace on $\mathcal{M}$ is normal (see [103, Theorem V.2.34]), the result follows from (i) and the density of the normal states in the states on the centre of $\mathcal{M}$.

(iii). The second part of the statement follows from Murray and von Neumann's comparison theorem for projections in von Neumann algebras, the properties of the centre-valued trace, and (i); see [104, Corollary V.2.8], for example. The first part follows from applying the second part both to p and q and to $1_{\mathcal{M}} - p$ and $1_{\mathcal{M}} - q$.

(iv). This proceeds in a very similar way to the proof that II_1 factors contain projections of arbitrary traces in $[0,1]$ (see [7, Theorem III.1.7.9 and Paragraph III.1.7.10], for example). As we have been unable to find a reference, we give the details for completeness.

Let P be the set of projections $p \in \mathcal{M}$ such that $\tau(p) \leq f(\tau)$ for all $\tau \in T(\mathcal{M})$. Then $P \neq \emptyset$ as $0 \in P$. Also, if (p_λ) is an increasing chain of projections in $\mathcal{M}$ with supremum $p \in \mathcal{M}$, then for any normal trace $\tau \in \mathcal{M}$ we have

$$\tau(p) = \lim_\lambda \tau(p_\lambda) \leq f(\tau). \tag{2.9}$$

As f is weak*-continuous, part (ii) gives $p \in P$. By Zorn's Lemma, there is a maximal $p \in P$. We will show $f(\tau) = \tau(p)$ for all $\tau \in T(\mathcal{M})$.

Suppose this is not the case. By the Krein–Milman theorem, there exists $\tau_0 \in \partial_e T(\mathcal{M})$ with $\tau_0(p) < f(\tau_0)$. Set $\epsilon \coloneqq \frac{1}{2}(f(\tau_0) - \tau_0(p))$. By (i), there is an isomorphism

$$Z(\mathcal{M}) \xrightarrow{\cong} C(\partial_e T(\mathcal{M}))\colon a \mapsto \big(\tau \mapsto \tau(a)\big). \tag{2.10}$$

In particular, $\partial_e T(\mathcal{M})$ is totally disconnected. Let $U \subseteq \partial_e T(\mathcal{M})$ be a clopen neighbourhood of τ_0 so that

$$\tau(p) < f(\tau) - \epsilon \tag{2.11}$$

for all $\tau \in U$, and let $z \in Z(\mathcal{M})$ be the projection corresponding to the characteristic function of U under the isomorphism in (2.10). Since $\tau(1_{\mathcal{M}} - p) \geq \epsilon$ for all $\tau \in U$, it follows that

$$\tau(z(1-p)) \geq \epsilon \tag{2.12}$$

for all $\tau \in \partial_e T(z\mathcal{M})$, and hence for all $\tau \in T(z\mathcal{M})$.

Fix $d \geq 1$ with $1/d < \epsilon$. Since $\mathcal{M}$ is type II_1, so is the corner $z\mathcal{M}$. By the proof of [104, Theorem V.1.35] (which handles the case $d = 2$), there is a unital embedding $M_d \to z\mathcal{M}$, and in particular, there is a projection $e \in z\mathcal{M}$ so that $\tau(e) = 1/d$ for all $\tau \in T(z\mathcal{M})$. By comparison of projections in the von Neumann algebra $z\mathcal{M}$, it follows that e is Murray–von Neumann subequivalent to $z(1-p)$. Hence after conjugating e by a unitary in $z\mathcal{M}$, we may assume e is orthogonal to zp. Since e is also orthogonal to the central projection $z^\perp$, it follows that e is orthogonal to p.

Set $p' \coloneqq p+e$. Since e is orthogonal to p, it follows that p' is a projection. For $\tau \in \partial_e X \backslash U$, we have $\tau(p') = \tau(p) \leq f(\tau)$, and for $\tau \in U$, we have $\tau(p') = \tau(p) + \frac{1}{d} < f(\tau)$. By the Krein–Milman theorem, it follows that $\tau(p') \leq f(\tau)$ for all $\tau \in T(\mathcal{M})$, and so $p' \in P$. This contradicts the maximality of p. □

Given a C^*-algebra A and $\tau \in T(A)$, let $\pi_\tau \colon A \to \mathcal{B}(\mathcal{H}_\tau)$ be the GNS representation associated to τ. The induced von Neumann algebra $\pi_\tau(A)''$ has a faithful normal trace induced by τ and hence is finite. Therefore, $\pi_\tau(A)''$ is the direct sum of a finite von Neumann algebra of type I and a von Neumann algebra of type II_1. Under the assumption that A has no finite dimensional quotients, $\pi_\tau(A)''$ is of type II_1 – this will be a common hypothesis throughout the paper.

The following well-known characterisation of the extreme points of $T(A)$ will be used frequently in the paper. A version of the result for not necessarily bounded tracial weights is given in [33, Theorem 6.7.3] – the result below is a special case.[24]

Proposition 2.9 (cf. [33, Theorem 6.7.3])**.** *If A is a unital C^*-algebra and $\tau \in T(A)$, then τ is an extreme point of $T(A)$ if and only if $\pi_\tau(A)''$ is a factor.*

The following lemma relates traces on the von Neumann algebra $\pi_\tau(A)''$ to traces on A. For a C^*-algebra A and a set of traces $X \subseteq T(A)$, we define

$$\pi_X \coloneqq \bigoplus_{\tau \in X} \pi_\tau \colon A \to \mathcal{B}\Big(\bigoplus_{\tau \in X} \mathcal{H}_\tau \Big) \tag{2.13}$$

to be the direct sum of the GNS representations associated with the traces in X.

Lemma 2.10. *Suppose A is a unital C^*-algebra and $X \subseteq T(A)$. If $\tau \in T(\pi_X(A)'')$, then $\tau \circ \pi_X$ is in the closed face generated by X.*

Proof. Let $F \subseteq T(A)$ be the weak*-closed face generated by X. By the density of the normal traces on $\pi_X(A)''$ in the traces on $\pi_X(A)''$ (Proposition 2.8(ii)), we may assume τ is normal. By Theorem 2.4(ii), it suffices to show that if $f \colon T(A) \to [0,1]$ is a continuous affine function vanishing on X, then $f(\tau \circ \pi_X) = 0$. Fix such an affine function f. By applying Proposition 2.7 to the strictly positive affine functions $f + \frac{1}{n}$ for each $n \in \mathbb{N}$, we get a bounded sequence $(a_n)_{n=1}^\infty \subseteq A_+$ such that

$$\lim_{n \to \infty} \sup_{\sigma \in T(A)} |\sigma(a_n) - f(\sigma)| = 0. \tag{2.14}$$

It follows that $\sigma(a_n) \to 0$ for all $\sigma \in X$. Since $a_n \geq 0$ and the sequence $(a_n)_{n=1}^\infty$ is bounded, this implies $\pi_X(a_n) \to 0$ strongly in $\pi_X(A)''$.[25] Since τ is normal, we have $\tau(\pi_X(a_n)) \to 0$, and hence using (2.14) with $\sigma = \tau \circ \pi_X$ implies that $f(\tau \circ \pi_X) = 0$. □

[24] Note that an extreme point of $T(A)$ is precisely a character of norm 1 in the sense of [33, Definition 6.7.1].

[25] Indeed, the topology on $\pi_X(A)''$ induced by the seminorms

$$\|a\|_{2,\sigma} \coloneqq \sigma(a^*a)^{1/2}, \qquad a \in \pi_X(A)'',\ \sigma \in X,$$

agrees with the strong topology on bounded sets – see [7, Proposition III.2.2.19], for example.

3. Tracially complete C^*-algebras

In this section, we introduce tracially complete C^*-algebras, which are the central objects of study in this paper. We discuss the motivating examples, including the tracial completion of a C^*-algebra, and prove some basic approximation and extension lemmas. We then construct inductive limits in the category of tracially complete C^*-algebras. In the final subsection, we examine the relationship between Ozawa's W^*-bundles ([86]) and tracially complete C^*-algebras.

3.1. **Definitions and basic properties.**

The definition of tracially complete C^*-algebras is based on uniform 2-norms.

Definition 3.1 (cf. [68, 97, 110, 86], for example)**.** For a C^*-algebra A and a set $X \subseteq T(A)$, define the *uniform* 2*-seminorm* on A by

$$\|a\|_{2,X} \coloneqq \sup_{\tau\in X} \sqrt{\tau(a^*a)}, \qquad a \in A. \tag{3.1}$$

The uniform 2-seminorm is indeed a seminorm, and the following inequality is easily verified:

$$\|ab\|_{2,X} \leq \min\{\|a\|\|b\|_{2,X}, \|a\|_{2,X}\|b\|\}, \qquad a, b \in A. \tag{3.2}$$

It is easily seen that $\|\cdot\|_{2,X}$ is a norm if and only if for every non-zero $a \in A_+$ there exists $\tau \in X$ with $\tau(a) > 0$. In this case, X is said to be a *faithful set of traces*, and we refer to $\|\cdot\|_{2,X}$ as the *uniform* 2*-norm* with respect to X.

The following properties of the uniform 2-norm are standard in the unique trace setting (i.e. when X is a singleton).

Proposition 3.2. *Suppose A is a C^*-algebra and $X \subseteq T(A)$ is a faithful set of traces.*

(i) *$\|\cdot\|$ is lower semi-continuous with respect to $\|\cdot\|_{2,X}$;*
(ii) *The unit ball of A is $\|\cdot\|_{2,X}$-closed;*
(iii) *A_+ is $\|\cdot\|_{2,X}$-closed.*

Proof. Fix $(a_n)_{n=1}^\infty \subseteq A$ and $a \in A$ with $\|a_n - a\|_{2,X} \to 0$. For $\tau \in X$, let $\pi_\tau \colon A \to \mathcal{B}(\mathcal{H}_\tau)$ denote the GNS representation of τ with associated cyclic vector $\xi_\tau \in \mathcal{H}_\tau$. If $b \in A$, then

$$\begin{aligned} \|(\pi_\tau(a_n) - \pi_\tau(a))\pi_\tau(b)\xi_\tau\| &= \|(a_n - a)b\|_{2,\tau} \\ &\leq \|a_n - a\|_{2,X}\|b\|. \end{aligned} \tag{3.3}$$

So $\pi_\tau(a_n)\eta$ converges to $\pi(a)\eta$ for all $\eta \in \pi_\tau(A)\xi_\tau$. As $\pi_\tau(A)\xi_\tau$ is dense in $\mathcal{H}_\tau$,

$$\|\pi_\tau(a)\| \leq \liminf_{n\to\infty} \|\pi_\tau(a_n)\| \leq \liminf_{n\to\infty} \|a_n\|. \tag{3.4}$$

The product of the π_τ over $\tau \in X$ is faithful, and hence isometric, since $\|\cdot\|_{2,X}$ is a norm on A. Therefore,

$$\|a\| = \sup_{\tau\in X} \|\pi_\tau(a)\| \leq \liminf_{n\to\infty} \|a_n\|. \tag{3.5}$$

This proves (i), and (ii) is an easy consequence of (i).

For (iii), suppose now that $a_n \geq 0$ for all $n \in \mathbb{N}$. Fix $\tau \in X$. By (3.3), we have

$$\langle \pi_\tau(a)\pi_\tau(b)\xi_\tau, \pi_\tau(b)\xi_\tau\rangle = \lim_{n\to\infty} \langle \pi_\tau(a_n)\pi_\tau(b)\xi_\tau, \pi_\tau(b)\xi_\tau\rangle \geq 0, \tag{3.6}$$

since $\pi_\tau(a_n) \geq 0$ for all $n \in \mathbb{N}$. The density of $\pi_\tau(A)\xi_\tau$ in $\mathcal{H}_\tau$ implies $\langle \pi_\tau(a)\eta, \eta\rangle \geq 0$ for all $\eta \in \mathcal{H}_\tau$, so $\pi_\tau(a) \geq 0$. As X is a faithful set of traces, the product of the π_τ over $\tau \in X$ is faithful. Since $\pi_\tau(a) \geq 0$ for all $\tau \in X$, it follows that $a \geq 0$. □

We will restrict to the case when $X \subseteq T(A)$ is weak*-compact since the weak*-closure of X defines the same uniform 2-norm as X. The following observation will allow us to further restrict to the case when X is convex. For a set $X \subseteq T(A)$, let $\overline{\text{co}}(X)$ denote the closed convex hull of X in A^*.

Proposition 3.3. *If A is a C^*-algebra and $X \subseteq T(A)$ is compact, then $\overline{\text{co}}(X) \subseteq T(A)$ and $\|a\|_{2,X} = \|a\|_{2,\overline{\text{co}}(X)}$ for all $a \in A$.*

Proof. Every $\tau \in \overline{\text{co}}(X)$ is a positive tracial functional with $\|\tau\| \leq 1$. To see the first claim, it remains to show that $\|\tau\| \geq 1$.

Let $(e_\lambda) \subseteq A$ be an approximate unit for A. Then $(\tau(e_\lambda))$ increases to 1 for all $\tau \in X$. By Dini's Theorem,[26] this convergence is uniform over τ. Fix $\epsilon > 0$ and let λ be such that $\tau(e_\lambda) > 1 - \epsilon$ for all $\tau \in X$. Then $\tau(e_\lambda) \geq 1 - \epsilon$ for all $\tau \in \overline{\text{co}}(X)$. Therefore, each $\tau \in \overline{\text{co}}(X)$ has norm 1.

Let $a \in A$. The set of all traces $\tau \in T(A)$ with $\tau(a^*a) \leq \|a\|_{2,X}^2$ is weak*-closed, convex, and contains X. Therefore, $\|a\|_{2,\overline{\text{co}}(X)} \leq \|a\|_{2,X}$. The reserve inequality is trivial as $X \subseteq \text{co}(X)$. □

A tracial von Neumann algebra (i.e. a von Neumann algebra with a distinguished faithful normal trace) can be characterised abstractly as a pair $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a C^*-algebra and τ is a faithful trace on $\mathcal{M}$ such that the unit ball of $\mathcal{M}$ is $\|\cdot\|_{2,\tau}$-complete (this follows from [100, Lemma A.3.3], for example). A morphism from a tracial von Neumann algebra $(\mathcal{M}, \tau_\mathcal{M})$ to another tracial von Neumann algebra $(\mathcal{N}, \tau_\mathcal{N})$ is a (necessarily normal) *-homomorphism $\phi\colon \mathcal{M} \to \mathcal{N}$ such that $\tau_\mathcal{N} \circ \phi = \tau_\mathcal{M}$. Our definition of tracially complete C^*-algebras and their morphisms are modelled on these definitions.

Definition 3.4. A *tracially complete C^*-algebra* is a pair $(\mathcal{M}, X)$ where $\mathcal{M}$ is a C^*-algebra and $X \subseteq T(\mathcal{M})$ is a compact convex set such that

(i) X is a faithful set of traces on $\mathcal{M}$, and
(ii) the unit ball of $\mathcal{M}$ is $\|\cdot\|_{2,X}$-complete.

Given two tracially complete C^*-algebras $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$, a *morphism* $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ between tracially complete C^*-algebras is a *-homomorphism $\phi\colon \mathcal{M} \to \mathcal{N}$ such that $\phi^*(Y) \subseteq X$; i.e. $\tau \circ \phi \in X$ whenever $\tau \in Y$.

Strictly speaking, X can be the empty set, which forces $\mathcal{M}$ to be the zero C^*-algebra (which is unital). Needless to say, this degenerate case is not of interest. Implicitly we always imagine X to be non-empty.

[26] Dini's Theorem is often stated for increasing sequences of continuous functions, but the standard proof is equally valid for increasing nets of continuous functions.

A major source of examples of tracially complete C^*-algebras are obtained by completing a C^*-algebra with respect to a compact convex set of traces. In a special case, if A is a C^*-algebra and τ is a trace on A, then the GNS-completion $(\pi_\tau(A)'', \tau)$ is a tracial von Neumann algebra which coincides with the tracial completion of A with respect to $\{\tau\}$. Tracial completions will be discussed further in Section 3.3 below.

Another significant source of examples are W^*-bundles, due to Ozawa ([86]).

Definition 3.5 ([86, Section 5])**.** Let K be a compact Hausdorff space.[27] A *W^*-bundle* over K is a unital C^*-algebra $\mathcal{M}$ together with a unital embedding $C(K) \subseteq Z(\mathcal{M})$ and a conditional expectation $E\colon \mathcal{M} \to C(X)$ such that

(i) E is tracial in the sense that $E(ab) = E(ba)$ for all $a, b \in \mathcal{M}$, and
(ii) the unit ball of $\mathcal{M}$ is complete with respect to the norm $\|\cdot\|_{2,E}$, defined by $\|a\|_{2,E} \coloneqq \|E(a^*a)\|^{1/2}$ for all $a \in \mathcal{M}$.

The *fibre* of $\mathcal{M}$ at a point $x \in K$ is $\pi_{\tau_x}(\mathcal{M})$, where $\tau_x = \mathrm{ev}_x \circ E$; this is equal to $\pi_{\tau_x}(\mathcal{M})''$ by [86, Theorem 11].

Every W^*-bundle gives rise to a tracially complete C^*-algebra.

Proposition 3.6. *Let $\mathcal{M}$ be a W^*-bundle over K with conditional expectation $E\colon \mathcal{M} \to C(K)$. Let X be the set of traces on $\mathcal{M}$ of the form*

$$\tau(a) = \int_K E(a)\,\mathrm{d}\mu, \qquad a \in \mathcal{M}, \tag{3.7}$$

for $\mu \in \mathrm{Prob}(K)$, the space of Radon probability measures on K. Then $(\mathcal{M}, X)$ is a tracially complete C^*-algebra and X is a Bauer simplex with extreme boundary $\partial_e X \cong K$.

Proof. There is a continuous affine embedding

$$\mathrm{Prob}(K) \to T(\mathcal{M})\colon \mu \mapsto \int_K E(\,\cdot\,)\,\mathrm{d}\mu. \tag{3.8}$$

If $X \subseteq T(\mathcal{M})$ denotes the image of this map, then X is compact and convex, and the map above restricts to a homeomorphism $K \to \partial_e X$. Since X is the closed convex hull of its extreme points, we have

$$\|a\|_{2,X} = \|a\|_{2,\partial_e X} = \|E(a^*a)\|^{1/2} = \|a\|_{2,E} \tag{3.9}$$

for all $a \in \mathcal{M}$. Since $\mathcal{M}$ is a W^*-bundle, it follows that $\|\cdot\|_{2,X}$ is a norm and the unit ball of $\mathcal{M}$ is $\|\cdot\|_{2,X}$-complete. □

We will give a partial converse of the previous result (also essentially due to Ozawa) in Section 3.6. The most straightforward examples of W^*-bundles are the trivial bundles.

Example 3.7. (cf. [86, Theorem 13 and the preceding paragraph]) Let K be a compact Hausdorff space and let $(\mathcal{N}, \tau_{\mathcal{N}})$ be a tracial von Neumann algebra. Define $C_\sigma(K, \mathcal{N})$ to be the set of $\|\cdot\|$-bounded, $\|\cdot\|_{2,\tau_{\mathcal{N}}}$-continuous functions $K \to \mathcal{N}$. This is a unital C^*-algebra.

[27] In [86], it is assumed K is metrisable, but this is not needed here. The non-metrisable case was also considered in [10] where ultraproducts of W^*-bundles were developed.

Let $\mathrm{Prob}(K)$ be the set of Radon probability measures on K, and for $\mu \in \mathrm{Prob}(K)$, define $\tau_\mu \in T\big(C_\sigma(K,\mathcal{N})\big)$ by

$$\tau_\mu(f) \coloneqq \int_K \tau_{\mathcal{N}}(f(x))\,\mathrm{d}\mu(x), \qquad f \in C_\sigma(K,\mathcal{N}). \tag{3.10}$$

Defining $X \coloneqq \{\tau_\mu : \mu \in \mathrm{Prob}(K)\}$, one finds that

$$\|f\|_{2,X} = \max_{x\in K} \|f(x)\|_{2,\tau_{\mathcal{N}}}, \qquad f \in C_\sigma(K,\mathcal{N}). \tag{3.11}$$

Then $\big(C_\sigma(K,\mathcal{N}), X\big)$ is a tracially complete C^*-algebra called the *trivial W^*-bundle* over K with fibre $(\mathcal{N},\tau_{\mathcal{N}})$.

For any compact space K, we note that $\mathcal{M} \coloneqq C(K)$ together with $X \coloneqq T(\mathcal{M}) \cong \mathrm{Prob}(X)$ is a tracially complete C^*-algebra where $\|\cdot\|$ and $\|\cdot\|_{2,X}$ agree. More generally, if M_n denotes the C^*-algebra of $n \times n$ matrices over $\mathbb{C}$, then $\mathcal{M}_n = C(K, M_n)$, together with $X = T(\mathcal{M}) \cong \mathrm{Prob}(X)$ is a tracially complete C^*-algebra where $\|\cdot\|$ and $\|\cdot\|_{2,X}$ are equivalent norms. These examples show that we cannot expect tracially complete C^*-algebras to behave like von Neumann algebras in general.

An important tool for investigating the structure of a tracially complete C^*-algebra $(\mathcal{M}, X)$ is the GNS representation $\pi_\tau \colon \mathcal{M} \to \mathcal{B}(\mathcal{H}_\tau)$ for $\tau \in X$. Each $\tau \in X$ induces a faithful trace on the von Neumann algebra $\pi_\tau(\mathcal{M})''$. We view these GNS representations as giving rise to 'fibres' on tracially complete C^*-algebras. We are intentially somewhat vague as to the formal definition of the fibre at τ: should it be $\pi_\tau(\mathcal{M})$ or $\pi_\tau(\mathcal{M})''$ and should we use all traces in X or restrict to $\partial_e X$ (as in the W^*-bundle case)? For non-extreme traces, $\pi_\tau(\mathcal{M})$ and $\pi_\tau(\mathcal{M})''$ can differ, even when $\mathcal{M}$ is a W^*-bundle (i.e. X is Bauer). We return to this discussion with Question 3.16.

For many of our latter results in this paper, we will want to restrict to the case that the von Neumann algebras $\pi_\tau(\mathcal{M})''$ are all type II_1.

Definition 3.8. A tracially complete C^*-algebra $(\mathcal{M}, X)$ is said to be *type* II_1 if $\pi_\tau(\mathcal{M})''$ is a type II_1 von Neumann algebra for every $\tau \in X$.

We now turn to proving some basic properties of tracially complete C^*-algebras. The following two results are both standard in the setting of tracial von Neumann algebras.

Proposition 3.9. *Every tracially complete C^*-algebra is unital.*

Proof. Suppose $(\mathcal{M}, X)$ is a tracially complete C^*-algebra and let (e_λ) be an approximate unit for $\mathcal{M}$. Then $(\tau(e_\lambda)) \subseteq \mathbb{R}$ is an increasing net converging to 1 for all $\tau \in X$, and since X is compact, Dini's Theorem implies this convergence is uniform in τ. As (e_λ) is an increasing net of positive contractions,

$$\|e_\lambda - e_\mu\|_{2,X}^2 = \sup_{\tau\in X} \tau\big((e_\lambda - e_\mu)^2\big) \leq \sup_{\tau\in X} \tau(e_\lambda - e_\mu), \tag{3.12}$$

whenever $\lambda \geq \mu$. Hence, (e_λ) is a $\|\cdot\|_{2,X}$-Cauchy net in the unit ball of $\mathcal{M}$. Since $\mathcal{M}$ is tracially complete, (e_λ) $\|\cdot\|_{2,X}$-converges to a positive contraction $e \in \mathcal{M}$ by Proposition 3.2(iii). For all $a \in \mathcal{M}$, working in the unitisation $\mathcal{M}^\dagger$ of $\mathcal{M}$ and extending each $\tau \in X$ to $\mathcal{M}^\dagger$,

$$\|(1_{\mathcal{M}^\dagger} - e)a\|_{2,X}^2 \leq \sup_{\tau\in X} \tau(1_{\mathcal{M}^\dagger} - e)\|a\|^2 = 0, \tag{3.13}$$

and hence e is a unit for $\mathcal{M}$. □

Proposition 3.10. *Every morphism of tracially complete C^*-algebras is unital and contractive with respect to the uniform 2-norms.*

Proof. Suppose $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ is a morphism between tracially complete C^*-algebras. To see ϕ is contractive, fix $\tau \in T(Y)$ and $a \in \mathcal{M}$. Then $\tau \circ \phi \in X$ by the definition of a morphism, so

$$\tau\big(\phi(a)^*\phi(a)\big) = (\tau \circ \phi)(a^*a) \leq \sup \|a\|_{2,X}^2. \tag{3.14}$$

Taking the supremum over τ yields $\|\phi(a)\|_{2,Y} \leq \|a\|_{2,X}$ for all $a \in \mathcal{M}$.

To see ϕ is unital, first note that $\phi(1_{\mathcal{M}}) \leq 1_{\mathcal{N}}$. If $\tau \in Y$, then $\tau \circ \phi \in X$, so $\tau(\phi(1_{\mathcal{M}})) = 1$. Therefore, $\big\|1_{\mathcal{N}} - \phi(1_{\mathcal{M}})\big\|_{2,Y} = 0$, and hence $\phi(1_{\mathcal{M}}) = 1_{\mathcal{N}}$. □

Note that if $(\mathcal{M}, X)$ is a tracially complete C^*-algebra and $\mathcal{N} \subseteq \mathcal{M}$ is a $\|\cdot\|_{2,X}$-closed unital C^*-subalgebra,[28] we may form a tracially complete C^*-algebra $(\mathcal{N}, Y)$ where

$$Y \coloneqq \{\tau|_{\mathcal{N}} : \tau \in X\} \subseteq T(\mathcal{N}). \tag{3.15}$$

We call $(\mathcal{N}, Y)$ a *tracially complete C^*-subalgebra* of $(\mathcal{M}, X)$. This suggests the following notion of an embedding of tracially complete C^*-algebras.

Definition 3.11. Let $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ be tracially complete C^*-algebras. A morphism $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ is called an *embedding* if $\phi^*(Y) = X$.

The next proposition justifies the terminology.

Proposition 3.12. *A morphism $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ is an embedding if and only if $\|\phi(a)\|_{2,Y} = \|a\|_{2,X}$ for all $a \in \mathcal{M}$. Further, if ϕ is an embedding, then ϕ is isometric in the operator norm.*

Proof. If ϕ is an embedding, then for $a \in \mathcal{M}$, we have

$$\|\phi(a)\|_{2,Y} = \sup_{\tau \in Y} \tau(\phi(a^*)\phi(a))^{1/2} = \sup_{\sigma \in X} \sigma(a^*a)^{1/2} = \|a\|_{2,X}. \tag{3.16}$$

To show the converse, suppose $\phi^*(Y) \neq X$ and fix $\tau_0 \in X \setminus \phi^*(Y)$. As $\phi^*(Y)$ is weak*-closed and convex, there exists a self-adjoint $a \in \mathcal{M}$ such that $\tau_0(a) > \sup_{\sigma \in \phi^*(Y)} \sigma(a)$ by the Hahn–Banach theorem. Replacing a by $a + \|a\|1_{\mathcal{M}}$, we may assume $a \in \mathcal{M}_+$. We then have

$$\|a^{1/2}\|_{2,X} \geq \tau_0(a) > \|\phi(a^{1/2})\|_{2,Y}, \tag{3.17}$$

so ϕ is not isometric in the uniform 2-norm.

When ϕ is an embedding, it is injective since $\|\cdot\|_{2,X}$ is a norm, and therefore, it is isometric in the operator norm. □

[28]Note that a C^*-subalgebra $\mathcal{N} \subseteq \mathcal{M}$ is $\|\cdot\|_{2,X}$-closed if and only if the unit ball of $\mathcal{N}$ is $\|\cdot\|_{2,X}$-closed in the unit ball of $\mathcal{M}$. The forward direction is immediate, and the backward direction follows from Lemma 3.27 below.

3.2. **Factoriality.** The following class of tracially complete C^*-algebras will be of greatest interest to us. This class includes the tracial completion of a unital C^*-algebras A with respect to the trace simplex $T(A)$, which is the main example of interest (see Proposition 3.23(iv)).

Definition 3.13. A tracially complete C^*-algebra $(\mathcal{M}, X)$ is said to be *factorial* if X is a closed face of $T(\mathcal{M})$.

When X is a singleton, a tracially complete C^*-algebra is a tracial von Neumann algebra $(\mathcal{M}, \tau)$, and in this case, $\{\tau\}$ is a face of $T(\mathcal{M})$ if and only if $\mathcal{M}$ is a factor.[29] Further, the trivial W^*-bundle $C_\sigma(K, \mathcal{N})$ from Example 3.7 is factorial as a tracially complete C^*-algebra if and only if the fibre $\mathcal{N}$ is a factor. An analogous result holds for non-trivial W^*-bundles (see Proposition 3.6). More generally, we have the following result.

Proposition 3.14. *A tracially complete C^*-algebra $(\mathcal{M}, X)$ is factorial if and only if $\pi_\tau(\mathcal{M})''$ is a factor for every extreme point τ of X.*

Proof. By Theorem 2.5, $(\mathcal{M}, X)$ is factorial if and only if $\partial_e X \subseteq \partial_e T(\mathcal{M})$. Also, by Proposition 2.9, $\partial_e X \subseteq \partial_e T(\mathcal{M})$ if and only if $\pi_\tau(\mathcal{M})''$ is a factor for all $\tau \in \partial_e X$. □

The following result shows the equivalence of the two versions of the question stated as the trace problem (Question 1.1). The proof of this proposition illustrates the utility of factoriality: it allows for powerful separation arguments coming from the classical result that closed faces in Choquet simplices are relatively exposed. The exposedness appears in the proof via an application of Theorem 2.4(ii).

Proposition 3.15. *If $(\mathcal{M}, X)$ is a tracially complete C^*-algebra, then every $\|\cdot\|_{2,X}$-continuous trace on $\mathcal{M}$ belongs to the closed face of $T(\mathcal{M})$ generated by X. In particular, if $(\mathcal{M}, X)$ is factorial, then X is precisely the set of all $\|\cdot\|_{2,X}$-continuous traces on $\mathcal{M}$.*

Proof. Since $\mathcal{M}$ is a unital C^*-algebra, $T(\mathcal{M})$ is a Choquet simplex (Theorem 2.6). By Theorem 2.4, it suffices to show that if $\tau_0 \in T(\mathcal{M})$ is $\|\cdot\|_{2,X}$-continuous and $f\colon T(\mathcal{M}) \to [0,1)$ is a continuous affine function with $f|_X = 0$, then $f(\tau_0) = 0$. Fix such a trace τ_0 and function f. By Proposition 2.7, applied to $f + \frac{1}{n}$, there is a sequence of positive contractions $(a_n)_{n=1}^\infty \subseteq \mathcal{M}$ such that

$$\sup_{\tau \in T(\mathcal{M})} |\tau(a_n) - f(\tau)| \to 0. \tag{3.18}$$

Since $f(\tau) = 0$ for all $\tau \in X$, we have $\|a_n^{1/2}\|_{2,X} \to 0$. As each a_n is contractive, it follows from (3.2) that $\|a_n\|_{2,X} \to 0$. Now, since τ_0 is $\|\cdot\|_{2,X}$-continuous, we have $\tau_0(a_n) \to 0$, and hence (3.18) implies $f(\tau_0) = 0$. □

Returning to our viewpoint that the GNS representations of tracially complete C^*-algebras should be viewed as giving rise to their fibres in some kind of affine bundle structure, the following question is natural. Ozawa's

[29] If $\mathcal{M}$ is a factor, then $T(\mathcal{M}) = \{\tau\}$, and in particular, $\{\tau\}$ is a face of $T(\mathcal{M})$. Conversely, if $\{\tau\}$ is a face of $T(\mathcal{M})$, then $\mathcal{M} \cong \pi_\tau(\mathcal{M})''$ is a factor since τ is extremal (see Proposition 2.9).

[86, Theorem 11] gives a positive answer for all W^*-bundles (factorial or not).

Question 3.16. Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra. Is it the case that $\pi_\tau(\mathcal{M})'' = \pi_\tau(\mathcal{M})$ for all $\tau \in \partial_e X$?

We end the subsection with a some examples. Given a finite von Neumann algebra $\mathcal{M}$, instead of specifying a faithful trace τ (which will exist when $\mathcal{M}$ has a separable predual) and viewing $\mathcal{M}$ as a (generally non-factorial) tracially complete C^*-algebra over a singleton set, one can work with all the traces. By [42, Proposition 3.1.6], $\|\cdot\|_{2,T(\mathcal{M})}$ is a complete norm on the unit ball of $\mathcal{M}$, so that $\big(\mathcal{M}, T(\mathcal{M})\big)$ is a tracially complete C^*-algebra which is evidently factorial. This example will play a technical role in obtaining CPoU from property Γ in Section 6.4.

Proposition 3.17. *Let $\mathcal{M}$ be a finite von Neumann algebra. Then the pair $\big(\mathcal{M}, T(\mathcal{M})\big)$ is a factorial tracially complete C^*-algebra.*

It is worth noting that when $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra, the tracially complete C^*-algebras $\big(\mathcal{M}, \{\tau\}\big)$ and $(\mathcal{M}, T(\mathcal{M}))$ can behave very differently, despite having the same underlying C^*-algebra. See Proposition 4.11, for example.

For later use, we note that matrix algebras over tracially complete C^*-algebras are also tracially complete C^*-algebras in a natural way. Let tr_d denote the unique trace on the C^*-algebra M_d of $d \times d$ matrices over $\mathbb{C}$. For a C^*-algebra A and a set $X \subseteq T(A)$, define

$$X \otimes \{\mathrm{tr}_d\} \coloneqq \{\tau \otimes \mathrm{tr}_d : \tau \in X\} \subseteq T(A \otimes M_d). \tag{3.19}$$

Proposition 3.18. *If $(\mathcal{M}, X)$ is a tracially complete C^*-algebra and $d \in \mathbb{N}$, then $(\mathcal{M} \otimes M_d, X \otimes \{\mathrm{tr}_d\})$ is a tracially complete C^*-algebra. Further, $(\mathcal{M}, X)$ is factorial if and only if $(\mathcal{M} \otimes M_d, X \otimes \{\mathrm{tr}_d\})$ is factorial.*

Proof. Since the map $T(\mathcal{M}) \to T(\mathcal{M} \otimes M_d)$ given by $\tau \mapsto \tau \otimes \mathrm{tr}_d$ is an affine homeomorphism, we have that $X \otimes \{\mathrm{tr}_d\}$ is a compact convex subset of $T(\mathcal{M} \otimes M_d)$ and is a face if and only if X is a face in $T(\mathcal{M})$. Thus it suffices to show that $X \otimes \{\mathrm{tr}_d\}$ is a faithful set of traces on $\mathcal{M} \otimes M_d$ and the unit ball of $\mathcal{M} \otimes M_d$ is $\|\cdot\|_{2, X \otimes \{\mathrm{tr}_d\}}$-complete.

Given $a \in \mathcal{M} \otimes M_d$, write $a = \sum_{i,j=1}^d a_{i,j} \otimes e_{i,j}$ for some $a_i, j \in \mathcal{M}$. For $\tau \in X$,

$$(\tau \otimes \mathrm{tr}_d)(a^*a) = \frac{1}{d}\tau\Big(\sum_{i,j=1}^d a_{i,j}^* a_{j,i}\Big). \tag{3.20}$$

If $a \neq 0$, then there exist i and j with $a_{i,j}^* a_{i,j} \neq 0$. As X is a faithful set of traces, there exists $\tau \in X$ such that $\tau(a_{i,j}^* a_{i,j}) \neq 0$. Therefore,

$$(\tau \otimes \mathrm{tr}_d)(a^*a) \geq \tau(a_{i,j}^* a_{i,j}) > 0, \tag{3.21}$$

as required.

Since $X \times \{\mathrm{tr}_d\}$ is a faithful set of traces on $\mathcal{M} \otimes M_d$, the unit ball of $\mathcal{M} \otimes M_d$ is $\|\cdot\|_{2,X\otimes\{\mathrm{tr}_d\}}$-closed by Proposition 3.2(ii). Furthermore,

$$\frac{1}{d}\max_{1\leq i,j\leq d}\|a_{i,j}\|_{2,X}^2 \leq \|a\|_{2,X\otimes\{\mathrm{tr}_d\}}^2 \leq \frac{1}{d}\sum_{i,j=1}^{d}\|a_{i,j}\|_{2,X}^2 \tag{3.22}$$

as an immediate consequence of (3.20). The $\|\cdot\|_{2,X\otimes\{\mathrm{tr}_d\}}$-completeness of the unit ball of $\mathcal{M} \otimes M_d$ now follows from the $\|\cdot\|_{2,X}$-completeness of the unit ball of $\mathcal{M}$. □

3.3. **Tracial completions.** In this section, we recall Ozawa's tracial completion construction from [86] and its important features. This is both the motivation for our study of tracially complete C^*-algebras and an important tool in the theory that we develop.

Definition 3.19 (cf. [86])**.** Let A be a C^*-algebra. For a compact convex set $X \subseteq T(A)$, the *tracial completion* of A with respect to X is the C^*-algebra[30]

$$\overline{A}^X \coloneqq \frac{\{(a_n)_{n=1}^\infty \in \ell^\infty(A) : (a_n)_{n=1}^\infty \text{ is } \|\cdot\|_{2,X}\text{-Cauchy}\}}{\{(a_n)_{n=1}^\infty \in \ell^\infty(A) : (a_n)_{n=1}^\infty \text{ is } \|\cdot\|_{2,X}\text{-null}\}}. \tag{3.23}$$

Remark 3.20. In Ozawa's work, the notation $\overline{A}^{\mathrm{u}}$ is used for the tracial completion with X being understood from the context. Ozawa also notes that $\overline{A}^{\mathrm{u}}$ can be viewed as the strict closure of the range of a certain representation of A on a Hilbert module. This leads to the alternative terminology *strict closure* for the uniform tracial completion and the alternative notation $\overline{A}^{\mathrm{st}}$ used in [86, 10]. Since it is important for us to keep track of the set X, we include it in the notation for tracial completions in this paper.

We define $\alpha_X \colon A \to \overline{A}^X$ to be the map given by sending $a \in A$ to the image of the constant sequence $(a, a, \dots)$. We also define a second norm on the tracial completion by $(a_n)_{n=1}^\infty \mapsto \lim_{n\to\infty}\|a_n\|_{2,X}$. It is immediate that this limit exists for any $\|\cdot\|_{2,X}$-Cauchy sequence $(a_n)_{n=1}^\infty \in \ell^\infty(A)$ and induces a well-defined norm on the quotient. Abusing notation slightly, we also write $\|\cdot\|_{2,X}$ for this second norm, the justification being that $\|\alpha_X(a)\|_{2,X} = \|a\|_{2,X}$ for all $a \in A$.[31]

Tracial completions with respect to a single trace are nothing but the von Neumann algebra generated by the associated GNS representation.

Proposition 3.21. *Let A be a C^*-algebra and $\tau \in T(A)$. Then, for $X \coloneqq \{\tau\}$, the tracial completion $\overline{A}^X$ can be canonically identified with the von Neumann algebra $\pi_\tau(A)''$, where π_τ is the GNS representation associated to τ; i.e. there is an isomorphism $\theta \colon \overline{A}^X \to \pi_\tau(A)''$ such that $\theta \circ \alpha_X = \pi_\tau$.*

[30]It is not hard to see – using (3.2) – that the $\|\cdot\|$-bounded, $\|\cdot\|_{2,X}$-Cauchy sequences in A form a C^*-subalgebra of $\ell^\infty(A)$ and that the $\|\cdot\|$-bounded, $\|\cdot\|_{2,X}$-null sequences form an ideal of this subalgebra.

[31]When $\|\cdot\|_{2,X}$ is a norm on A, α_X is an embedding and it is natural to identify A with $\alpha_X(A)$. If one desires, one can quotient A by the ideal of $\|\cdot\|_{2,X}$-null elements first before performing the tracial completion (replacing X with the set of induced traces on the quotient). In this way, one can reduce to the case that α_X is an embedding.

Proof. As the unit ball of $\pi_\tau(A)''$ is $\|\cdot\|_{2,\tau}$-complete (see [100, Lemma A.3.3], for example), there is a well-defined *-homomorphism

$$\tilde{\theta}\colon \{(a_n)_{n=1}^\infty \in \ell^\infty(A) : (a_n)_{n=1}^\infty \text{ is } \|\cdot\|_{2,X}\text{-Cauchy}\} \to \pi_\tau(A)'' \tag{3.24}$$

given $\tilde{\theta}((a_n)) = \lim \pi_\tau(a_n)$, where the limit is taken in the norm $\|\cdot\|_{2,\tau}$. The kernel of this map consists of the $\|\cdot\|_{2,X}$-null sequences, so the first isomorphism theorem gives an injective *-homomorphism θ with $\theta\circ\alpha_X = \pi_\tau$. As the unit ball of $\overline{A}^X$ is complete in the 2-norm and θ is isometric with respect to the 2-norms, the image of the unit ball of θ is $\|\cdot\|_{2,\tau}$-closed and contains the unit ball of $\pi_\tau(A)$. By Kaplansky's density theorem, the unit ball of $\pi_\tau(A)$ is $\|\cdot\|_{2,\tau}$-dense in the unit ball of $\pi_\tau(A)''$, and hence θ is surjective, proving that θ is an isomorphism. □

Very similarly, tracial completions with respect to finite dimensional simplices also come from GNS representations.

Example 3.22. When X is a finite dimensional simplex, the tracial completion is $\pi_\tau(A)''$, where τ is the average of the traces in $\partial_e X$, and so this tracial completion is again a von Neumann algebra.

For each trace $\tau \in X$, we define an induced trace $\widetilde{\tau}$ on $\overline{A}^X$ by setting $\widetilde{\tau}(a) \coloneqq \lim_{n\to\infty} \tau(a_n)$ for any representative $\|\cdot\|_{2,X}$-Cauchy sequence $(a_n)_{n=1}^\infty \in \ell^\infty(A)$. The inequality $|\tau(a)| \leq \|a\|_{2,X}$, which holds for all $a \in A$, ensures that this limit exists and that $\widetilde{\tau}$ is well-defined on the tracial completion. The induced trace satisfies $\widetilde{\tau}(\alpha_X(a)) = \tau(a)$ for all $a \in A$ and is the unique $\|\cdot\|_{2,X}$-continuous trace with this property since $\alpha_X(A)$ is $\|\cdot\|_{2,X}$-dense in the tracial completion.[32]

The following proposition establishes the basic properties of tracial completions.

Proposition 3.23. *Let A be a C^*-algebra and let X be a compact convex subset of $T(A)$. Let $\widetilde{X} \subseteq T(\overline{A}^X)$ denote the set of all traces on $\overline{A}^X$ that are induced by traces in X.*

(i) *With the notation in the paragraph preceding the proposition, the seminorm $\|\cdot\|_{2,\widetilde{X}}$ coincides with the norm $\|\cdot\|_{2,X}$ on $\overline{A}^X$.*
(ii) *The map $X \to \widetilde{X}$ sending a trace to its induced trace is an affine homeomorphism, where X and $\widetilde{X}$ are equipped with their respective weak* topologies.*
(iii) *The pair $(\overline{A}^X, \widetilde{X})$ is a tracially complete C^*-algebra.*
(iv) *The completion $(\overline{A}^X, \widetilde{X})$ is factorial if and only if X is a face in $T(A)$.*
(v) *Let $Y \subset X$ be a compact convex set and let $\tilde{Y}$ denote the image of Y in $\tilde{X}$. Then $(\overline{A}^Y, Y)$ and $(\overline{\overline{A}^X}^{\tilde{Y}}, \tilde{Y})$ are isomorphic via an*

[32] By construction, $\widetilde{\tau}$ is $\|\cdot\|_{2,X}$-continuous on $\|\cdot\|$-bounded subsets of $\overline{A}^X$; continuity on all of $\overline{A}^X$ follows from Proposition 3.23(i) (or from (3.27) in the proof).

isomorphism θ such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha_Y} & \overline{A}^Y \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \theta} \\ \overline{A}^X & \xrightarrow[\alpha_{\widetilde{Y}}]{} & \overline{\overline{A}^X}^{\widetilde{Y}} \end{array} \tag{3.25}$$

commutes.

(vi) *Given $\tau \in X$ with extension $\widetilde{\tau}$ to $\overline{A}^X$, write $\pi_\tau \colon A \to \pi_\tau(A)''$ and $\pi_{\widetilde{\tau}} \colon \overline{A}^X \to \pi_{\widetilde{\tau}}(\overline{A}^X)''$ for the respective GNS representations. Then $\alpha_X \colon A \to \overline{A}^X$ induces an isomorphism $\theta \colon \pi_\tau(A)'' \to \pi_{\widetilde{\tau}}(\overline{A}^X)''$ such that*

$$\begin{array}{ccc} A & \xrightarrow{\pi_\tau} & \pi_\tau(A)'' \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \theta} \\ \overline{A}^X & \xrightarrow[\pi_{\widetilde{\tau}}]{} & \pi_{\widetilde{\tau}}(\overline{A}^X)'' \end{array} \tag{3.26}$$

commutes.

Proof. (i): Let $a \in \overline{A}^X$ be represented by a $\|\cdot\|_{2,X}$-Cauchy sequence $(a_n)_{n=1}^\infty \in \ell^\infty(A)$. Let $\tau \in X$. Then

$$\|a\|_{2,\widetilde{\tau}}^2 = \lim_{n\to\infty} \tau(a_n^* a_n) \leq \lim_{n\to\infty} \|a_n\|_{2,X}^2 = \|a\|_{2,X}^2. \tag{3.27}$$

Hence $\|a\|_{2,\widetilde{X}} \leq \|a\|_{2,X}$ for all $a \in \overline{A}^X$, and this implies $\|\cdot\|_{2,\widetilde{X}}$ is $\|\cdot\|_{2,X}$-continuous. Since $\|\alpha_X(a)\|_{2,X} = \|a\|_{2,X} = \|\alpha_X(a)\|_{2,\widetilde{X}}$ for all $a \in A$ and $\alpha_X(A)$ is $\|\cdot\|_{2,X}$-dense in $\overline{A}^X$, we deduce $\|a\|_{2,X} = \|a\|_{2,\widetilde{X}}$ for all $a \in \overline{A}^X$.

(ii): It is clear that the map $\tau \mapsto \widetilde{\tau}$ is affine and bijective. As X is compact and $\widetilde{X}$ is Hausdorff, it suffices to show continuity of this map. Let $(\tau_\lambda)_\lambda$ be a net in X that converges to $\tau \in X$ and let $\widetilde{\tau_\lambda}, \widetilde{\tau} \in \widetilde{X}$ denote the $\|\cdot\|_{2,X}$-continuous traces induced by τ_λ and τ, respectively. We must show that $\widetilde{\tau_\lambda} \to \widetilde{\tau}$.

Let $a \in \overline{A}^X$ and let $\epsilon > 0$. Pick $b \in A$ such that $\|a - \alpha_X(b)\|_{2,X} < \epsilon/3$. Pick λ_0 such that for $\lambda \geq \lambda_0$, we have $\tau_\lambda(b) \approx_{\epsilon/3} \tau(b)$. Then for $\lambda \geq \lambda_0$,

$$\widetilde{\tau}(a) \approx_{\epsilon/3} \widetilde{\tau}(\alpha_X(b)) = \tau(b) \approx_{\epsilon/3} \tau_\lambda(b) = \widetilde{\tau_\lambda}(\alpha_X(b)) \approx_{\epsilon/3} \widetilde{\tau_\lambda}(a), \tag{3.28}$$

showing continuity of the extension map.

(iii): It follows from (ii) that $\widetilde{X}$ is a compact convex subset of $T(\overline{A}^X)$. By the construction of $\overline{A}^X$, the seminorm $\|\cdot\|_{2,X}$ is a norm and the unit ball of $\overline{A}^X$ is $\|\cdot\|_{2,X}$-complete, so by (i), the unit ball of $\overline{A}^X$ is also $\|\cdot\|_{2,\widetilde{X}}$-complete.

(iv): Suppose that X is a face in $T(A)$. To show that $\widetilde{X}$ is a face of $T(\overline{A}^X)$, suppose we can write $\tau \in \widetilde{X}$ as $\tau = \frac{1}{2}(\tau_1 + \tau_2)$ for some $\tau_1, \tau_2 \in T(\overline{A}^X)$. Since $(\alpha_X(e_\lambda))$ converges to the unit of $\overline{A}^X$ for any approximate unit (e_λ) of A, it follows that $\tau \circ \alpha_X$ is a trace on A. As X is a face of $T(A)$, $\tau_1 \circ \alpha_X, \tau_2 \circ \alpha_X \in X$. Also, since $\tau_1, \tau_2 \leq 2\tau$ and τ is $\|\cdot\|_{2,\widetilde{X}}$-continuous, it follows that τ_1 and

τ_2 are also $\|\cdot\|_{2,X}$-continuous. Thus τ_1 and τ_2 are the $\|\cdot\|_{2,\widetilde{X}}$-continuous extensions of $\tau_1 \circ \alpha_X, \tau_2 \circ \alpha_X$ respectively, so $\tau_1, \tau_2 \in \widetilde{X}$.

Conversely, suppose $(\overline{A}^X, \widetilde{X})$ is factorial and $\tau_1, \tau_2 \in T(A)$ satisfy $\tau := \frac{1}{2}(\tau_1 + \tau_2) \in \widetilde{X}$. Then τ is $\|\cdot\|_{2,X}$-continuous, and hence so are τ_1 and τ_2 as they are dominated by 2τ. Now, τ, τ_1, and τ_2 extend to $\|\cdot\|_{2,X}$-continuous traces $\widetilde{\tau}$, $\widetilde{\tau}_1$, and $\widetilde{\tau}_2$ on $\overline{A}^X$, respectively, using that the unit ball of $\alpha_X(A)$ is $\|\cdot\|_{2,\widetilde{X}}$-dense in the unit ball of $\overline{A}^X$. Then $\widetilde{\tau} \in \widetilde{X}$ and $\widetilde{\tau} = \frac{1}{2}(\widetilde{\tau}_1 + \widetilde{\tau}_2)$. Since $(\overline{A}^X, \widetilde{X})$ is factorial, we have $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ are in $\widetilde{X}$, so τ_1 and τ_2 are in X.

(v): The map $\alpha_X \colon A \to \overline{A}^X$ is $\|\cdot\|_{2,Y}$-$\|\cdot\|_{2,\tilde{Y}}$-isometric. Therefore, it sends $\|\cdot\|_{2,Y}$-Cauchy and $\|\cdot\|_{2,Y}$-null sequences in A to $\|\cdot\|_{2,\tilde{Y}}$-Cauchy and $\|\cdot\|_{2,\tilde{Y}}$-null sequences in $\overline{A}^Y$ (respectively). Accordingly, it induces θ making (3.25) commute. By commutativity of this diagram and the inequality $\|\cdot\|_{2,\tilde{Y}} \leq \|\cdot\|_{2,X}$ on $\overline{A}^X$, the image of the unit ball of $\overline{A}^Y$ under θ is dense in the unit ball of $\overline{\overline{A}^X}^{\tilde{Y}}$. Since θ is isometric with respect to the uniform 2-norms and the unit ball of $\overline{A}^Y$ is $\|\cdot\|_{2,Y}$-complete, θ is surjective.

(vi): This follows from (v) using Proposition 3.21 to identify uniform tracial completions at τ and $\tilde{\tau}$ with the von Neumann algebras generated by the associated GNS representations.

□

Notation 3.24. With Proposition 3.23 now established, we will typically identify X with $\widetilde{X}$ henceforth. Furthermore, for a trace $\tau \in X$, we will write τ in place of $\widetilde{\tau}$ for the induced trace on $\overline{A}^X$.

Not surprisingly, tracial completions satisfy a universal property allowing for trace-preserving u.c.p. maps to be extended by continuity to tracial completions.

Proposition 3.25. *Let A be a C^*-algebra and let X be a compact convex subset of $T(A)$. If $(\mathcal{N}, Y)$ is a tracially complete C^*-algebra and $\theta \colon A \to \mathcal{N}$ is a u.c.p. map such that $\tau \circ \theta \in X$ for all $\tau \in Y$, then there is a unique u.c.p. map $\overline{\theta} \colon \overline{A}^X \to \mathcal{N}$ of tracially complete C^*-algebras such that $\overline{\theta} \circ \alpha_X = \theta$ and $\tau \circ \overline{\theta} \in X$ for all $\tau \in Y$. Further, if θ is a *-homomorphism, then so is $\overline{\theta}$.*

Proof. For uniqueness, consider a u.c.p. map $\phi \colon \overline{A}^X \to \mathcal{N}$ with $\tau \circ \phi \in X$ for all $\tau \in Y$. Then for $\tau \in Y$ and $a \in \overline{A}^X$,

$$\tau(\phi(a)^*\phi(a)) \leq \tau(\phi(a^*a)) \leq \|a\|_{2,X}^2, \tag{3.29}$$

where the first inequality follows from the Schwarz inequality for u.c.p. maps (cf. [88, Proposition 3.3]) and the second inequality is a consequence of the hypothesis on traces. Accordingly, ϕ is $\|\cdot\|_{2,X}$-$\|\cdot\|_{2,Y}$-continuous. By the $\|\cdot\|_{2,X}$-density of $\alpha_A(A)$ in $\overline{A}^X$, it follows that any extension of θ as in the proposition must be unique.

We turn to the proof of existence. Since $\theta^*(Y) \subseteq X$, the same computation used above shows $\|\theta(a)\|_{2,Y} \leq \|a\|_{2,X}$ for all $a \in A$. Hence if $(a_n)_{n=1}^\infty \subseteq A$ is a $\|\cdot\|$-bounded, $\|\cdot\|_{2,X}$-Cauchy sequence in A, then $(\theta(a_n))_{n=1}^\infty$ is a $\|\cdot\|$-bounded, $\|\cdot\|_{2,Y}$-Cauchy sequence in $\mathcal{N}$. Therefore, we obtain a

well-defined map $\overline{\theta}\colon \overline{A}^X \to \mathcal{N}$ by $\overline{\theta}(a) \coloneqq \lim_{n\to\infty} \theta(a_n)$, where $(a_n)_{n=1}^\infty \subseteq A$ is a $\|\cdot\|$-bounded, $\|\cdot\|_{2,X}$-Cauchy sequence representing $a \in \overline{A}^X$ and where the limit is taken in the norm $\|\cdot\|_{2,Y}$. By construction, $\overline{\theta} \circ \alpha_X = \theta$. Using Propositions 3.2(iii) and 3.18, a density argument gives that $\overline{\theta}$ is a u.c.p. map. Likewise when θ is a *-homomorphism, a density argument, this time using continuity of multiplication in the uniform 2-norm on norm-bounded sets, shows that so too is $\overline{\theta}$.

It remains to check that $\tau \circ \overline{\theta} \in X$ for all $\tau \in Y$. Note that $\|\overline{\theta}(a)\|_{2,Y} \leq \|a\|_{2,X}$ for all $a \in \overline{A}^X$. Therefore, if $\tau \in Y$, then $\tau \circ \overline{\theta}$ is $\|\cdot\|_{2,X}$-continuous. Now since $\tau \circ \overline{\theta} \circ \alpha_X = \tau \circ \theta \in X$, we have $\tau \circ \overline{\theta} \in X$. □

3.4. **Dense subalgebras of tracially complete C^*-algebras.** We will frequently need a version of Kaplansky's density theorem for tracially complete C^*-algebras (Lemma 3.27 and Proposition 3.28 below). This is obtained using the well-known matrix amplification trick to reduce to the self-adjoint case. At the core is the following estimate.

Lemma 3.26. *Let A be a commutative C^*-algebra and $\tau \in T(A)$. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with constant $M > 0$. If $a, b \in A$ are self-adjoint, then $\|f(a) - f(b)\|_{2,\tau} \leq M\|a - b\|_{2,\tau}$.*

Proof. View $A \cong C_0(K)$ and $a, b \in C_0(K)$ for a locally compact Hausdorff space K. Since f is M-Lipschitz continuous, we have

$$|f(a(t)) - f(b(t))| \leq M|a(t) - b(t)| \tag{3.30}$$

for all $t \in K$. It follows that

$$|f(a) - f(b)|^2 \leq M^2|a - b|^2 \tag{3.31}$$

in A, and the claim follows by applying τ. □

We isolate the following quantitative version of Kaplansky's density theorem for later use.

Lemma 3.27. *Let $(\mathcal{M}, X)$ be a tracially complete C^*-algebra and let $A \subseteq \mathcal{M}$ be a C^*-subalgebra. If $b \in \mathcal{M}$ is a contraction, $\epsilon > 0$, and there is an $a \in A$ with $\|a - b\|_{2,X} < \epsilon$, then there is a contraction $a' \in A$ with $\|a' - b\|_{2,X} < 3\epsilon$.*

Proof. We first prove the result in the case where b is self-adjoint. By replacing a with its real part, we may assume a is self-adjoint as well. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(t) \coloneqq \begin{cases} -1 & t < -1; \\ t, & t \in [-1, 1]; \\ 1, & t > 1. \end{cases} \tag{3.32}$$

We will show $\|f(a) - b\|_{2,X} < 3\epsilon$, so that the result (in this case) holds by setting $a' \coloneqq f(a)$.

Fix $\tau \in X$ and let $E\colon \pi_\tau(\mathcal{M})'' \to \{\pi_\tau(a)\}''$ be the τ-preserving conditional expectation, which is contractive with respect to each of the norms $\|\cdot\|$ and $\|\cdot\|_{2,\tau}$.[33] Set $a_1 \coloneqq \pi_\tau(a)$, and note that $E(a_1) = a_1$. Therefore, for

[33]The expectation is given explicitly by $E(\pi_\tau(c))\xi_\tau = P\pi_\tau(c)\xi_\tau$ for $c \in \mathcal{M}$, where $\xi_\tau \in \mathcal{H}_\tau$ is the canonical cyclic vector and P is the projection of $\mathcal{H}_\tau$ onto the closure of $\pi_\tau(a)\mathcal{H}_\tau$ – see [15, Lemma 1.5.11], for example, for more details.

$b_1 \coloneqq E(\pi_\tau(b))$,

$$\|a_1 - b_1\|_{2,\tau} \leq \|\pi_\tau(a) - \pi_\tau(b)\|_{2,\tau} < \epsilon, \tag{3.33}$$

and Lemma 3.26 applied to $\{\pi_\tau(a)\}''$ implies

$$\|f(a_1) - f(b_1)\|_{2,\tau} \leq \|a_1 - b_1\|_{2,\tau} < \epsilon. \tag{3.34}$$

Since b_1 is a self-adjoint contraction, $f(b_1) = b_1$.

Hence, in the norm $\|\cdot\|_{2,\tau}$, we have strict approximations

$$\pi_\tau(f(a)) = f(a_1) \approx_\epsilon f(b_1) = b_1 \approx_\epsilon a_1 = \pi_\tau(a) \approx_\epsilon \pi_\tau(b). \tag{3.35}$$

Therefore, $\|\pi_\tau(f(a)) - \pi_\tau(b)\|_{2,\tau} < 3\epsilon$. This completes the proof for the case when b is self-adjoint.

In the general case, view $\mathcal{M} \otimes M_2$ as a tracially complete C^*-algebra as in Proposition 3.18. By the first part of the proof, there is a self-adjoint contraction $a'' = (a''_{ij}) \in A \otimes M_2$ such that

$$\left\| a'' - \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} \right\|_{2,X\otimes\{\mathrm{tr}_2\}} < 3\epsilon. \tag{3.36}$$

Since a'' is self-adjoint, $a''_{21} = a''^*_{12}$. For any $\tau \in X$, we compute that

$$\begin{aligned} \|a''_{12} - b\|^2_{2,\tau} &\leq \frac{1}{2}\tau\big(|a''_{11}|^2 + 2|a''_{12} - b|^2 + |a''_{22}|^2\big) \\ &= \left\| a'' - \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} \right\|^2_{2,\tau\otimes\{\mathrm{tr}_2\}} \\ &< (3\epsilon)^2. \end{aligned} \tag{3.37}$$

Hence, taking $a' \coloneqq a''_{12}$, we have $\|a' - b\|_{2,X} < 3\epsilon$. □

The following version of Kaplansky's density theorem follows immediately from the previous lemma.

Proposition 3.28 (cf. [80, Theorem 4.3.3])**.** *Let $(\mathcal{M}, X)$ be a tracially complete C^*-algebra and let $A \subseteq \mathcal{M}$ be a $\|\cdot\|_{2,X}$-dense C^*-subalgebra. Then the unit ball of A is $\|\cdot\|_{2,X}$-dense in the unit ball of $\mathcal{M}$.*

As an application, a tracially complete C^*-algebra is the tracial completion of any of its $\|\cdot\|_{2,X}$-dense subalgebras.

Corollary 3.29. *Suppose $(\mathcal{M}, X)$ is a tracially complete C^*-algebra and let $A \subseteq \mathcal{M}$ be a $\|\cdot\|_{2,X}$-dense C^*-subalgebra.*

(i) *$X_A \coloneqq \{\tau|_A : \tau \in X\} \subseteq T(A)$ is a compact convex set and the inclusion $A \hookrightarrow \mathcal{M}$ induces an isomorphism $(\overline{A}^{X_A}, X_A) \to (\mathcal{M}, X)$.*

(ii) *If $(\mathcal{M}, X)$ is factorial, then X_A is a closed face in $T(A)$.*

(iii) *If $(\mathcal{N}, Y)$ is another tracially complete C^*-algebra and $\phi\colon A \to \mathcal{N}$ is a *-homomorphism with $\phi^*(Y) \subseteq X_A$, then ϕ has a unique extension to $\tilde{\phi}\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$.*

Proof. (i): Proposition 3.28 implies the unit ball of A is $\|\cdot\|_{2,X}$-dense in the unit ball of $\mathcal{M}$. Therefore, if $\tau \in X$, then $\tau|_A$ has norm 1, and hence $X_A \subseteq T(A)$. As X is compact and convex so too is X_A. By Proposition 3.25, the inclusion $A \hookrightarrow \mathcal{M}$ extends to a morphism $\theta\colon (\overline{A}^{X_A}, X_A) \to (\mathcal{M}, X)$ of tracially complete C^*-algebras.

Note that θ is an embedding in the sense of Definition 3.11, and hence is isometric by Proposition 3.12. Further, the range of θ contains the $\|\cdot\|_{2,X}$-dense subalgebra $A \subseteq \mathcal{M}$. Since the unit ball of $\overline{A}^{X_A}$ is $\|\cdot\|_{2,X_A}$-complete, Kaplansky's density theorem (Proposition 3.28) implies θ maps the unit ball of $\overline{A}^{X_A}$ onto the unit ball of $\mathcal{M}$, and in particular θ is surjective. Finally, for $\tau \in X$, $\tau \circ \theta^{-1}$ is a $\|\cdot\|_{2,X}$-continuous trace on $\overline{A}^{X_A}$ extending $\tau|_A$, so $\tau \in X_A$. This shows θ^{-1} is also a morphism of tracially complete C^*-algebras.

(ii): In light of (i), this follows from Proposition 3.23(iv).

(iii): This follows from (i) and Proposition 3.25. □

3.5. **Constructions.** There is a recipe for producing constructions on tracially complete C^*-algebras.

(i) Perform the corresponding (spatial) C^*-construction on the underlying C^*-algebras.
(ii) Identify the suitable collection of traces on the newly constructed C^*-algebra
(iii) Take the tracial completion with respect to these traces.

We illustrate this process here with direct sums, tensor products, and sequential inductive limits. Products and reduced products (including ultraproducts) are constructed in Section 5.1. In the case of direct sums, step (iii) above is redundant as the C^*-direct sum is already tracially complete.

Definition 3.30. Let $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ be tracially complete C^*-algebras. Let $X \oplus Y$ denote the set of traces of the form $\lambda\tau_X + (1-\lambda)\tau_Y$ for $\tau_X \in X$, $\tau_Y \in Y$ and $0 \leq \lambda \leq 1$. The direct sum of $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ is the tracially complete C^*-algebra $(\mathcal{M} \oplus \mathcal{N}, X \oplus Y)$.

Note that the tracially complete direct sum of two finite von Neumann algebras $(\mathcal{M}, T(\mathcal{M})$ and $(\mathcal{N}, T(\mathcal{N}))$ is the finite von Neumann algebra $(\mathcal{M} \oplus \mathcal{N}, T(\mathcal{M} \oplus N))$. As the extreme traces on $X \oplus Y$ are precisely the union of the extreme traces on X and the extreme traces on Y, the direct sum $(\mathcal{M}, X) \oplus (\mathcal{N}, Y)$ is factorial if and only if both $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ are factorial.

Next up are tensor products.

Definition 3.31. Let $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ be tracially complete C^*-algebras. Let $\mathcal{M} \otimes \mathcal{N}$ denote the minimal C^*-tensor product of $\mathcal{M}$ and $\mathcal{N}$ and let $X \otimes Y \subseteq T(\mathcal{M} \otimes \mathcal{N})$ be the closed convex hull of the traces $\sigma \otimes \rho$ for $\sigma \in X$ and $\rho \in Y$. Define $\mathcal{M} \bar{\otimes} \mathcal{N}$ to be the tracial completion of $\mathcal{M} \otimes \mathcal{N}$ with respect to $X \otimes Y$. The tensor product $(\mathcal{M}, X) \bar{\otimes} (\mathcal{N}, Y)$ is the pair $(\mathcal{M} \bar{\otimes} \mathcal{N}, X \otimes Y)$.

Proposition 3.32. *If $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ are factorial tracially complete C^*-algebras, then $(\mathcal{M}, X) \bar{\otimes} (\mathcal{N}, Y)$ is also factorial.*

Proof. Let $F \subseteq T(\mathcal{M} \otimes \mathcal{N})$ be the closed face generated by $X \otimes Y$. By Proposition 3.23(iv), it is enough to show that $F = X \otimes Y$. In fact, by the Krein–Milman theorem, it is enough to show $\partial_e F \subseteq X \otimes Y$. To this end, let $\tau_0 \in \partial_e F$ be given. Since F is a closed face in $T(\mathcal{M} \otimes \mathcal{N})$, we have $\tau_0 \in \partial_e T(\mathcal{M} \otimes \mathcal{N})$. By [10, Proposition 3.5], there are traces $\sigma_0 \in T(\mathcal{M})$ and $\rho_0 \in T(\mathcal{N})$ such that $\tau_0 = \sigma_0 \otimes \rho_0$. Now, it is enough to show $\sigma_0 \in X$ and $\rho_0 \in Y$ – we will only show the former as the latter follows by symmetry.

By Theorem 2.4(ii), to show $\sigma_0 \in X$, it is enough to show $f(\sigma_0) = 0$ for every continuous affine function $f\colon T(\mathcal{M}) \to [0,1]$ with $f|_X = 0$. Let f be such a function and apply Proposition 2.7 to produce a self-adjoint $a \in \mathcal{M}$ with $\sigma(a) = f(\sigma)$ for all $\sigma \in T(\mathcal{M})$. Define

$$G \coloneqq \{\tau \in T(\mathcal{M} \otimes \mathcal{N}) : \tau(a \otimes 1_{\mathcal{N}}) = 0\}. \tag{3.38}$$

For any $\sigma \in T(\mathcal{M})$ and $\rho \in T(\mathcal{N})$, we have $(\sigma \otimes \rho)(a \otimes 1_{\mathcal{N}}) = f(\sigma) \geq 0$. By [10, Proposition 3.5] and the Krein–Milman theorem,

$$T(\mathcal{M} \otimes \mathcal{N}) = \overline{\mathrm{co}}\left\{\sigma \otimes \rho : \sigma \in T(\mathcal{M}),\ \rho \in T(\mathcal{N})\right\}. \tag{3.39}$$

Hence $\tau(a \otimes 1_{\mathcal{N}}) \geq 0$ for all $\tau \in T(\mathcal{M} \otimes \mathcal{N})$. It follows that G is a closed face in $T(\mathcal{M} \otimes \mathcal{N})$. For all $\sigma \in X$ and $\rho \in Y$,

$$(\sigma \otimes \rho)(a \otimes 1_{\mathcal{N}}) = \sigma(a) = f(\sigma) = 0, \tag{3.40}$$

and hence $X \otimes Y \subseteq G$. Then, as G is a closed face containing $X \otimes Y$, we have $F \subseteq G$, and so $\tau_0 \in G$. Since $\tau_0 = \sigma_0 \otimes \rho_0$, this shows that $f(\sigma_0) = \sigma_0(a) = 0$. □

Finally, we construct sequential inductive limits in the category of tracially complete C^*-algebras.

Definition 3.33. Let

$$\cdots \to (\mathcal{M}_n, X_n) \xrightarrow{\phi_n^{n+1}} (\mathcal{M}_{n+1}, X_{n+1}) \xrightarrow{\phi_{n+1}^{n+2}} (\mathcal{M}_{n+2}, X_{n+2}) \to \cdots \tag{3.41}$$

be a sequential inductive system of tracially complete C^*-algebras. Form the C^*-inductive limit $A \coloneqq \varinjlim(\mathcal{M}_n, \phi_n^{n+1})$ and let $\hat{\phi}_n^{\infty}\colon \mathcal{M}_n \to A$ be the canonical unital *-homomorphism. The inductive system (3.41) induces a projective system

$$\cdots \longleftarrow X_n \xleftarrow{(\phi_n^{n+1})^*} X_{n+1} \xleftarrow{(\phi_{n+1}^{n+2})^*} X_{n+2} \longleftarrow \cdots \tag{3.42}$$

of compact convex sets. Set

$$X \coloneqq \{\tau \in T(A) : \tau \circ \hat{\phi}_n^{\infty} \in X_n \text{ for all } n \geq 1\} \subseteq T(A) \tag{3.43}$$

and note that X is a compact convex subset of $T(A)$. Further, the maps $(\hat{\phi}_n^{\infty})^*\colon X \to X_n$ induce an affine homeomorphism

$$X \xrightarrow{\cong} \varprojlim\left(X_n, (\phi_n^{n+1})^*\right) \tag{3.44}$$

(and in particular, X is non-empty whenever all of the X_n are non-empty; see [35, Theorem VIII.3.5], for example). Define $\mathcal{M} \coloneqq \overline{A}^X$ and

$$\varinjlim\left((\mathcal{M}_n, X_n), \phi_n^{n+1}\right) \coloneqq (\mathcal{M}, X) \tag{3.45}$$

Finally, write $\phi_n^{\infty}\colon \mathcal{M}_n \to \mathcal{M}$ for the unital *-homomorphism obtained by composing $\hat{\phi}_n^{\infty}$ with the canonical map $\alpha_X\colon A \to \mathcal{M}$.

We now show the definition of inductive limit in Definition 3.33 satisfies the required universal property and hence is an inductive limit in the category of tracially complete C^*-algebras.

Proposition 3.34. *Let $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ be a sequence of tracially complete C^*-algebras and $\big(\phi_n^{n+1}\colon (\mathcal{M}_n, X_n) \to (\mathcal{M}_{n+1}, X_{n+1})\big)_{n=1}^{\infty}$ be a sequence of morphisms. Set*

$$(\mathcal{M}, X) \coloneqq \varinjlim \big((\mathcal{M}_n, X_n), \phi_n^{n+1}\big) \tag{3.46}$$

as in Definition 3.33 and write $\phi_n^{\infty}\colon \mathcal{M}_n \to \mathcal{M}$ for the canonical *-homomorphisms as above. Then $(\mathcal{M}, X)$ is a tracially complete C^*-algebra, ϕ_n^{∞} is a morphism for $n \geq 1$, and together, they define the inductive limit of $\big((\mathcal{M}_n, X_n), \phi_n^{n+1}\big)_{n=1}^{\infty}$ in the category of tracially complete C^*-algebras. If each X_n is metrisable, then so is X. Further, if each $(\mathcal{M}_n, X_n)$ is factorial, then so is $(\mathcal{M}, X)$.

Proof. Define A, X, and $\hat{\phi}_n^{\infty}$ as in Definition 3.33, so that $(\mathcal{M}, X) = (\overline{A}^X, X)$. By Proposition 3.23(iii), $(\mathcal{M}, X)$ is a tracially complete C^*-algebra. By construction, $(\hat{\phi}_n^{\infty})^*(X) \subseteq X_n$, so $(\phi_n^{\infty})^*(X) \subseteq X_n$ and each ϕ_n^{∞} is a morphism. As sequential projective limits of metrisable spaces are metrisable, X is metrisable if each X_n is metrisable. If each $(\mathcal{M}_n, X_n)$ is factorial, then X_n is a face in $T(\mathcal{M}_n)$ for all $n \geq 1$. It follows from (3.43) that X is a face in $T(A)$. Proposition 3.23(iv) then implies that $(\mathcal{M}, X)$ is factorial.

To show that $(\mathcal{M}, X)$ is the inductive limit of $\big((\mathcal{M}_n, X_n), \phi_n^{n+1}\big)$, suppose we have a tracially complete C^*-algebra $(\mathcal{N}, Y)$ and morphisms

$$\psi_n\colon (\mathcal{M}_n, X_n) \to (\mathcal{N}, Y), \qquad n \in \mathbb{N}, \tag{3.47}$$

such that $\psi_{n+1} \circ \phi_n^{n+1} = \psi_n$. Then, working in the category of unital C^*-algebras, we obtain a unique unital *-homomorphism $\hat{\psi}\colon A \to \mathcal{N}$ such that $\psi_n = \hat{\psi} \circ \hat{\phi}_n^{\infty}$ for each $n \in \mathbb{N}$. For $\tau \in Y$, we have

$$\tau \circ \hat{\psi} \circ \hat{\phi}_n^{\infty} = \tau \circ \psi_n \in X_n \tag{3.48}$$

for all $n \geq 1$, so by (3.43), $\tau \circ \hat{\psi} \in X$. This shows $\hat{\psi}^*(Y) \subseteq X$. Accordingly, there is a unique morphism $\psi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ with $\hat{\psi} = \psi \circ \alpha_X$ by Proposition 3.25. Moreover

$$\psi_n = \hat{\psi} \circ \hat{\phi}_n^{\infty} = \psi \circ \alpha_X \circ \hat{\phi}_n^{\infty} = \psi \circ \phi_n^{\infty}, \tag{3.49}$$

as required. Uniqueness of ψ follows from uniqueness in Proposition 3.25 as any morphism $\psi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ with $\psi \circ \phi_n = \psi_n$ for all n satisfies $\psi \circ \alpha_X = \hat{\psi}$. □

For each metrisable Choquet simplex X, we now use an inductive limit construction to produce the concrete model $(\mathcal{R}_X, X)$ of a tracially complete C^*-algebra covered by Theorem C as discussed in the overview of results. This is achieved by mimicking construction of a simple AF algebra A for which $T(A) = X$, found in [6, 54], and can be deduced from these results by considering the tracial completion of such an AF algebras. The details are slightly easier in the tracially complete setting since $K_0(\mathcal{R}) \cong \mathbb{R}$ and one need not worry about simplicity.

Example 3.35. Let X be a metrisable Choquet simplex. By [55, Theorem 11.6], we can write X as the inverse limit of a system of finite dimensional simplices

$$X_1 \xleftarrow{\alpha_2^1} X_2 \xleftarrow{\alpha_3^2} \cdots, \tag{3.50}$$

where the connecting maps are continuous and affine. We will construct an inductive system of tracially complete C^*-algebras realising this data.

Set $\mathcal{M}_n \coloneqq \mathcal{R}^{\oplus \partial_e X_n}$, where $\mathcal{R}$ is the hyperfinite II_1 factor. Note that $T(\mathcal{M}_n)$ can be canonically identified with X_n and $\big(\mathcal{M}_n, T(\mathcal{M}_n)\big)$ is tracially complete and factorial. Next, we can choose a *-homomorphism $\phi_n^{n+1}\colon \mathcal{M}_n \to \mathcal{M}_{n+1}$ that induces the map α_{n+1}^n. To do this explicitly, write $\partial_e X_{n+1} = \{x_1, \dots, x_k\}$ and $\partial_e X_n = \{y_1, \dots, y_l\}$. For each $i = 1, \dots, k$, write $\alpha_{n+1}^n(x_i)$ as the convex combination $\sum_{j=1}^l \lambda_{i,j} y_j$. Fixing i, find a partition of unity $p_{i,1}, \dots, p_{i,l} \in \mathcal{R}$ of projections such that $\tau(p_{i,j}) = \lambda_{i,j}$ for each j. For each i and j, choose any unital *-homomorphism $\psi_{i,j}\colon \mathcal{R} \to p_{i,j}\mathcal{R}p_{i,j}$ (which exists since $\mathcal{R}$ has full fundamental group, a result which goes back to Murray and von Neumann in [83]). Then define $\phi_n^{n+1}\colon \mathcal{R}^{\oplus l} \to \mathcal{R}^{\oplus k}$ by

$$\phi_n^{n+1}(a_1 \oplus \cdots \oplus a_l) \coloneqq \Big(\sum_{j=1}^l \psi_{1,j}(a_j), \dots, \sum_{j=1}^l \psi_{k,j}(a_j) \Big), \tag{3.51}$$

and note that $(\phi_n^{n+1})^* = \alpha_{n+1}^n$.

Define $(\mathcal{R}_X, X') \coloneqq \varinjlim \big((\mathcal{M}_n, X_n), \phi_n^{n+1}\big)$. As $(\phi_n^{n+1})^* = \alpha_{n+1}^n$, (3.44) provides an isomorphism $X \to X'$, so after identifying X with X' via this isomorphism, $(\mathcal{R}_X, X)$ is the desired factorial tracially complete C^*-algebra.

A priori, the construction outlined above depends not only on the Choquet simplex X but also on the choice of inverse limit and the choices made when defining connecting maps. However, it will follow from Theorem 9.13 that the tracially complete C^*-algebra $(\mathcal{R}_X, X)$ depends only on X. Moreover, as X varies over all metrisable Choquet simplices, these will provide models for the classifiable tracially complete C^*-algebras (see Theorem 9.15, which contains Theorem C).

Remark 3.36. The infinite tensor product $\bigotimes_{n=1}^\infty (\mathcal{M}_n, X_n)$ of a countable family $(\mathcal{M}_n, X_n)$ of tracially complete C^*-algebras can now be constructed as the inductive limit of the finite tensor products $\bigotimes_{i=1}^n (\mathcal{M}_i, X_i)$ with the obvious connecting maps.

3.6. **W^*-bundles.** In Proposition 3.6, we showed Ozawa's W^*-bundles can be viewed as tracially complete C^*-algebras. We show here that by a reformulation of a theorem due to Ozawa, any factorial tracially complete C^*-algebra $(\mathcal{M}, X)$ with X a Bauer simplex can be given the structure of a W^*-bundle whose fibres are factors. This is essentially due to Ozawa in [86]. Recall from Section 2.1 that if X is a Bauer simplex, then there is an affine homeomorphism

$$X \xrightarrow{\cong} \mathrm{Prob}(\partial_e X)\colon \tau \mapsto \mu_\tau, \tag{3.52}$$

where μ_τ is the measure with barycentre τ.

Theorem 3.37 (cf. [86, Theorem 3])**.** *If $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra such that X is a Bauer simplex with $K \coloneqq \partial_e X$, then there is a unique embedding $C(K) \subseteq Z(\mathcal{M})$ such that*

$$\tau(fa) = \int_K f(\sigma)\sigma(a)\,\mathrm{d}\mu_\tau(\sigma), \qquad f \in C(K),\ a \in \mathcal{M},\ \tau \in X. \tag{3.53}$$

Further, the map $E\colon \mathcal{M} \to C(K)$ given by $E(a)(\tau) \coloneqq \tau(a)$ is a conditional expectation endowing $\mathcal{M}$ with the structure of a W^*-bundle.

Proof. We first show the embedding of $C(K)$ is unique, assuming it exists. Let $\phi, \psi\colon C(K) \to Z(\mathcal{M})$ be two embeddings satisfying (3.53). Fix $f \in C(K)$. If $\tau \in K$, then μ_τ is the point mass at τ, so (3.53) gives

$$\tau(\phi(f)\psi(f)) = f(\tau)\tau(\psi(f)) = f(\tau)^2. \tag{3.54}$$

Similarly, $\tau(\phi(f)^2) = \tau(\psi(f)^2) = f(\tau)^2$. Now, if $f \in C(K)$ is self-adjoint, we have

$$\|\phi(f) - \psi(f)\|_{2,\tau}^2 = \tau\big((\phi(f) - \psi(f))^2\big) = 0 \tag{3.55}$$

for all $\tau \in \partial_e X$, and hence for all $\tau \in X$. Therefore, $\phi(f) = \psi(f)$ for all self-adjoint $f \in C(K)$, and hence for all $f \in C(K)$.

Note that once we know the embedding of $C(K)$ exists, then E is necessarily a tracial conditional expectation onto $C(K)$ such that $\|\cdot\|_{2,X} = \|\cdot\|_{2,E}$. In particular, the unit ball of $\mathcal{M}$ is $\|\cdot\|_{2,E}$-complete.

The hardest portion of the proof is the existence of the embedding of $C(K)$. This is constructed in [86, Theorem 3]. Note that there is a standing assumption of metrisability in [86]. The metrisability in this result is only needed so that $\partial_e X$ is Borel, but in our case, this is clear since $\partial_e X$ is closed. □

The use of factoriality in Theorem 3.37 is subtle – it enters in the proof implicitly through the computation of the centre of $\pi_\tau(\mathcal{M})''$ for $\tau \in X$ in [86, Lemma 10]. In more detail, if $\tau \in X$, and $z \in Z(\pi_\tau(\mathcal{M})'')$, then we may define $z\tau \in \mathcal{M}^*$ by $(z\tau)(a) \coloneqq \langle \pi_\tau(a) z\xi_\tau, \xi_\tau\rangle$. Then $z\tau$ is a tracial linear functional with $|z\tau| \leq \|z\|\tau$. The assumption that X is a face implies $z\tau$ belongs to the span of X – this is necessary to define the representing measure $\mu_{z\tau}$ on $\partial_e X$ as needed in the proof of [86, Lemma 10]. As the following example demonstrates, Theorem 3.37 can fail without factoriality even for the one-dimensional simplex. The salient point is that in a W^*-bundle the fibres are fairly independent of each other, whereas in a non-factorial tracially complete algebra $(\mathcal{M}, X)$, the GNS representation π_τ with respect to the extreme traces $\tau \in \partial_e X$ may not be.

Example 3.38. Let $\mathcal{M} \coloneqq \mathcal{R} \oplus \mathcal{R} \oplus \mathcal{R}$, and let $\tau_i \in T(\mathcal{M})$ be given by $\tau_i(a_1, a_2, a_3) \coloneqq \tau_{\mathcal{R}}(a_i)$ for $i = 1, 2, 3$. Define $\sigma_1, \sigma_2 \in T(\mathcal{M})$ by

$$\sigma_1 \coloneqq \frac{1}{2}(\tau_1 + \tau_2) \qquad \text{and} \qquad \sigma_2 \coloneqq \frac{1}{2}(\tau_1 + \tau_3) \tag{3.56}$$

Let $Y \coloneqq \operatorname{co}\{\sigma_1, \sigma_2\} \subseteq T(\mathcal{M})$. Then $(\mathcal{M}, Y)$ is a tracially complete C^*-algebra and Y is a Bauer simplex. Note that $(\mathcal{M}, Y)$ is not factorial – in fact, neither extreme point of Y is an extreme point of $T(\mathcal{M})$. Let $K \coloneqq \partial_e Y = \{\sigma_1, \sigma_2\}$.

We will show there is no embedding of $C(K) \cong \mathbb{C}^2$ into $Z(\mathcal{M}) \cong \mathbb{C}^3$ satisfying (3.53). Note that any embedding satisfying (3.53) is unital. Let $f_1, f_2 \in C(K)$ be given by $f_1(\sigma_1) \coloneqq 1$ and $f_1(\sigma_2) \coloneqq 0$, and $f_2 \coloneqq 1_{C(K)} - f_1$. Exploiting the symmetry of τ_2 and τ_3, it suffices to show that the three unital embeddings $j_1, j_2, j_3\colon C(K) \to Z(\mathcal{M})$ determined by

$$j_1(f_1) = (1_{\mathcal{R}}, 0, 0),\ j_2(f_1) = (1_{\mathcal{R}}, 1_{\mathcal{R}}, 0),\ \text{and}\ j_3(f_1) = (0, 1_{\mathcal{R}}, 0) \tag{3.57}$$

fail (3.53).

In the first case, for $a \coloneqq (0, 1_{\mathcal{R}}, 0)$, $\sigma_1(j_1(f_1)a) = 0$, but

$$\int_K f_1(\sigma)\sigma(a)\,\mathrm{d}\mu_{\sigma_1} = f_1(\sigma_1)\sigma_1(a) = \frac{1}{2}. \tag{3.58}$$

In the second case, for $a \coloneqq (1_{\mathcal{R}}, 0, 0)$, $\sigma_2(j_2(f_2)a) = 0$, but

$$\int_K f_2(\sigma)\sigma(a)\,\mathrm{d}\mu_{\sigma_2} = f_2(\sigma_2)\sigma_2(a) = \frac{1}{2}. \tag{3.59}$$

In the third case, for $a \coloneqq (1_{\mathcal{R}}, 0, 0)$, $\sigma_1(j_3(f_1)a) = 0$, but

$$\int_K f_1(\sigma)\sigma(a)\,\mathrm{d}\mu_{\sigma_1} = f_1(\sigma_1)\sigma_1(a) = \frac{1}{2}. \tag{3.60}$$

So (3.53) fails in all three cases.

4. Amenability for tracially complete C^*-algebras

In this section, we show how to combine fibrewise amenability to obtain a global uniform 2-norm completely positive approximation property. The main result is Theorem 4.9, characterising morphisms into tracially complete C^*-algebras which are *tracially nuclear* – the appropriate notion of amenability which feeds into classification. Theorem 1.2 will follow as a special case of Theorem 4.9.

4.1. Definition and basic properties.

Our notion of amenability for tracially complete C^*-algebras is given by the following version of the completely positive approximation property.

Definition 4.1. Let A be a C^*-algebra and let $(\mathcal{N}, Y)$ be a tracially complete C^*-algebra. We say that a c.p. map $\theta\colon A \to \mathcal{N}$ is *tracially nuclear* if there are nets of finite dimensional C^*-algebras F_λ and c.p. maps

$$A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\phi_\lambda} \mathcal{N} \tag{4.1}$$

such that for all $a \in A$,

$$\lim_\lambda \|\phi_\lambda(\psi_\lambda(a)) - \theta(a)\|_{2,Y} = 0. \tag{4.2}$$

Further, we say $(\mathcal{N}, Y)$ is *amenable* if $\mathrm{id}_{\mathcal{N}}$ is tracially nuclear.

As usual, we may restrict to sequences in Definition 4.1 when A is separable.[34]

Using the canonical inclusion and projection maps, it immediate that amenability passes to direct sums.

Proposition 4.2. *For tracially complete C^*-algebras $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ the direct sum $(\mathcal{M} \oplus \mathcal{N}, X \oplus Y)$ is amenable if and only if both $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ are amenable.*

[34]This requires having a uniform bound on the norms of ϕ_λ and ψ_λ, which is always possible by Proposition 4.4.

Recall that if $\mathcal{N}$ is a von Neumann algebra, then a c.p. map $\theta\colon A \to \mathcal{N}$ is *weakly nuclear* if there are c.p. maps as in (4.1) such that for all $a \in \mathcal{N}$, $\phi_\lambda(\psi_\lambda(a)) \to a$ in the weak*-topology on $\mathcal{N}$. When τ is a faithful normal trace, $(\mathcal{N}, \{\tau\})$ is a tracially complete C^*-algebra, tracial nuclearity and weak nuclearity agree (and likewise, semidiscreteness of $\mathcal{N}$ agrees with amenability of $(\mathcal{N}, \{\tau\})$. We defer the proof to Proposition 4.5 as we first need some prerequisite results.

The following standard lemma allows us to reduce problems about tracial nuclearity to the unital case. We use $A^\dagger$ for the forced unitisation of A; i.e. when A is unital, we add a new unit to form $A^\dagger \cong A \oplus \mathbb{C}$.

Lemma 4.3. *Suppose A is a C^*-algebra, $(\mathcal{N}, Y)$ is a tracially complete C^*-algebra, and $\theta\colon A \to \mathcal{N}$ is a c.p.c. map. Then θ is tracially nuclear if and only if the unitisation $\theta^\dagger\colon A^\dagger \to \mathcal{N}$ is also tracially nuclear.*

Proof. First note that θ factorises as the inclusion $A \to A^\dagger$ followed by $\theta^\dagger$, so that tracial nuclearity of $\theta^\dagger$ implies that of θ.

Conversely, suppose that θ is tracially nuclear. Let (e_λ) be an approximate unit for A consisting of positive contractions and define $\theta_\lambda\colon A^\dagger \to \mathcal{N}$ by

$$\theta_\lambda(a + \alpha 1_{A^\dagger}) \coloneqq \theta(e_\lambda a e_\lambda) + \alpha 1_{\mathcal{N}}. \tag{4.3}$$

Then θ_λ is unital. Further, θ_λ is tracially nuclear as it is the sum of the tracially nuclear maps

$$a + \alpha 1_{A^\dagger} \mapsto \theta(e_\lambda(a + \alpha 1_{A^\dagger})e_\lambda) \tag{4.4}$$

and

$$a + \alpha 1_{A^\dagger} \mapsto \alpha(1_{\mathcal{N}} - \theta(e_\lambda^2)) \tag{4.5}$$

Further, since each θ_λ is unital and $\|\theta_\lambda(a) - \theta(a)\| \to 0$ for all $a \in A$, we have $\|\theta_\lambda(a) - \theta^\dagger(a)\| \to 0$ for all $a \in A^\dagger$. As each θ_λ is tracially nuclear, so is θ. □

As with the analogous versions of the completely positive approximation property for both C^*-algebras and von Neumann algebras, we can arrange for a uniform bound on the norms of the ϕ_λ and ψ_λ in Definition 4.1. The proof here follows the proof of the von Neumann algebraic version in [15, Proposition 3.8.2] taking care to avoid the use of Borel functional calculus, which, in general, does not exist in tracially complete C^*-algebras.

Proposition 4.4. *Suppose that A is a C^*-algebra, $(\mathcal{N}, Y)$ is a tracially complete C^*-algebra, and $\theta\colon A \to \mathcal{N}$ is a tracially nuclear map. Then there are nets*

$$A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\phi_\lambda} \mathcal{N} \tag{4.6}$$

as in Definition 4.1 with $\|\psi_\lambda\| \leq \|\theta\|$ and $\phi_\lambda(1_{F_\lambda}) = 1_{\mathcal{N}}$. Further, if A and θ are unital, we may arrange each ψ_λ to be unital.

Proof. Rescaling θ, we may assume that $\|\theta\| \leq 1$. Then by adding a unit to A and using Lemma 4.3, we may assume A and θ are unital. We will construct u.c.p. maps ϕ_λ and ψ_λ as in (4.6) approximately factorising θ.

Since θ is assumed to be tracially nuclear, there are nets of finite dimensional C^*-algebras F_λ and c.p. maps

$$A \xrightarrow{\psi''_\lambda} F_\lambda \xrightarrow{\phi'''_\lambda} \mathcal{N} \tag{4.7}$$

such that for all $a \in A$,

$$\lim_\lambda \|\phi'''_\lambda(\psi''_\lambda(a)) - \theta(a)\|_{2,Y} = 0. \tag{4.8}$$

By [15, Lemma 2.2.5], there is a u.c.p. map $\psi'_\lambda \colon A \to F_\lambda$ such that

$$\psi''_\lambda(a) = \psi''_\lambda(1_A)^{1/2}\psi'_\lambda(a)\psi''_\lambda(1_A)^{1/2}, \qquad a \in A. \tag{4.9}$$

Define $\phi''_\lambda \colon F_\lambda \to \mathcal{N}$ by

$$\phi''_\lambda(x) \coloneqq \phi'''_\lambda\big(\psi''_\lambda(1_A)^{1/2}x\psi''_\lambda(1_A)^{1/2}\big), \qquad x \in F_\lambda, \tag{4.10}$$

and note that $\phi''_\lambda \circ \psi'_\lambda = \phi'''_\lambda \circ \psi''_\lambda$. Therefore, we have

$$\lim_\lambda \|\phi''_\lambda(\psi'_\lambda(a)) - \theta(a)\|_{2,Y} = 0, \qquad a \in A. \tag{4.11}$$

Define a continuous function $f \colon \mathbb{R} \to [0,1]$ by

$$f(t) \coloneqq \begin{cases} 1, & t \le 1; \\ 2-t, & 1 < t < 2; \\ 0, & t \ge 2. \end{cases} \tag{4.12}$$

Let $b_\lambda \coloneqq f(\phi''_\lambda(1_{F_\lambda}))$ and define $\phi'_\lambda \colon F_\lambda \to \mathcal{N}$ by

$$\phi'_\lambda(x) \coloneqq b_\lambda\phi''_\lambda(x)b_\lambda, \qquad x \in F_\lambda. \tag{4.13}$$

By elementary calculus, $0 \le tf(t)^2 \le 1$ for all $t \in [0,\infty)$. Therefore, $\|\phi'_\lambda(1_{F_\lambda})\| \le 1$, and ϕ'_λ is a c.p.c. map. Also, for all $t \in \mathbb{R}$, we have $0 \le 1 - f(t) \le |t-1|$, and so

$$0 \le 1_{\mathcal{N}} - b_\lambda \le |\phi''_\lambda(1_{F_\lambda}) - 1_{\mathcal{N}}|. \tag{4.14}$$

Since θ and ψ'_λ are unital, (4.11) and (4.14) imply

$$\lim_\lambda \|1_{\mathcal{N}} - b_\lambda\|_{2,Y} = 0. \tag{4.15}$$

Fix $a \in A$. Since the elements b_λ are contractions, we have

$$\|\phi'_\lambda(\psi'_\lambda(a)) - b_\lambda\theta(a)b_\lambda\|_{2,Y} \le \|\phi''_\lambda(\psi'_\lambda(a)) - \theta(a)\|_{2,Y}, \tag{4.16}$$

so $\lim_\lambda \|\phi'_\lambda(\psi'_\lambda(a)) - b_\lambda\theta(a)b_\lambda\|_{2,Y} = 0$ by (4.11). Moreover, as multiplication in $\mathcal{N}$ is $\|\cdot\|_{2,Y}$-continuous on $\|\cdot\|$-bounded sets, we have

$$\lim_\lambda \|b_\lambda\theta(a)b_\lambda - \theta(a)\|_{2,Y} = 0, \qquad a \in A, \tag{4.17}$$

by (4.15). Therefore,

$$\lim_\lambda \|\phi'_\lambda(\psi'_\lambda(a)) - \theta(a)\|_{2,Y} = 0, \qquad a \in A. \tag{4.18}$$

While ψ'_λ was arranged to be unital, the map ϕ'_λ need not be. We modify the maps once more to correct this. Let ρ be a state on A and define $\psi_\lambda \colon A \to F_\lambda \oplus \mathbb{C}$ by $\psi_\lambda(a) \coloneqq (\psi'_\lambda(a), \rho(a))$. Then ψ_λ is u.c.p. as both ψ'_λ and ρ are. Also, define $\phi_\lambda \colon F_\lambda \oplus \mathbb{C} \to \mathcal{N}$ by

$$\phi_\lambda(x,\alpha) \coloneqq \phi'_\lambda(x) + \alpha(1_{\mathcal{N}} - \phi'_\lambda(1_{F_\lambda})) \tag{4.19}$$

Then ϕ_λ is u.c.p. as ϕ'_λ is c.p.c. By (4.18), to see that

$$\lim_\lambda \|\phi_\lambda(\psi_\lambda(a)) - \theta(a)\|_{2,Y} = 0, \qquad a \in A, \tag{4.20}$$

it suffices to show that $\lim_\lambda \|1_\mathcal{N} - \phi_\lambda(1_{F_\lambda})\|_{2,Y} = 0$. However, this follows immediately from taking $a \coloneqq 1_A$ in (4.18) using that the maps ψ'_λ and θ are both unital. □

We now return to the promised connection between tracial and weak nuclearity.

Proposition 4.5. *Let A be a C^*-algebra and $(\mathcal{N}, \tau)$ be a tracial von Neumann algebra. A c.p. map $\theta\colon A \to (\mathcal{N}, \{\tau\})$ is tracially nuclear if and only if it is weakly nuclear. In particular, $(\mathcal{N}, \tau)$ is semidiscrete as a tracial von Neumann algebra if and only if $(\mathcal{N}, \{\tau\})$ is amenable as a tracially complete C^*-algebra.*

Proof. The key observation is that the strong operator topology on the operator norm unit ball of $\mathcal{N}$ coincides with the topology induced by $\|\cdot\|_{2,\tau}$ (see [7, Proposition III.2.2.17], for example). From here, assume that θ is weakly nuclear. By scaling θ, we may assume $\|\theta\| \leq 1$. Then, as in the proof of Lemma 4.3, the unitisation $\theta^\dagger\colon A^\dagger \to \mathcal{N}$ is weakly nuclear. Hence we may assume that A and θ are unital. By [15, Proposition 3.8.2], there are nets of finite dimensional C^*-algebras and c.p.c. maps

$$A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\phi_\lambda} \mathcal{N} \tag{4.21}$$

such that $\phi_\lambda(\psi_\lambda(a)) \to a$ weak*, and in particular in the weak operator topology for all $a \in A$. Since the set of c.p.c. maps $A \to \mathcal{N}$ which admit c.p.c. factorisations through a finite dimensional C^*-algebra form a convex set ([15, Lemma 2.3.6 and Remark 2.3.7]), the Hahn–Banach theorem, in the form of [15, Lemma 3.8.1], shows that there is a net of c.p.c. maps $\theta_\lambda\colon A \to \mathcal{N}$ such that $\theta_\lambda(a) \to \theta(a)$ in the strong-operator topology for all $a \in A$, and each θ_λ admits a c.p.c. factorisation through a finite dimensional C^*-algebra. As each θ_λ is contractive, this implies $\|\theta(a) - \theta_\lambda(a)\|_{2,\tau} \to 0$ for all $a \in A$, and hence θ is tracially nuclear.

Conversely, suppose that θ is tracially nuclear. Again, after scaling θ, we may assume that θ is contractive. By Proposition 4.4, there are nets of finite dimensional C^*-algebras and c.p.c. maps

$$A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\phi_\lambda} \mathcal{N} \tag{4.22}$$

such that $\|\theta(a) - \phi_\lambda(\psi_\lambda(a))\|_{2,\tau} \to 0$ for all $a \in A$. Then $\phi_\lambda(\psi_\lambda(a)) \to a$ in the strong operator topology, and hence in the weak operator topology. As the weak operator topology and weak* topology on $\mathcal{N}$ agree on bounded sets, the convergence holds weak*, and this implies that θ is weakly nuclear. □

It is known that if A is an exact C^*-algebra and $\mathcal{N}$ is a von Neumann algebra, then any weakly nuclear map $A \to \mathcal{N}$ is nuclear – see [59, Remark 3.4] (as observed there, this characterises exactness of A). This suggests the following question, which we have been unable to answer even with the additional assumptions that $(\mathcal{N}, Y)$ is factorial and has CPoU. The difficulty

is that the passage from weak nuclearity to nuclearity involves taking point-weak*-limit points.[35] We do not have the required compactness for this in the setting of tracially complete C^*-algebras.

Question 4.6. Suppose A is an exact C^*-algebra and $(\mathcal{N}, Y)$ is a tracially complete C^*-algebra. Is every tracially nuclear map $A \to \mathcal{N}$ nuclear?

4.2. **Fibrewise amenability.** We work towards showing that amenability of tracially complete C^*-algebras can be detected in its tracial von Neumann algebra completions. The basic strategy is to use a convexity argument similar to the one in the fundamental result from [34, 24, 25] stating that if A is a C^*-algebra such that A^{**} is semidiscrete, then A is nuclear (see [15, Proposition 2.3.8], for example).

We set up the convexity argument in a somewhat general setting for later use in Section 6.4. Both the statement and proof are abstracted from an argument in the proof of Ozawa's [86, Theorem 3] computing the centre of tracial completions of C^*-algebras. A similar Hahn–Banach argument appears in the proof of (8)⇒(1) in [48, Theorem 2.5].

Lemma 4.7. *Let $\mathcal{C}$ be a convex subset of a real vector space and let $(\mathcal{M}, X)$ be a tracially complete C^*-algebra. Suppose we are given $m \in \mathbb{N}$ and a finite collection $f_1, \dots, f_m \colon \mathcal{C} \to \mathcal{M}$ of affine functions such that for all $\tau \in X$ and $\epsilon > 0$, there is $c_\tau \in \mathcal{C}$ such that*

$$\max_{1 \le i \le m} \|f_i(c_\tau)\|_{2,\tau} < \epsilon. \tag{4.23}$$

Then for all $\epsilon > 0$, there is $c \in \mathcal{C}$ such that

$$\max_{1 \le i \le m} \|f_i(c)\|_{2,X} < \epsilon. \tag{4.24}$$

Proof. Let Λ be the directed set of pairs $(\mathcal{T}, \epsilon)$ where $\mathcal{T} \subseteq X$ is a non-empty finite set and $\epsilon > 0$, equipped with the ordering $(\mathcal{T}_1, \epsilon_1) \le (\mathcal{T}_2, \epsilon_2)$ if and only if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\epsilon_1 \ge \epsilon_2$.

Fix $\lambda \coloneqq (\mathcal{T}_\lambda, \epsilon_\lambda) \in \Lambda$ and let $\sigma_\lambda \coloneqq |\mathcal{T}_\lambda|^{-1} \sum_{\tau \in \mathcal{T}_\lambda} \tau \in X$ denote the average of the traces in $\mathcal{T}_\lambda$. By assumption, there is $c_\lambda \in \mathcal{C}$ such that

$$|\mathcal{T}_\lambda|^{1/2} \max_{1 \le i \le m} \|f_i(c_\lambda)\|_{2,\sigma_\lambda} < \epsilon_\lambda. \tag{4.25}$$

It follows from the definition of σ_λ that

$$\max_{\tau \in \mathcal{T}_\lambda} \max_{1 \le i \le m} \|f_i(c_\lambda)\|_{2,\tau} < \epsilon_\lambda. \tag{4.26}$$

For $\lambda \in \Lambda$ and $i = 1, \dots, m$, define $h_{i,\lambda} \colon X \to \mathbb{R}$ by

$$h_{i,\lambda}(\tau) \coloneqq \|f_i(c_\lambda)\|_{2,\tau}^2 \tag{4.27}$$

and note that $h_{i,\lambda} \in \mathrm{Aff}(X)$. For all $i = 1, \dots, m$, $h_{i,\lambda} \to 0$ pointwise on X by (4.26), and hence Proposition 2.1 implies $h_{i,\lambda} \to 0$ weakly. View

[35] The proof in [59, Remark 3.4] uses Kirchberg's $\mathcal{O}_2$-embedding theorem, but this can be avoided using the following slight variation of their proof. Let $\theta \colon A \to \mathcal{N}$ be weakly nuclear. Fix nets of finite dimensional C^*-algebras F_λ and c.p.c. maps $\psi_\lambda \colon A \to F_\lambda$ and $\phi_\lambda \colon F_\lambda \to \mathcal{N}$ such that $\phi_\lambda \circ \psi_\lambda$ converges to θ point-weak*. Let $\pi \colon A \to \mathcal{B}(\mathcal{H})$ be a faithful representation of A and use Arveson's extension theorem to find c.p.c. maps $\tilde{\psi}_\lambda \colon \mathcal{B}(\mathcal{H}) \to F_\lambda$ with $\tilde{\psi}_\lambda \circ \pi = \psi_\lambda$. Let $\rho \colon \mathcal{B}(\mathcal{H}) \to \mathcal{N}$ be a point-weak* limit point of $\phi \circ \tilde{\psi}_\lambda$ and note that $\theta = \rho \circ \pi$. As A is exact, π is nuclear, and hence so is θ.

$h_\lambda \coloneqq (h_{i,\lambda}, \dots, h_{m,\lambda})$ as a net in the Banach space $\mathrm{Aff}(X)^{\oplus m}$, and note that $h_\lambda \to 0$ weakly.

Fix $\epsilon > 0$. By the Hahn–Banach theorem, there are $l \in \mathbb{N}$, $\lambda_1, \dots, \lambda_l \in \Lambda$, and real numbers $\alpha_1, \dots, \alpha_l \geq 0$ such that $\sum_{k=1}^l \alpha_k = 1$ and

$$\max_{1 \leq i \leq m} \Big\| \sum_{k=1}^l \alpha_k h_{i,\lambda_k} \Big\|_\infty < \epsilon^2. \tag{4.28}$$

Define $c \coloneqq \sum_{k=1}^l \alpha_k c_{\lambda_k} \in \mathcal{C}$. For $i = 1, \dots, m$, using the triangle inequality and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|f_i(c)\|_{2,X} &= \Big\| \sum_{k=1}^l \alpha_k f_i(c_{\lambda_k}) \Big\|_{2,X} \\ &\leq \sum_{k=1}^l \alpha_k \|f_i(c_{\lambda_k})\|_{2,X} \\ &= \sum_{k=1}^l \big(\alpha_k^{1/2}\big)\big(\alpha_k^{1/2} \|f_i(c_{\lambda_k})\|_{2,X}\big) \qquad (4.29) \\ &\leq \Big(\sum_{k=1}^l \alpha_k \Big)^{1/2} \Big(\sum_{k=1}^l \alpha_k \|f_i(c_{\lambda_k})\|_{2,X}^2 \Big)^{1/2} \\ &= \Big(\sum_{k=1}^l \alpha_k \|f_i(c_{\lambda_k})\|_{2,X}^2 \Big)^{1/2} \end{aligned}$$

Then for $i = 1, \dots, m$ and $\tau \in X$, we have

$$\|f_i(c)\|_{2,\tau}^2 \overset{(4.27),(4.29)}{\leq} \sum_{k=1}^l \alpha_k h_{i,\lambda_k}(\tau) \overset{(4.28)}{<} \epsilon^2. \tag{4.30}$$

Hence $\|f_i(c)\|_{2,X} < \epsilon$ for all $i = 1, \dots, m$. □

The following lemma is standard. For example, it follows from the proof of [59, Lemma 1.1] by quoting the Choi–Effros lifting theorem in place of the projectivity of order zero maps in the last paragraph.

Lemma 4.8. *If F and B are C^*-algebras with F finite dimensional, $\pi\colon B \to \mathcal{B}(H)$ is a *-homomorphism, and $\phi\colon F \to \pi(B)''$ is a c.p. map, then there is a net of c.p. maps $\phi_\lambda\colon F \to B$ such that $\|\phi_\lambda\| \leq \|\phi\|$ for all λ and*

$$\phi(b) = \text{strong}^*\text{-}\lim_\lambda \pi(\phi_i(b)), \qquad b \in B. \tag{4.31}$$

Further, if ϕ is unital, we may arrange for each ϕ_λ to be unital.

We are now in a position to give the 'fibrewise' characterisation of tracially nuclear *-homomorphisms (Theorem 4.9) from which Theorem 1.2 follows immediately (by taking $A \coloneqq \mathcal{M}$ and $\theta \coloneqq \mathrm{id}_{\mathcal{M}}$). Note that the following theorem also implies the first part of Theorem D from the overview.

The equivalence of (ii) and (iii) in Theorem 4.9 is essentially obtained by Brown in [12, Theorem 3.2.2] (which is a variation on Connes' theorem), so the main implication we need to show is (ii) implies (i). This will follow from the Hahn–Banach argument of Lemma 4.7.

Theorem 4.9. *Let A be a C^*-algebra, let $(\mathcal{M}, X)$ be a tracially complete C^*-algebra, and let $\theta\colon A \to \mathcal{M}$ be a c.p. map. The following are equivalent:*

(i) *θ is tracially nuclear;*
(ii) *for all $\tau \in X$, $\pi_\tau \circ \theta\colon A \to \pi_\tau(\mathcal{M})''$ is weakly nuclear.*

If θ is a *-homomorphism then these are also equivalent to:

(iii) *for every $\tau \in X$ with $\tau \circ \theta \neq 0$, the trace $\|\tau \circ \theta\|^{-1} \cdot \tau \circ \theta \in T(A)$ is uniformly amenable in the sense of* [12, Definition 3.2.1].[36]

Proof. To see (i) implies (ii), note that if θ is tracially nuclear, then for all $\tau \in X$, $\pi_\tau \circ \theta$ is tracially nuclear as a map into the tracial von Neumann algebra $(\pi_\tau(\mathcal{M})'', \tau)$. As noted in the remarks following Definition 4.1, for maps into tracial von Neumann algebras, weak and tracial nuclearity are equivalent, so (ii) follows.

When θ is a *-homomorphism, the equivalence of (ii) and (iii) can be reduced to the case that A and θ are unital by adding a unit to A and using Lemma 4.3. For $\tau \in X$, since there is a normal trace-preserving conditional expectation $\pi_\tau(\mathcal{M})'' \to \pi_\tau(\theta(A))''$ (see [15, Lemma 1.5.10], for example), $\pi_{\tau\circ\theta} = \pi_\tau \circ \theta$ is weakly nuclear when viewed as a map into $\pi_\tau(\mathcal{M})''$ if and only if it is weakly nuclear when viewed as a map into $\pi_{\tau\circ\theta}(A)'' = \pi_\tau(\theta(A))''$. The equivalence of (ii) and (iii) then follows from the equivalence of (1) and (6) in [12, Theorem 3.2.2].

It remains to show that (ii) implies (i). Assume that θ is contractive. Fix a finite set $\mathcal{F} \subseteq A$ and $\epsilon > 0$. Let $\mathcal{C}$ denote the set of all c.p. maps $A \to \mathcal{M}$ which factor by c.p. maps through a finite dimensional C^*-algebra and note that $\mathcal{C}$ is convex (cf. [15, Lemma 2.3.6]). We will apply Lemma 4.7 to the affine functions

$$f_a\colon \mathcal{C} \to \mathcal{M}\colon \eta \mapsto \eta(a) - \theta(a), \qquad a \in \mathcal{F}. \tag{4.32}$$

For $\tau \in X$, $\pi_\tau \circ \theta$ is weakly nuclear by (ii), so there are a finite dimensional C^*-algebra F_τ and c.p.c. maps

$$A \xrightarrow{\psi_\tau} F_\tau \xrightarrow{\bar{\phi}_\tau} \pi_\tau(\mathcal{M})'' \tag{4.33}$$

such that

$$\|\bar{\phi}_\tau(\psi_\tau(a)) - \pi_\tau(\theta(a))\|_{2,\tau} < \epsilon, \qquad a \in \mathcal{F}. \tag{4.34}$$

By Lemma 4.8, there is then a c.p.c. map $\phi_\tau\colon F_\tau \to \mathcal{M}$ such that

$$\|\phi_\tau(\psi_\tau(a)) - \theta(a)\|_{2,\tau} < \epsilon, \qquad a \in \mathcal{F}. \tag{4.35}$$

Then $\eta_\tau := \phi_\tau \circ \psi_\tau \in \mathcal{C}$ and $\|f_a(\eta_\tau)\|_{2,\tau} < \epsilon$ for all $a \in \mathcal{F}$. By Lemma 4.7, there is $\eta \in \mathcal{C}$ such that $\|f_a(\eta)\|_{2,X} < \epsilon$ for all $a \in \mathcal{F}$. Unpacking notation, η is a c.p. map factoring through a finite dimensional C^*-algebra, and

$$\|\eta(a) - \theta(a)\|_{2,X} < \epsilon, \qquad a \in \mathcal{F}, \tag{4.36}$$

so θ is tracially nuclear. □

[36] Brown implicitly only defines uniformly amenable for traces on unital C^*-algebras in [15] (the definition there requires u.c.p. maps out of A). We extend this definition to the non-unital case, by saying that $\tau \in T(A)$ is uniformly amenable if its unitisation, $\tau^\dagger \in T(A^\dagger)$, is uniformly amenable.

Combining the previous result with Connes' theorem gives a tracially complete analogue of the fact that a von Neumann algebra completion of a nuclear C^*-algebra is semidiscrete. Let $T_{\mathrm{am}}(A) \subseteq T(A)$ denote the set of amenable traces on A.

Corollary 4.10. *If A is an exact C^*-algebra with $T_{\mathrm{am}}(A)$ compact,*[37] *then the tracial completion of A with respect to $T_{\mathrm{am}}(A)$ is an amenable factorial tracially complete C^*-algebra. In particular, if A is a nuclear C^*-algebra with $T(A)$ compact, then the tracial completion of A with respect to $T(A)$ is an amenable factorial tracially complete C^*-algebra.*

Proof. By [66, Lemma 3.4], $T_{\mathrm{am}}(A)$ is a closed face in $T(A)$. Therefore, the tracial completion of A with respect to $T_{\mathrm{am}}(A)$ is a factorial tracially complete C^*-algebra by Proposition 3.23(iv).

As A is exact, A is locally reflexive by [65, Remark (11)], and hence by [12, Theorem 4.3.3], all amenable traces on A are uniformly amenable. By the equivalence of (1) and (5) in [12, Theorem 3.2.2], it follows that $\pi_\tau(A)''$ is semidiscrete for $\tau \in T_{\mathrm{am}}(A)$. Using Proposition 3.23(vi), we also have that $\pi_\tau(\overline{A}^{T_{\mathrm{am}}(A)})''$ is semidiscrete for all $\tau \in T_{\mathrm{am}}(A)$. By Theorem 4.9, this implies that $\big(\overline{A}^{T_{\mathrm{am}}(A)}, T_{\mathrm{am}}(A)\big)$ is amenable. This proves the first sentence of the theorem. The second sentence follows since $T(A) = T_{\mathrm{am}}(A)$ when A is nuclear (see [12, Theorem 4.2.1], for example). □

Without exactness, it need not be the case that all amenable traces are uniformly amenable. Indeed, given a sequence (k_n) of natural numbers converging to ∞, let $A \coloneqq \prod M_{k_n}$. For a free ultrafilter ω on $\mathbb{N}$, the trace $\tau_\omega((x_n)) \coloneqq \lim_{n\to\omega} \mathrm{tr}_{k_n}(x_n)$ is an amenable trace which is not uniformly amenable.[38] This observation leads to the following characterisation of those finite von Neumann algebras which are amenable as tracially complete C^*-algebras over their trace space.

Proposition 4.11. *Let $\mathcal{M}$ be a semidiscrete finite von Neumann algebra. Then $\big(\mathcal{M}, T(\mathcal{M})\big)$ is amenable as a tracially complete C^*-algebra if and only if its type II_1 summand has only finitely many extremal traces and it has no type I_n summand for sufficiently large n.*

Proof. Suppose $\mathcal{M}$ satisfies the stated condition. Then the type I part $\mathcal{M}_{\mathrm{I}}$ of $\mathcal{M}$ is a finite direct sum of matrices over abelian C^*-algebras (see [104, Theorem V.1.27], for example) and hence nuclear so has the completely positive approximation property in norm. Hence $\big(\mathcal{M}_{\mathrm{I}}, T(\mathcal{M}_{\mathrm{I}})\big)$ is amenable as a tracially complete C^*-algebra. Let $e_1, \ldots, e_m$ denote the minimal central projections of the type II_1 part $\mathcal{M}_{\mathrm{II}}$ of $\mathcal{M}$ so that each $\mathcal{M}e_i = \mathcal{M}_{\mathrm{II}}e_i$ is a semidiscrete factor, whence $\big(\mathcal{M}e_i, T(\mathcal{M}e_i)\big)$ is an amenable tracially complete C^*-algebra (by Proposition 4.5). Since the finite direct sum of amenable tracially complete C^*-algebras is amenable (Proposition 4.2), it follows that $\big(\mathcal{M}, T(\mathcal{M})\big)$ is amenable.

[37] Note that $T_{\mathrm{am}}(A)$ is closed in $T(A)$ by [12, Proposition 3.5.1], so the compactness is automatic if A is unital.

[38] This is because the tracial von Neumann ultraproduct $\prod^\omega M_{n_k}$ is not amenable, as it contains a copy of $L(F_2)$..

Conversely, supposing the condition does not hold, we can find a sequence (n_k) converging to ∞ and orthogonal central projections (e_k) in $\mathcal{M}$ such that there is a unital embedding $M_{n_k} \to \mathcal{M}e_k$.[39] In this way, we have an embedding of the infinite product C^*-algebra $\prod M_{n_k}$ in $\mathcal{M}$. For each k, let τ_k be a trace on $\mathcal{M}$ with $\tau_k(e_k) = 1$ and for a free ultrafilter ω on $\mathbb{N}$, let $\tau_\omega \coloneqq \lim_{k\to\omega} \tau_k$, which exists by compactness of the trace space of $\prod M_{n_k}$. Then τ_ω restricts to the trace $(x_n) \mapsto \lim_{n\to\omega} \operatorname{tr}_{n_k}(x_n)$ on $\prod M_{n_k}$. This restricted trace is not uniformly amenable (see Footnote 38). As uniform amenability of traces passes to subalgebras (this is immediate from the approximation form of the definition in [12, Definition 3.2.1]), τ_ω is not uniformly amenable, and hence $\big(\mathcal{M}, T(\mathcal{M})\big)$ is not amenable.

□

In particular, the tracially complete C^*-algebra $\big(\ell^\infty(\mathcal{R}), T(\ell^\infty(\mathcal{R}))\big)$ is not amenable even though $\ell^\infty(\mathcal{R})$ is semidiscrete as a von Neumann algebra.

We end this section by applying the characterisation of tracial nuclearity, to show that it can be tested on C^*-subalgebras which are dense in the uniform 2-norm. A naive argument (showing that the witnesses of tracial nuclearity for $\phi|_A$ extend to witnesses of tracial nuclearity for ϕ) is not possible, as one cannot a priori assume uniform $\|\cdot\|_{2,X}$-boundedness of the c.p.c. approximations in the definition of tracial nuclearity.

Lemma 4.12. *Suppose $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ are tracially complete C^*-algebras and $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ is a morphism. If $A \subseteq \mathcal{M}$ is a $\|\cdot\|_{2,X}$-dense C^*-subalgebra, then ϕ is tracially nuclear if and only if $\phi|_A$ is tracially nuclear.*

Proof. For all $\tau \in X$, the inclusion $A \hookrightarrow \mathcal{M}$ induces an isomorphism $\pi_\tau(A)'' \to \pi_\tau(\mathcal{M})''$ (by combining Corollary 3.29(i) and Proposition 3.23(vi), for example). Since a trace on a C^*-algebra is uniformly amenable if and only if it generates a semidiscrete von Neumann algebra (see [12, Theorem 3.2.2(2)⇔(3)]), we have that for each $\tau \in Y$, $\tau\circ\phi$ is uniformly amenable if and only if $\tau \circ \phi|_A$ is uniformly amenable. The result follows from Theorem 4.9. □

5. Reduced products and central sequences

In this section, we define the reduced product $\prod^\omega(\mathcal{M}_n, X_n)$ associated with a sequence of tracially complete C^*-algebras $\big((\mathcal{M}_n, X_n)\big)_{n=1}^\infty$ and a free filter ω on $\mathbb{N}$.

Aided by the language of reduced products and motivated by the analogous conditions in von Neumann algebra theory, we introduce two properties of (factorial) tracially complete C^*-algebras: the McDuff property and property Γ. Both of these properties have been studied in the setting of C^*-algebras with their uniform 2-norm in [22, 23], where they are referred to as the *uniform McDuff property* and *uniform property* Γ, respectively. We view tracially complete C^*-algebras as the natural framework to develop these properties, and the definitions given here extend the definitions from

[39] This uses the fact that for any II_1 von Neumann algebra $\mathcal{N}$, and any $n \in \mathbb{N}$, one can find a unital embedding $M_n \to \mathcal{N}$. This goes back to Murray and von Neumann. The proof for $n = 2$ from [104, Proposition V.1.35] can be readily modified to cover general n.

[22, 23] when restricted to tracial completions of C^*-algebras with respect to the full trace simplex. Moreover, in the unique trace case, these properties reduce to the homonymous properties for II_1 factors.

5.1. **Reduced products.** Reduced products provide an algebraic setting for manipulating properties involving approximations. The most common constructions are ultrapowers with respect to a free ultrafilter on the natural numbers $\mathbb{N}$ and sequence algebras consisting of the algebra of bounded sequences modulo the ideal of c_0-sequences.

For many basic applications, ultrapowers and sequence algebras can be used interchangeably, but each has its technical advantages. In settings where traces are considered, ultrapowers are often more natural, as the resulting set of limit traces is already convex; on the other hand, working with sequence algebras allows the reparameterisation argument of [46, Theorem 4.3] (see Theorem 5.11), which will be used in our classification result in Section 9. In order to allow for both constructions simultaneously, we will work with reduced products defined with respect to a free filter.

For the remainder of the paper, ω will denote a free filter on the natural numbers $\mathbb{N}$. We recall that a filter ω on the natural numbers is free if and only if it contains all cofinite sets (see [15, Appendix A], for example, for a general discussion of filters). The following selection theorem of Kirchberg will be used frequently. It is most often stated for ultrafilters, but the result (and proof) is equally valid for general filters. For the readers convenience, we include the details.

Lemma 5.1 (Kirchberg's ϵ-test, cf. [67, Lemma A.1])**.** *Let ω be a free filter on $\mathbb{N}$. Let $(X_n)_{n=1}^{\infty}$ be a sequence of non-empty sets and for $k, n \in \mathbb{N}$, let $f_n^{(k)} \colon X_n \to [0,\infty]$ be a function. Define functions $f^{(k)} \colon X_1 \times X_2 \times \cdots \to [0,\infty]$ by*

$$f^{(k)}(x_1, x_2, \dots) \coloneqq \limsup_{n\to\omega} f_n^{(k)}(x_n). \tag{5.1}$$

If for every $\epsilon > 0$ and $k_0 \in \mathbb{N}$, there exists $x \in X_1 \times X_2 \times \cdots$ such that $f^{(k)}(x) < \epsilon$ for $k = 1, \dots, k_0$, then there exists $y \in X_1 \times X_2 \times \cdots$ such that $f^{(k)}(y) = 0$ for all $k \in \mathbb{N}$.

Proof. For each $r \in \mathbb{N}$, there exists $x^{(r)} = (x_1^{(r)}, x_2^{(r)}, \dots) \in X_1 \times X_2 \times \cdots$ such that $f^{(k)}(x^{(r)}) < \frac{1}{r}$ for $k = 1, \dots, r$. By (5.1), there exists $I_r \in \omega$ such that $f_n^{(k)}(x_n^{(r)}) < \frac{1}{r}$ for all $n \in I_r$ and $k = 1, \dots, r$. As ω is a free filter we may assume that $I_r \subseteq \{r, r+1, \dots\}$.

For each $n \in \mathbb{N}$, if n lies in $\bigcup_{r=1}^{\infty} I_r$, then let $r_n \in \mathbb{N}$ be maximal such that $n \in I_{r_n}$ (noting that $n \notin I_r$ for $r > n$) and set $y_n \coloneqq x_n^{(r_n)}$. Otherwise, define $y_n \in X_n$ arbitrarily. Fix $k, r \in \mathbb{N}$ with $k \leq r$. Then for all $n \in I_r$, it follows that $r_n \geq r$, and hence

$$f_n^{(k)}(y_n) < \frac{1}{r_n} \leq \frac{1}{r}. \tag{5.2}$$

Thus

$$f^{(k)}(y) = \limsup_{n\to\omega} f_n(y_n) \leq \sup_{n\in I_r} f_n(y_n) \leq \frac{1}{r}. \tag{5.3}$$

Since this holds for all $r \geq k$, we obtain $f^{(k)}(y) = 0$, as required. □

We now formally define reduced products.

Definition 5.2. For a sequence $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ of tracially complete C^*-algebras, define a C^*-algebra[40]

$$\prod^{\omega} \mathcal{M}_n \coloneqq \prod_{n=1}^{\infty} \mathcal{M}_n \big/ \big\{(a_n)_{n=1}^{\infty} : \lim_{n\to\omega} \|a_n\|_{2,X_n} = 0\big\}. \tag{5.4}$$

For every sequence of traces $(\tau_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X_n$ and every ultrafilter ω' containing ω, there is a trace defined on $\prod^{\omega} \mathcal{M}_n$ by $a \mapsto \lim_{n\to\omega'} \tau_n(a_n)$, where $(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{M}_n$ is any sequence representing a – such traces are called *limit traces*. Let $\sum^{\omega} X_n$ be the closed convex hull of the set of limit traces on $\prod^{\omega} \mathcal{M}_n$.

Then the pair

$$\prod^{\omega} (\mathcal{M}_n, X_n) \coloneqq \Big(\prod^{\omega} \mathcal{M}_n, \sum^{\omega} X_n \Big) \tag{5.5}$$

is called the *reduced product* of the sequence $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ with respect to ω (and the *ultraproduct* when ω is an ultrafilter). In the case when ω is the Fréchet filter, we write the reduced product as

$$\prod^{\infty} (\mathcal{M}_n, X_n) \coloneqq \Big(\prod^{\infty} \mathcal{M}_n, \sum^{\infty} X_n \Big). \tag{5.6}$$

Our first goal is to prove that the reduced product of a sequence of tracially complete C^*-algebras (with respect to a given free filter ω) is itself a tracially complete C^*-algebra. Before doing that, we isolate a useful lemma that will be used frequently in our analysis of reduced products.

Lemma 5.3. *Let $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ be a sequence of tracially complete C^*-algebras. If $a \in \prod^{\omega} (\mathcal{M}_n, X_n)$ be represented by the sequence $(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{M}_n$, then*

$$\|a\|_{2,\sum^{\omega} X_n} = \limsup_{n\to\omega} \|a_n\|_{2,X_n}. \tag{5.7}$$

Proof. Given a sequence of traces $(\tau_n)_{n=1}^{\infty}$ and an ultrafilter $\omega' \supseteq \omega$, let τ be the associated limit trace. Since $\|a_n\|_{2,\tau_n} \leq \|a_n\|_{2,X_n}$ for all $n \in \mathbb{N}$, we have

$$\|a\|_{2,\tau} = \lim_{n\to\omega'} \|a_n\|_{2,\tau_n} \leq \lim_{n\to\omega'} \|a_n\|_{2,X_n} \leq \limsup_{n\to\omega} \|a_n\|_{2,X_n}. \tag{5.8}$$

Hence $\|a\|_{2,\sum^{\omega} X_n} \leq \limsup_{n\to\omega} \|a_n\|_{2,X_n}$.

Conversely, let $\omega' \supseteq \omega$ be an ultrafilter with

$$\lim_{n\to\omega'} \|a_n\|_{2,X_n} = \limsup_{n\to\omega} \|a_n\|_{2,X_n}. \tag{5.9}$$

[40]Here, $\prod_{n=1}^{\infty} \mathcal{M}_n$ denotes the ℓ^{∞}-product. Using (3.2), it is easy to see that

$$\big\{(a_n)_{n=1}^{\infty} : \lim_{n\to\omega} \|a_n\|_{2,X_n} = 0\big\}$$

is an ideal of $\prod_{n=1}^{\infty} \mathcal{M}_n$, so this quotient is a C^*-algebra.

For every $n \in \mathbb{N}$, there exists $\tau_n \in X_n$ with $\|a_n\|_{2,\tau_n} > \|a_n\|_{2,X_n} - 2^{-n}$. Let τ be the limit trace corresponding to the sequence $(\tau_n)_{n=1}^{\infty}$ and the ultrafilter ω'. Then

$$\|a\|_{2,\sum^\omega X_n} \geq \|a\|_{2,\tau} = \lim_{n\to\omega'} \|a_n\|_{2,\tau_n} \geq \limsup_{n\to\omega} \|a_n\|_{2,X_n}. \tag{5.10}$$ □

We now prove that a reduced product of a sequence of tracially complete C^*-algebras is a tracially complete C^*-algebra.

Proposition 5.4 (cf. [23, Lemma 1.6])**.** *Let $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ be a sequence of tracially complete C^*-algebras. Then $\prod^\omega (\mathcal{M}_n, X_n)$ is a tracially complete C^*-algebra.*

Proof. By Lemma 5.3 and the definition of $\prod^\omega \mathcal{M}_n$, it is clear that $\|\cdot\|_{2,\sum^\omega X_n}$ is a norm. It remains to show that the $\|\cdot\|$-closed unit ball is $\|\cdot\|_{2,\sum^\omega X_n}$-complete.

Let $(a^{(k)})_{k=1}^{\infty}$ be a $\|\cdot\|_{2,\sum^\omega X_n}$-Cauchy sequence in the unit ball of $\prod^\omega \mathcal{M}_n$, and for each $k \in \mathbb{N}$, fix a sequence $(a_n^{(k)})_{n=1}^{\infty}$ of contractions which represents $a^{(k)}$. Set

$$\epsilon^{(k)} \coloneqq \sup\big\{\|a^{(l)} - a^{(l')}\|_{2,\sum^\omega X_n} : l, l' \geq k\big\}, \tag{5.11}$$

noting that $\epsilon^{(k)} \to 0$ as $k \to \infty$ since $(a^{(k)})_{k=1}^{\infty}$ is $\|\cdot\|_{2,\sum^\omega X_n}$-Cauchy. Define functions $f_n^{(k)} \colon \{b \in \mathcal{M} : \|b\| \leq 1\} \to [0,2]$ by

$$f_n^{(k)}(b) \coloneqq \max\big\{\|b - a_n^{(k)}\|_{2,X_n} - \epsilon^{(k)}, 0\big\}. \tag{5.12}$$

For $k_0 \in \mathbb{N}$ and $k = 1, \ldots, k_0$, since $\|a^{(k_0)} - a^{(k)}\|_{2,\sum^\omega X_n} \leq \epsilon^{(k)}$, we have $\limsup_{n\to\omega} f_n^{(k)}(a_n^{(k_0)}) = 0$ by Lemma 5.3. Therefore, Kirchberg's ϵ-test (Lemma 5.1) gives a sequence $(a_n)_{n=1}^{\infty}$ of contractions representing an element a in the unit ball of $\prod^\omega \mathcal{M}_n$ such that $\limsup_{n\to\omega} f_n^{(k)}(a_n) = 0$ for all $k \in \mathbb{N}$. By Lemma 5.3 and (5.12), this means that

$$\|a - a^{(k)}\|_{2,\sum^\omega X_n} \leq \epsilon^{(k)} \to 0 \tag{5.13}$$

as $k \to \infty$. Hence the unit ball of $\prod^\omega \mathcal{M}_n$ is $\|\cdot\|_{2,\sum^\omega X_n}$-complete. □

A particularly relevant case of a reduced product is when $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a constant sequence; i.e. for some tracially complete C^*-algebra $(\mathcal{M}, X)$, we have $(\mathcal{M}_n, X_n) = (\mathcal{M}, X)$ for all $n \in \mathbb{N}$. In this case, we write

$$(\mathcal{M}^\omega, X^\omega) \coloneqq \prod^\omega (\mathcal{M}_n, X_n) \tag{5.14}$$

and call $(\mathcal{M}^\omega, X^\omega)$ the *reduced power* of $(\mathcal{M}, X)$ with respect to the free filter ω, or the *ultrapower* if ω is an ultrafilter. Again, when ω is the Fréchet filter, we write $(\mathcal{M}^\infty, X^\infty)$ in place of $(\mathcal{M}^\omega, X^\omega)$.

Note that there is a natural embedding $\iota_{(\mathcal{M},X)} \colon (\mathcal{M}, X) \to (\mathcal{M}^\omega, X^\omega)$ of tracially complete C^*-algebras given by identifying $\mathcal{M}$ with constant sequences in $\mathcal{M}^\omega$. Typically, the map $\iota_{(\mathcal{M},X)}$ will be suppressed and we will view $\mathcal{M}$ as a subalgebra of $\mathcal{M}^\omega$ – this is the case for example when considering the central sequence algebra $\mathcal{M}^\omega \cap \mathcal{M}'$.

For a C^*-algebra A with $T(A)$ compact, we write A^ω for the reduced power of A in the norm $\|\cdot\|_{2,T(A)}$, which is defined in a way analogous to

Definition 5.2, and we write $T_\omega(A)$ for the limit traces on A^ω induced by $T(A)$. These uniform tracial reduced powers of C^*-algebras appear explicitly in connection with the Toms–Winter conjecture in [23] (working with ultrafilters ω) and in the abstract approach to classification [16] (working with the Fréchet filter).[41] In our formalism, the pair $\big(A^\omega, \overline{\mathrm{co}}(T_\omega(A))\big)$ is a tracially complete C^*-algebra and, in fact, if $(\mathcal{M}, X)$ is the tracial completion of A with respect to $T(A)$, then the canonical map $\alpha_X\colon A \to \mathcal{M}$ induces an isomorphism

$$\big(A^\omega, \overline{\mathrm{co}}(T_\omega(A))\big) \to (\mathcal{M}^\omega, X^\omega), \tag{5.15}$$

defined on representative sequences by $(a_n)_{n=1}^\infty \mapsto (\alpha_X(a_n))_{n=1}^\infty$. This follows from a more general result on the compatibility of tracial completions: Proposition 5.6) below. We first introduce some more notation.

Definition 5.5. Let $(A_n)_{n=1}^\infty$ be a sequence of C^*-algebras and let $X_n \subseteq T(A_n)$ be a compact convex set for each $n \geq 1$. We write

$$\prod^\omega A_n \coloneqq \prod_{n=1}^\infty A_n \big/ \big\{(a_n)_{n=1}^\infty : \lim_{n\to\omega} \|a_n\|_{2,X_n} = 0\big\}. \tag{5.16}$$

Let $\sum^\omega X_n$ be the closed convex hull of the limit traces on $\prod^\omega A_n$ defined by sequences $(\tau_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n$. In the case of a constant sequence, say $A_n = A$ and $X_n = X$, we write $A^\omega = \prod^\omega A_n$ and $X^\omega = \sum^\omega X_n$.

Proposition 5.6. *Let $(A_n)_{n=1}^\infty$ be a sequence of C^*-algebras, let $X_n \subseteq T(A_n)$ be a compact convex set for each $n \geq 1$, let $(\mathcal{M}_n, X_n)$ be the tracial completion of A_n with respect to X_n as in Definition 3.19, and let $\alpha_n\colon A_n \to \mathcal{M}_n$ be the canonical map for $n \geq 1$. Then $\big(\prod^\omega A_n, \sum^\omega X_n\big)$ is a tracially complete C^*-algebra and there is an isomorphism of tracially complete C^*-algebras*

$$\prod^\omega \alpha_n\colon \Big(\prod^\omega A_n, \sum^\omega X_n\Big) \xrightarrow{\cong} \Big(\prod^\omega \mathcal{M}_n, \sum^\omega X_n\Big) \tag{5.17}$$

defined at the level of representative sequences by $(a_n)_{n=1}^\infty \mapsto (\alpha_n(a_n))_{n=1}^\infty$.

Proof. It is easy to see that $\alpha \coloneqq \prod^\omega \alpha_n$ is a *-homomorphism and is isometric with respect to the uniform 2-norms, so it suffices to show that α is surjective. Fix $b \in \prod^\omega \mathcal{M}_n$ and represent b by a bounded sequence $(b_n)_{n=1}^\infty$. By the construction of the tracial completion $(\mathcal{M}_n, X_n)$, there is, for each $n \geq 1$, $a_n \in A_n$ such that $\|\alpha_n(a_n) - b_n\|_{2,X_n} < \frac{1}{n}$ and $\|a_n\| \leq \|b_n\|$. Then the sequence $(a_n)_{n=1}^\infty$ defines an element $a \in \prod^\omega A_n$ such that $\alpha(a) = b$. □

In the case of reduced powers, an important additional observation is that the isomorphism defined in Proposition 5.6 is well behaved on the central sequences.

Proposition 5.7. *Let A be a C^*-algebra, let $X \subseteq T(A)$ be compact and convex, let (A^ω, X^ω) be the uniform tracial reduced power, let $\iota : A \to A^\omega$ be given by constant sequences, let $(\mathcal{M}, X)$ be the tracial completion of A with*

[41] Various related constructions appeared earlier. A suitable quotient of the norm ultrapower, which is isomorphic to the tracial ultrapower, appeared in [68, 110], and all these ideas have their spiritual origins in Matui and Sato's work [75, 76]. Uniform tracial ultrapowers of W^*-bundles also were used in [10].

respect to X, and let $\alpha_X : A \to \mathcal{M}$ be the canonical map. The isomorphism $\alpha^\omega : (A^\omega, X^\omega) \to (\mathcal{M}^\omega, X^\omega)$ defined at the level of representative sequences by $(a_n)_{n=1}^\infty \mapsto (\alpha_X(a_n))_{n=1}^\infty$ satisfies

$$\alpha^\omega(A^\omega \cap \iota(A)') = \mathcal{M}^\omega \cap \mathcal{M}'. \tag{5.18}$$

More generally, for any $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}$, there is a $\|\cdot\|$-separable subset $S_0 \subseteq A$ such that $\alpha(A^\omega \cap \iota(S_0)') \subseteq \mathcal{M}^\omega \cap S'$.

Proof. By Proposition 5.6, α^ω is an isomorphism. Since

$$\alpha^\omega(\iota(A)) = \alpha_X(A) \subseteq \mathcal{M}^\omega, \tag{5.19}$$

we have

$$\alpha^\omega(A^\omega \cap \iota(A)') = \mathcal{M}^\omega \cap \alpha_X(A)'. \tag{5.20}$$

By Lemma 5.3, $\iota_{(\mathcal{M},X)}$ is an isometry for the respective uniform 2-norms. Hence $\alpha_X(A)$ is $\|\cdot\|_{2,X^\omega}$-dense in $\mathcal{M}$. In a tracially complete C^*-algebra, left and right multiplication by a fixed element are continuous with respect to the uniform 2-norm by (3.2). Therefore, any element of $\mathcal{M}^\omega$ that commutes with $\alpha_X(A)$ must also commute with $\mathcal{M}$. Hence $\mathcal{M}^\omega \cap \alpha_X(A)' = \mathcal{M}^\omega \cap \mathcal{M}'$.

Let $S \subseteq \mathcal{M}$ be $\|\cdot\|_{2,X}$-separable. Since $\alpha_X(A)$ is $\|\cdot\|_{2,X^\omega}$-dense in $\mathcal{M}$, there is a countable subset $S_0 \subseteq A$ such that $\overline{\alpha_X(S_0)}^{\|\cdot\|_{2,X}} \supseteq S$. Then

$$\alpha^\omega(A^\omega \cap \iota(S_0)') = \mathcal{M}^\omega \cap \alpha_X(S_0)' = \mathcal{M}^\omega \cap (\overline{\alpha_X(S_0)}^{\|\cdot\|_{2,X}})' \subseteq \mathcal{M}^\omega \cap S', \tag{5.21}$$

as claimed. □

As with C^*-norm reduced products (see [15, Lemma 3.9.5]), matrix amplifications commute with reduced products in the sense of the following theorem. The proof is essentially the same as in the C^*-algebra setting.

Proposition 5.8. *For any sequence $\big((\mathcal{M}_n, X_n)\big)_{n=1}^\infty$ of tracially complete C^*-algebras with reduced product $(\mathcal{M}, X)$ and $d \in \mathbb{N}$, there is a natural isomorphism*

$$\big(\mathcal{M} \otimes M_d, X \otimes \{\mathrm{tr_d}\}\big) \xrightarrow{\cong} \prod^\omega \big(\mathcal{M}_n \otimes M_d, X_n \otimes \{\mathrm{tr_d}\}\big). \tag{5.22}$$

defined on representing sequences by $(a_n)_{n=1}^\infty \otimes b \mapsto (a_n \otimes b)_{n=1}^\infty$.

Proof. The natural map

$$\phi\colon \Big(\prod_{n=1}^\infty \mathcal{M}_n\Big) \otimes M_d \to \prod_{n=1}^\infty (\mathcal{M}_n \otimes M_d) \tag{5.23}$$

is an isomorphism of C^*-algebras (see the proof of [15, Lemma 3.9.4], for example). Now, suppose $(\tau_n)_{n=1}^\infty \in \prod_{n=1}^\infty T(\mathcal{M}_n)$ and ω' is an ultrafilter containing ω, and let τ denote the trace on $\prod_{n=1}^\infty \mathcal{M}_n$ given by

$$\tau(a) \coloneqq \lim_{n\to\omega'} \tau_n(a_n), \qquad a = (a_n)_{n=1}^\infty \in \prod\nolimits_{n=1}^\infty \mathcal{M}_n. \tag{5.24}$$

Then for $a = (a_n)_{n=1}^\infty \in \prod_{n=1}^\infty (\mathcal{M}_n \otimes M_d)$, we have

$$\lim_{n\to\omega'} (\tau_n \otimes \mathrm{tr}_d)(a_n) = (\tau \otimes \mathrm{tr}_d)(\phi^{-1}(a)). \tag{5.25}$$

Therefore, ϕ descends to an isomorphism as in (5.22). □

We now turn to the question of traces on reduced products. It is well known that the tracial ultrapower of II_1 factor is also a II_1 factor. We have been unable to answer the general question for tracially complete C^*-algebras. Under the additional hypothesis of CPoU (see Section 6), a positive answer to the following question is given in Theorem 7.5. In particular, the answer is affirmative in the presence of property Γ (see Section 5.3) by Theorem 1.4.

Question 5.9. If $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of type II_1 factorial tracially complete C^*-algebras, is the reduced power $\big(\prod^{\omega} \mathcal{M}_n, \sum^{\omega} X_n\big)$ also factorial?

Remark 5.10. In an earlier draft of this paper, we asked this question without the type II_1 assumption. A counterexample in the type I setting was given by Vaccaro in [113]. It is based on a family of C^*-algebras introduced in [89]: $\mathcal{M}_n$ is the continuous sections of a bundle over the complex projective space $\mathbb{C}P^n$ with fibre M_2, and $X_n = T(\mathcal{M}_n)$.

We end this subsection with a reparameterisation theorem which will allow us to prove existence results for morphisms given both existence and uniqueness results for approximate morphisms. This should be regarded as an abstract version of the standard intertwining arguments commonly used in C^*-algebra theory. An operator norm version of this result appears in [46, Theorem 4.3], which, in turn, is a sequential version of [90, Proposition 1.3.7], attributed to Kirchberg.

If $(\mathcal{N}, Y)$ is a tracially complete C^*-algebra and $r\colon \mathbb{N} \to \mathbb{N}$ is a function such that $\lim_{n\to\infty} r(n) = \infty$, then there is an induced endomorphism r^* of $(\mathcal{N}^\infty, Y^\infty)$ given on representing sequences by

$$r^*\big((b_n)_{n=1}^{\infty}\big) \coloneqq (b_{r(n)})_{n=1}^{\infty}, \qquad (b_n)_{n=1}^{\infty} \in \ell^\infty(\mathcal{N}). \tag{5.26}$$

Equivalently, viewing $\ell^\infty(\mathcal{N})$ as bounded functions $\mathbb{N} \to \mathcal{N}$, r^* is the map induced by

$$\ell^\infty(\mathcal{N}) \to \ell^\infty(\mathcal{N})\colon f \mapsto f \circ r. \tag{5.27}$$

Note that the map r^* would typically not be well-defined if $(\mathcal{N}^\infty, Y^\infty)$ were replaced with an ultrapower – this is the main reason for working with general reduced products of tracially complete C^*-algebras.

In applications of the following theorem, the metric space S will typically be either a separable C^*-algebra with the operator norm or a tracially complete C^*-algebra which is separable in its uniform 2-norm.

Theorem 5.11 (Intertwining via reparameterisation)**.** *Let $(\mathcal{N}, Y)$ be a tracially complete C^*-algebra, let S be a separable metric space, and let $\phi\colon S \to \mathcal{N}^\infty$ be a $\|\cdot\|_{2,Y^\infty}$-continuous function. Suppose also that every unitary in $\mathcal{N}^\infty$ lifts to a unitary in $\ell^\infty(\mathcal{N})$. If for every function $r\colon \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} r(n) = \infty$, we have $r^* \circ \phi$ is approximately unitarily equivalent to ϕ, then there is a $\|\cdot\|_{2,Y^\infty}$-continuous function $\psi\colon S \to (\mathcal{N}, Y)$ such that $\iota_{(\mathcal{N},Y)} \circ \psi$ is unitarily equivalent to ϕ.*

The proof is a minor modification of the C^*-algebra version obtained in [46, Theorem 4.3], where the reduced power is taken in the C^*-norm. In the C^*-norm setting, the condition on unitaries is automatic since every

approximate unitary is close to a genuine unitary. We do not know if the condition on unitaries in $(\mathcal{N}^\infty, Y^\infty)$ is necessary. In all of our applications of Theorem 5.11, $(\mathcal{N}, Y)$ will satisfy CPoU (see Section 6), and hence the condition on unitaries will follow from Corollary 7.11.

Proof of Theorem 5.11. Define a sequence of functions $\phi_n \colon S \to Y$ by choosing a $\|\cdot\|$-preserving lift $(\phi_n(b))_{n=1}^\infty \in \ell^\infty(\mathcal{N})$ of $\phi(b) \in \mathcal{N}^\infty$ for every $b \in S$. We claim that for every finite subset $\mathcal{F} \subseteq S$, every $\epsilon > 0$, and every $m \in \mathbb{N}$, there exists $k \geq m$ such that for every $n \geq k$, there exists a unitary $u \in \mathcal{N}$ such that

$$\max_{a \in F} \|u^* \phi_n(a) u - \phi_k(a)\|_{2,Y} < \epsilon, \quad a \in \mathcal{F}. \tag{5.28}$$

Suppose for a contradiction that the claim is false. Then there exist a finite set $\mathcal{F}_0 \subseteq S$, $\epsilon_0 > 0$, $m_0 \in \mathbb{N}$, and a sequence of natural numbers $(n_k)_{k=m_0}^\infty$ with $n_k \geq k$ such that

$$\max_{a \in \mathcal{F}_0} \|u^* \phi_{n_k}(a) u - \phi_k(a)\|_{2,Y} \geq \epsilon_0 \tag{5.29}$$

for all unitaries $u \in \mathcal{N}$ and all $k \geq m_0$. Let $r : \mathbb{N} \to \mathbb{N}$ be given by $r(k) \coloneqq n_k$ for $k \geq m_0$ and define $r(k)$ arbitrarily for $k < m_0$. Then $\lim_{k\to\infty} r(k) = \infty$. By our hypothesis, ϕ and $r^* \circ \phi$ are approximately unitarily equivalent. Therefore, there exists a unitary $u \in \mathcal{N}^\infty$ such that

$$\|u^*(r^* \circ \phi)(a) u - \phi(a)\|_{2,Y^\infty} < \epsilon_0 \tag{5.30}$$

for all $a \in \mathcal{F}_0$. By our hypothesis, we may lift u to a sequence of unitaries $(u_k)_{k=1}^\infty$ in $\mathcal{N}$. By Lemma 5.3, we have

$$\limsup_{k\to\infty} \|u_k^* \phi_{n_k}(a) u_k - \phi_k(a)\|_{2,Y} < \epsilon_0 \tag{5.31}$$

for all $a \in \mathcal{F}_0$. Then (5.31) contradicts (5.29) for some sufficiently large $k \in \mathbb{N}$. This proves the claim. We now use this to construct ψ.

Let $(\mathcal{F}_i)_{i=1}^\infty$ be an increasing sequence of finite subsets whose union is dense in S. Applying the claim recursively, we obtain an increasing sequence of natural numbers $(k_n)_{n=1}^\infty$ and a sequence of unitaries $(u_n)_{n=1}^\infty$ in $\mathcal{N}$ such that

$$\|u_n^* \phi_{k_n}(a) u_n - \phi_{k_{n-1}}(a)\|_{2,Y} < 2^{-n} \tag{5.32}$$

for all $a \in \mathcal{F}_n$. Set $v_n \coloneqq u_n u_{n-1} \cdots u_1$ (and put $v_0 \coloneqq 1_{\mathcal{N}}$). Then, since $v_n = u_n v_{n-1}$ and $\|v_{n-1}\| \leq 1$, we have

$$\|v_n^* \phi_{k_n}(a) v_n - v_{n-1}^* \phi_{k_{n-1}}(a) v_{n-1}\|_{2,Y} < 2^{-n} \tag{5.33}$$

for all $n \in \mathbb{N}$ and $a \in \mathcal{F}_n$.

By construction, for every $a \in \bigcup_{i=1}^\infty \mathcal{F}_i$, the sequence $(v_n^* \phi_{k_n}(a) v_n)_{n=1}^\infty$ is $\|\cdot\|_{2,Y}$-Cauchy. Indeed, if $a \in \mathcal{F}_{i_0}$, then for $n > m > i_0$, we have

$$\|v_m^* \phi_{k_m}(a) v_m - v_n^* \phi_{k_n}(a) v_n\|_{2,Y} < \sum_{i=n+1}^{m} 2^{-i} < 2^{-n}. \tag{5.34}$$

Let $b \in S$ and $\epsilon > 0$. Since $\bigcup_{i=1}^\infty \mathcal{F}_i$ is dense in S and $\phi : S \to (\mathcal{N}^\infty, Y^\infty)$ is $\|\cdot\|_{2,Y^\infty}$-continuous, there exists $i_0 \in \mathbb{N}$ and some $a \in F_{i_0}$ with $\|\phi(b) - \phi(a)\|_{2,Y^\infty} < \epsilon$. Hence, by Lemma 5.3, we have

$$\limsup_{k\to\infty} \|\phi_k(b) - \phi_k(a)\|_{2,Y} < \epsilon. \tag{5.35}$$

Choose $N_1 \in \mathbb{N}$ with $\|\phi_k(b) - \phi_k(a)\|_{2,Y} < \epsilon$ for all $k > N_1$, and choose $N_2 \in \mathbb{N}$ with $\|v_m^*\phi_{k_m}(a)v_m - v_n^*\phi_{k_n}(a)v_n\|_{2,Y} < \epsilon$ for all $n, m > N_2$. Then a simple 3ϵ-argument gives that

$$\|v_m^*\phi_{k_m}(b)v_m - v_n^*\phi_{k_n}(b)v_n\|_{2,Y} < 3\epsilon \tag{5.36}$$

whenever $n, m > \max(N_1, N_2)$. Hence $(v_n^*\phi_{k_n}(b)v_n)_{n=1}^\infty$ is $\|\cdot\|_{2,Y}$-Cauchy for all $b \in S$. Moreover, we have $\|v_n^*\phi_{k_n}(b)v_n\| \leq \|\phi_n(b)\| \leq \|\phi(b)\|$ for all $n \in N$. Since $(\mathcal{N}, Y)$ is a tracially complete C^*-algebra, $(v_n^*\phi_{k_n}(b)v_n)_{n=1}^\infty$ converges in the uniform 2-norm for all $b \in S$. Hence we may define $\psi : S \to (\mathcal{N}, Y)$ by $\psi(b) \coloneqq \lim_{n\to\infty} v_n^*\phi_{k_n}(b)v_n$.

By construction, $\iota_{(\mathcal{N},Y)} \circ \psi$ is unitarily equivalent to $r^* \circ \phi$, where $r : \mathbb{N} \to \mathbb{N}$ is given by $r(n) \coloneqq k_n$, via the unitary represented by the sequence $(v_{k_n})_{n=1}^\infty$. Since $r^* \circ \phi$ is unitarily equivalent to ϕ by hypothesis (using Kirchberg's ϵ-test to replace approximate unitary equivalence with unitary equivalence), we conclude that $\iota_{(\mathcal{N},Y)} \circ \psi$ is unitarily equivalent to ϕ. □

5.2. **The McDuff property.** Central sequences in II_1 factors have been studied beginning with the foundational work of Murray and von Neumann ([83]) when they used them to distinguish the hyperfinite II_1 factor from any free group factor. The systematic study of central sequences was later instigated by two breakthrough results of McDuff: the existence of infinitely many (and in fact, uncountably many) non-isomorphic II_1 factors ([77, 78]) and, of particular relevance to us, her characterisation those II_1 factors $\mathcal{M}$ with separable predual which tensorially absorb the hyperfinite II_1 factor (i.e. $\mathcal{M} \cong \mathcal{M}\bar{\otimes}\mathcal{R}$) as those $\mathcal{M}$ with non-abelian central sequence algebras. Motivated by this last result, a II_1 factor $\mathcal{M}$ is said to have the *McDuff property* if there are approximately central unital embeddings of matrix algebras into $\mathcal{M}$.[42]

The McDuff property of II_1 factors has been of considerable interest to C^*-algebraists working in the classification programme because of its relation to $\mathcal{Z}$-stability at both conceptual and technical levels ([111, 75, 68, 110, 97, 22, 16]).

In this section, we generalise the McDuff property to the setting of tracially complete C^*-algebras and show that the McDuff property is equivalent to $\mathcal{R}$-stability in the separable case (Theorem 5.18). We begin with a formal definition of the McDuff property before establishing some useful technical reformulations and permanence properties.

Definition 5.12 (cf. [22, Definition 4.2])**.** Let $(\mathcal{M}, X)$ be a tracially complete C^*-algebra. We say that $(\mathcal{M}, X)$ is *McDuff* if for any $\|\cdot\|_{2,X}$-separable set $S \subseteq \mathcal{M}$ and $k \geq 1$, there is a unital embedding $M_k \to \mathcal{M}^\omega \cap S'$.[43]

[42]From a ultrapower viewpoint, McDuff's work shows that the central sequence algebra $\mathcal{M}^\omega \cap \mathcal{M}'$ is non-abelian if and only if it is type II_1 (and so admits unital embeddings of all matrix algebras). For II_1 factors with non-separable predual, the equivalence of tensorial absorption of $\mathcal{R}$ and the McDuff property no longer holds ([44, Theorem 1.5]). Experience has shown that the formulation in terms of approximately central matrix embeddings is the correct way to extend the McDuff property to the non-separable predual situation (where one should of course work with ultrapowers over uncountable sets). Indeed, the ultrapower $\mathcal{M}^\omega$ of a McDuff II_1 factor $\mathcal{M}$ with separable predual has the McDuff property but is not stable under tensoring by $\mathcal{R}$ – see [16, Footnote 65], which uses [50].

[43]Of course, when $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable, we can just take $S = \mathcal{M}$.

The *uniform McDuff property* was defined for a separable C^*-algebra A with $T(A)$ compact in [22, Definition 4.2] as the existence of unital embeddings $M_k \to A^\omega \cap A'$ for all $k \geq 1$. By Proposition 5.7, this is consistent with our definition.

Proposition 5.13. *Let A be a separable C^*-algebra with $T(A)$ compact. Then A is uniformly McDuff in the sense of* [22] *if and only if its tracial completion with respect to $T(A)$ is McDuff in the sense of Definition 5.12.*

We now establish some equivalent reformulations of the McDuff property. Note that the equivalence of (i) and (ii) in the following theorem shows that the McDuff property is independent of choice of free filter ω. The result is standard in the setting of II_1 factors, and the same techniques work in this context.

Proposition 5.14 (cf. [10, Proposition 3.11])**.** *Let $(\mathcal{M}, X)$ be a tracially complete C^*-algebra. The following are equivalent:*

(i) *$(\mathcal{M}, X)$ is McDuff.*
(ii) *Given a finite set $\mathcal{F} \subseteq \mathcal{M}$ and $\epsilon > 0$, there is a contraction $v \in \mathcal{M}$ such that*

$$\max_{a \in \mathcal{F}} \|[v, a]\|_{2,X} < \epsilon, \ \|v^*v + vv^* - 1_{\mathcal{M}}\|_{2,X} < \epsilon, \text{ and } \|v^2\|_{2,X} < \epsilon. \tag{5.37}$$

(iii) *For each $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}^\omega$, there is a contraction $v \in \mathcal{M}^\omega \cap S'$ such $v^2 = 0$ and $vv^* + vv^* = 1_{\mathcal{M}^\omega}$.*
(iv) *For each $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}^\omega$, there is $k \geq 2$ and a unital embedding $M_k \to \mathcal{M}^\omega \cap S'$.*
(v) *For each $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}^\omega$, there is a unital embedding $\mathcal{R} \to \mathcal{M}^\omega \cap S'$.*

Proof. (i)⇒(ii): Given a finite set $\mathcal{F} \subseteq \mathcal{M}$, fix a unital embedding $\phi\colon M_2 \to \mathcal{M}^\omega \cap \mathcal{F}'$. Let $(v_k)_{k=1}^\infty \subseteq \mathcal{M}$ be a sequence of contractions representing $\phi(e_{1,2})$ and set $v := v_k$ for some suitable index k.

(ii)⇒(iii): Let $\{s^{(i)} : i \in \mathbb{N}\}$ be a countable dense set in S. For each $i \in \mathbb{N}$, let $(s_n^{(i)})_{n=1}^\infty$ be a sequence representing $s^{(i)}$. For each $n \in \mathbb{N}$, set $\epsilon_n := 2^{-n}$ and $\mathcal{F}_n := \{s_n^{(i)} : i \leq n\}$. Let $v_n \in \mathcal{M}$ be given as in (ii) with $(\mathcal{F}_n, \epsilon_n)$ in place of $(\mathcal{F}, \epsilon)$. Then the sequence $(v_n)_{n=1}^\infty$ induces a contraction $v \in \mathcal{M}^\omega \cap S'$ with $v^2 = 0$ and $v^*v + vv^* = 1_{\mathcal{M}^\omega}$.

(iii)⇒(iv): Let $v \in \mathcal{M}^\omega \cap S'$ be as in (iii). Since v is a contraction and $v^2 = 0$, we have v^*v and vv^* are orthogonal positive contractions. As $v^*v + vv^* = 1_{\mathcal{M}}$, v^*v and vv^* are projections, and in particular, v is a partial isometry. It follows that the C^*-subalgebra generated by v is spanned by $1_{\mathcal{M}}$, v, v^*, and v^*v. As this subalgebra is non-commutative and has dimension at most four, it is isomorphic to M_2. This verifies (iv) with $k = 2$.

(iv)⇒(v): Fix the subset S from (v). Using (iv), let $k_1 \geq 2$ and let $\phi_1\colon M_{k_1} \to \mathcal{M}^\omega \cap S'$ be a unital embedding. Let $S_1 := C^*(\phi_1(M_{k_1}) \cup S)$ and use (iv) again to produce an integer $k_2 \geq 2$ and a unital embedding $\phi_2\colon M_{k_2} \to \mathcal{M}^\omega \cap S_1'$. Continuing inductively, there are integers $k_n \geq 2$ and unital embeddings $\phi_n\colon M_{k_n} \to \mathcal{M}^\omega \cap S'$ with commuting ranges. The ϕ_n

induce a unital embedding

$$\mathcal{R} \cong \overline{\bigotimes}_{n=1}^{\infty}(M_{k_n}, \mathrm{tr}_{k_n}) \to \mathcal{M}^\omega \cap S'. \tag{5.38}$$

(v)⇒(i): This follows as there is a unital embedding $M_k \to \mathcal{R}$ for all $k \geq 1$. □

Remark 5.15. Other equivalent properties to McDuffness can be given by strengthening Definition 5.12 to allow S to be any $\|\cdot\|_{2,X^\omega}$-separable subset of $\mathcal{M}^\omega$ or weakening any of (iii), (iv) or (v) to only have $S \subseteq \mathcal{M}$. In all cases the argument (for the non-trivial direction) is by reindexing.

For example, to go from the weakening of (iv) to (iv), given a $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}$, let $T \subseteq \mathcal{M}$ be a countable set consisting of the sequence entries of lifts of a countable dense subset of S. Then, given a unital embedding $\phi : M_k \to \mathcal{M}^\omega \cap T'$, we may lift ϕ to a sequence of c.p.c. maps $\phi_n : M_k \to \mathcal{M}$ which are asymptotically multiplicative, unital, and commute with T. An appropriate reindexing (ϕ_{m_n}) will provide a unital embedding $\phi : M_k \to \mathcal{M}^\omega \cap S'$.

As a corollary of the equivalence between (i) and (ii) above, we obtain the following permanence properties.

Corollary 5.16. *Inductive limits and reduced products of McDuff tracially complete C^*-algebras are McDuff.*

The following result gives a large supply of examples of McDuff tracially complete C^*-algebras. The result extends [23, Proposition 2.3], and the proof is very similar.

Proposition 5.17 (cf. [23, Proposition 2.3])**.** *If A is a separable $\mathcal{Z}$-stable C^*-algebra and $X \subseteq T(A)$ is a compact convex set, then $(\overline{A}^X, X)$ is a McDuff tracially complete C^*-algebra.*

Proof. Let ω be a free ultrafilter on $\mathbb{N}$, let $A_\omega \coloneqq \ell^\infty(A)/c_\omega(A)$ be the operator norm ultrapower of A, and let $\mathcal{M} \coloneqq \overline{A}^X$. By [67, Proposition 4.4(4)], there is a unital embedding $\phi\colon \mathcal{Z} \to (A_\omega \cap A')/A^\perp$, where

$$A^\perp \coloneqq \{a \in A_\omega : ab = ba = 0 \text{ for all } b \in A\}. \tag{5.39}$$

The natural map $\alpha_X\colon A \to \mathcal{M}$ induces a *-homomorphism $q\colon A_\omega \to \mathcal{M}^\omega$. Since α_X maps the unit ball of A onto a $\|\cdot\|_{2,X}$-dense subset of the unit ball of $\mathcal{M}$, we have q is surjective and $q(A_\omega \cap A') \subseteq \mathcal{M}^\omega \cap \mathcal{M}'$.

Following the proof of [23, Lemma 1.10], we show $q(A^\perp) = 0$. Assume $b \in A^\perp$ with $0 \leq b \leq 1$. Let $\tau \in X^\omega \subseteq T(\mathcal{M}^\omega)$ be given and note that

$$\tau \circ q|_A = \tau|_{\mathcal{M}} \circ \alpha_X. \tag{5.40}$$

Combining Corollary 3.29(i) with (5.40) shows that $\tau \circ q|_A$ has norm 1. Fix $\epsilon > 0$ and a positive contraction $e \in A$ with $\tau(q(e)) \geq 1 - \epsilon$. Since $b \in A^\perp$ and $e \in A$ are orthogonal positive contractions, their sum is also a positive contraction. Therefore,

$$0 \leq \tau(q(b)) = \tau(q(b+e)) - \tau(q(e)) \leq 1 - (1-\epsilon) = \epsilon. \tag{5.41}$$

Since $\epsilon > 0$ was arbitrary, $\tau(q(b)) = 0$ for all $\tau \in X^\omega$, and since $q(b) \geq 0$, this implies $q(b) = 0$. Hence $q(A^\perp) = 0$.

Let $\bar{q}\colon (A_\omega \cap A')/A^\perp \to \mathcal{M}^\omega \cap \mathcal{M}'$ be the *-homomorphism determined by q. Then $\bar{q}\circ\phi\colon \mathcal{Z} \to \mathcal{M}^\omega \cap \mathcal{M}'$ is a unital *-homomorphism. As $\mathcal{M}^\omega \cap \mathcal{M}'$ is a $\|\cdot\|_{2,X^\omega}$-closed, unital C^*-subalgebra of $\mathcal{M}^\omega$, we may view $\mathcal{M}^\omega \cap \mathcal{M}'$ as a tracially complete C^*-algebra as in the comments preceding Definition 3.11. Since $\mathcal{Z}$ has a unique trace τ and $\pi_\tau(\mathcal{Z})'' \cong \mathcal{R}$, Proposition 3.25 allows us to extend $\bar{q}\circ\phi$ to a unital embedding $\mathcal{R} \to \mathcal{M}^\omega \cap \mathcal{M}'$. Since M_k embeds unitally in $\mathcal{R}$ for all $k \in \mathbb{N}$, we see that $\mathcal{M}$ has the McDuff property. □

The following result gives a tensor product characterisation of McDuff tracially complete C^*-algebras in the separable setting, analogous to McDuff's original result for II_1 factors from [79]. Our proof is an adaptation of the argument found in [10, Proposition 3.11] in the setting of W^*-bundles. These follow the framework of the analogous results for absorption of a strongly self-absorbing C^*-algebra (see [111, Theorem 2.2] or [93, Theorem 7.2.2]), which are powered by an Elliott intertwining argument.

Theorem 5.18. *Suppose $(\mathcal{M}, X)$ is a tracially complete C^*-algebra such that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. Then $(\mathcal{M}, X)$ is uniformly McDuff if and only if $(\mathcal{M}, X)\bar{\otimes}(\mathcal{R}, \mathrm{tr}_\mathcal{R}) \cong (\mathcal{M}, X)$.*

Proof. Since $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable and McDuff, there exists a unital embedding $\phi : \mathcal{R} \to \mathcal{M}^\omega \cap \mathcal{M}'$. Take a set-theoretic lift of ϕ to a sequence $(\phi_n\colon \mathcal{R} \to \mathcal{M})_{n=1}^\infty$ of maps. We define commuting unital embeddings

$$\alpha, \beta : \mathcal{R} \to (\mathcal{M}\bar{\otimes}\mathcal{R})^\omega \cap (\mathcal{M} \otimes 1_\mathcal{R})' \tag{5.42}$$

at the level of representative sequences by

$$\alpha(x) \coloneqq (1_\mathcal{M} \otimes x)_{n=1}^\infty \quad \text{and} \quad \beta(x) \coloneqq (\phi_n(x) \otimes 1_\mathcal{R})_{n=1}^\infty, \tag{5.43}$$

respectively.

Let M_{2^∞} be the UHF algebra with supernatural number 2^∞, which we view as a $\|\cdot\|_{2,\mathrm{tr}_\mathcal{R}}$-dense subalgebra of $\mathcal{R}$. As M_{2^∞} is nuclear, there exists an embedding

$$\gamma : \mathcal{M} \otimes M_{2^\infty} \otimes M_{2^\infty} \to (\mathcal{M}\bar{\otimes}\mathcal{R})^\omega \tag{5.44}$$

such that

$$\gamma(a \otimes b \otimes c) = (a \otimes 1_\mathcal{R})\alpha(b)\beta(c) = (a \otimes b)\beta(c) \tag{5.45}$$

for all $a \in \mathcal{M}$ and $b, c \in M_{2^\infty}$. Since M_{2^∞} has a unique trace, it follows that γ is trace-preserving[44] and, accordingly, extends to an embedding of the tracial completion

$$\bar{\gamma} : \mathcal{M}\bar{\otimes}\mathcal{R}\bar{\otimes}\mathcal{R} \to (\mathcal{M}\bar{\otimes}\mathcal{R})^\omega \tag{5.46}$$

with

$$\bar{\gamma}(a \otimes b \otimes c) = (a \otimes 1_\mathcal{R})\alpha(b)\beta(c) = (a \otimes b)\beta(c) \tag{5.47}$$

for all $a \in \mathcal{M}$ and $b, c \in \mathcal{R}$.

[44]To ease notation, set $A \coloneqq \mathcal{M} \otimes \mathcal{M}_{2^\infty}$ and $(\mathcal{N}, Y) = (\mathcal{M}, X)\bar{\otimes}(\mathcal{R}, \mathrm{tr}_\mathcal{R})$, and view A as a $\|\cdot\|_{2,Y}$-dense subalgebra of $\mathcal{N}$. Then γ is a map $A \otimes \mathcal{M}_{2^\infty} \to (\mathcal{N} \otimes \mathcal{R})^\omega$ given by $\gamma(a \otimes c) = a\beta(c)$. It suffices to show $\gamma^*(\tau) = \tau|_A \otimes \mathrm{tr}_{2^\infty}$ for all $\tau \in Y^\infty$. Fix $\tau \in Y^\infty$ and positive $a \in A$. If $\tau(a) = 0$, then $\tau(a\beta(c)) = 0 = \tau(a)\mathrm{tr}_{2^\infty}(c)$ by the Cauchy–Schwarz inequality. If $\tau(a) \neq 0$, then $c \mapsto \tau(\gamma(a \otimes c))/\tau(a)$ is a trace on M_{2^∞} and hence is the unique trace tr_{2^∞}. Therefore, for all $a \in A$ and $c \in M_{2^\infty}$, we have $\tau(\gamma(a \otimes c)) = \tau(a)\mathrm{tr}_{2^\infty}(c)$.

Recall that $\mathcal{R}$ has an approximately inner half-flip, in the sense that there exists unitaries $(\bar{u}_m)_{m=1}^{\infty}$ in $\mathcal{R}\bar{\otimes}\mathcal{R}$ such that for all $b \in \mathcal{R}$

$$\lim_{m\to\infty} \|\bar{u}_m^*(b \otimes 1_{\mathcal{R}})\bar{u}_m - 1_{\mathcal{R}} \otimes b\|_{2,\mathrm{tr}_{\mathcal{R}}\otimes\mathrm{tr}_{\mathcal{R}}} = 0. \tag{5.48}$$

Moreover, since $\mathcal{R}\bar{\otimes}\mathcal{R}$ is a von Neumann algebra, each unitary $\bar{u}_m$ is of the form $\exp(2\pi i\bar{h}_m)$ for some self-adjoint $\bar{h}_m \in \mathcal{R}\bar{\otimes}\mathcal{R}$.

Set $u_m \coloneqq \bar{\gamma}(1_{\mathcal{M}} \otimes \bar{u}_m) \in (\mathcal{M}\bar{\otimes}\mathcal{R})^{\omega} \cap (\mathcal{M} \otimes 1_{\mathcal{R}})'$ (it is in this commutant because $\bar{\gamma}(\mathcal{M} \otimes 1_{\mathcal{R}} \otimes 1_{\mathcal{R}}) = \mathcal{M} \otimes 1_{\mathcal{R}}$). Then the two embeddings $\mathcal{M}\bar{\otimes}\mathcal{R} \to (\mathcal{M}\bar{\otimes}\mathcal{R})^{\omega}$ given by

$$a\otimes b \mapsto \bar{\gamma}(a\otimes b\otimes 1_{\mathcal{R}}) = a\otimes b \text{ and } a\otimes b \mapsto \bar{\gamma}(a\otimes 1_{\mathcal{R}}\otimes b) = (a\otimes 1)\beta(b) \tag{5.49}$$

are approximately unitary equivalent in $\|\cdot\|_{2,(X\otimes\{\mathrm{tr}_{\mathcal{R}}\})^{\omega}}$, via the sequence $(u_m)_{m=1}^{\infty}$. Since $\beta(b) \in (\mathcal{M} \otimes 1_{\mathcal{R}})^{\omega}$, it follows that

$$\lim_{m\to\infty} \inf\left\{\|u_m^* x u_m - z\|_{2,(X\otimes\{\mathrm{tr}_{\mathcal{R}}\})^{\omega}} : \begin{matrix} z \in (\mathcal{M} \otimes 1_{\mathcal{R}})^{\omega} \\ \|z\| \le 1 \end{matrix}\right\} = 0 \tag{5.50}$$

for all contractions $x \in \mathcal{M} \otimes \mathcal{R}$. We also have $u_m = \exp(2\pi i h_m)$, where $h_m = \bar{\gamma}(1_{\mathcal{M}} \otimes \bar{h}_m)$. Since we can lift the self adjoint $h_m \in (\mathcal{M}\bar{\otimes}\mathcal{R})^{\omega}$ to a sequence of self-adjoint elements $(h_{m,n})_{n=1}^{\infty}$ in $\ell^{\infty}(\mathcal{M}\bar{\otimes}\mathcal{R})$, we can find a sequence of unitaries $(u_{m,n})_{n=1}^{\infty}$ in $\mathcal{M} \otimes \mathcal{R}$ representing u_m.

We are now ready to construct an isomorphism $\psi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{R}$. Let $(x_k)_{k=1}^{\infty}$ be a $\|\cdot\|_{2,X}$-dense sequence in the unit ball of $\mathcal{M}$ and $(y_k)_{k=1}^{\infty}$ be a $\|\cdot\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}}$-dense sequence in the unit ball of $\mathcal{M} \otimes \mathcal{R}$. We shall iteratively produce unitaries w_k in $\mathcal{M} \otimes \mathcal{R}$ and contractions $z_k^{(j)} \in \mathcal{M}$ for $k \in \mathbb{N}$ and $1 \le j \le k$. The construction begins with $w_0 \coloneqq 1$. Fix $k \ge 1$ and suppose that $w_s, z_s^{(r)} \in \mathcal{M}$ for $1 \le r \le s < k$ have already been constructed.

By (5.50), there exists contractions $\bar{z}^{(1)}, \dots, \bar{z}^{(k)} \in (\mathcal{M}\otimes 1_{\mathcal{R}})^{\omega}$ and $m_k \in \mathbb{N}$ such that

$$\|u_{m_k}^* w_{k-1}^* y_j w_{k-1} u_{m_k} - \bar{z}_k^{(j)} \otimes 1_{\mathcal{R}}\|_{2,(X\otimes\{\mathrm{tr}_{\mathcal{R}}\})^{\omega}} < \frac{1}{k}. \tag{5.51}$$

for $j = 1, \dots, k-1$. Let $(u_{m_k,n})_{n=1}^{\infty}$ be a sequence of unitaries in $\mathcal{M} \otimes \mathcal{R}$ representing u_{m_k} and let $(z_{k,n}^{(j)})_{n=1}^{\infty}$ be sequences of contractions representing each $\bar{z}_k^{(j)}$. Since $u_{m_k} \in (\mathcal{M}\bar{\otimes}\mathcal{R})^{\omega} \cap (\mathcal{M} \otimes 1_{\mathcal{R}})'$, we can choose $n_k \in \mathbb{N}$ such that

$$\|u_{m_k,n_k}^* w_{k-1}^* y_j w_{k-1} u_{m_k,n_k} - z_{k,n_k}^{(j)} \otimes 1_{\mathcal{R}}\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}} < \frac{1}{k}, \tag{5.52}$$

$$\|[u_{m_k,n_k}, x_j \otimes 1_{\mathcal{R}}]\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}} < 2^{-k}, \tag{5.53}$$

for all $1 \le j \le k$, and

$$\|[u_{m_k,n_k}, z_s^{(r)} \otimes 1_{\mathcal{R}}]\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}} < 2^{-k}, \tag{5.54}$$

for all $1 \le r \le s < k$. Set $w_k \coloneqq w_{k-1}u_{m_k,n_k}$ and set $z_k^{(j)} \coloneqq z_{k,n_k}^{(j)}$ for $j = 1, \dots, k$.

For $j \in \mathbb{N}$, the sequence $(w_k(x_j \otimes 1_{\mathcal{R}})w_k^*)_{k=1}^{\infty}$ is $\|\cdot\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}}$-Cauchy by (5.53) as $\sum_{k=1}^{\infty} 2^{-k}$ converges. Since $(x_j)_{j=1}^{\infty}$ is a $\|\cdot\|_{2,X}$-dense sequence in the unit ball of $\mathcal{M}$, a 3ϵ-argument gives that $(w_k(a\otimes 1_{\mathcal{R}})w_k^*)_{n=1}^{\infty}$ is $\|\cdot\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}}$-Cauchy for all $a \in \mathcal{M}$. As $(w_k(a\otimes 1_{\mathcal{R}})w_k^*)_{n=1}^{\infty}$ is $\|\cdot\|$-bounded, we may define a map $\psi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{R}$ by $\psi(a) \coloneqq \lim_{k\to\infty} w_k(a \otimes 1_{\mathcal{R}})w_k^*$. Since the w_k

are unitaries, we see that ψ is an injective *-homomorphism and we have $(\tau \otimes \mathrm{tr}_{\mathcal{R}}) \circ \psi = \tau$ for all $\tau \in X$. Surjectivity follows from the $\|\cdot\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}}$-density of $(y_j)_{j=1}^{\infty}$ in the unit ball of $\mathcal{M}$, because for $k > j$ we have

$$\begin{aligned} \|y_j - \psi(z_k^{(j)})\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}} &\overset{(5.54)}{\leq} \sum_{r=k+1}^{\infty} 2^{-r} + \|w_k(z_{k,n_k}^{(j)} \otimes 1_{\mathcal{R}})w_k^*\| \\ &\overset{(5.52)}{<} \sum_{r=k+1}^{\infty} 2^{-r} + \frac{1}{k}, \end{aligned} \tag{5.55}$$

which converges to zero as $k \to \infty$. Indeed, as the unit ball of $\mathcal{M}$ is $\|\cdot\|_{2,X}$-complete and ψ preserves the uniform 2-norm, its image must be $\|\cdot\|_{2,X\otimes\{\mathrm{tr}_{\mathcal{R}}\}}$-closed. □

5.3. **Property Γ.** In its original formulation, a II_1 factor $\mathcal{M}$ has property Γ if there is an approximately central net of trace-zero unitaries in $\mathcal{M}$. Since $\mathcal{R} \cong \overline{\bigotimes}_{n=1}^{\infty}(\mathcal{R}, \mathrm{tr}_{\mathcal{R}})$, the hyperfinite II_1 factor has property Γ. Consequently, all McDuff factors have property Γ. On the other hand, Murray and von Neumann's 14ϵ-argument shows that the factors associated to free groups do not have property Γ ([83, Lemma 6.2.1]).

Dixmier extended the work of Murray and von Neumann, proving that property Γ was equivalent to the existence of systems of approximately central projections that sum to the unit ([32]). It is through this reformulation that most structural consequence of property Γ are obtained for II_1 factors (see [26, 27, 49, 91]), and so it was the basis for the definition of *uniform property* Γ for C^*-algebras introduced in [23] and studied further in [22].

Here, we define property Γ for tracially complete C^*-algebras. It is an immediate consequence of Proposition 5.7 and a reindexing argument (see Remark 5.24 below) that a C^*-algebra A with $T(A)$ compact has uniform property Γ in the sense of [23, Definition 2.1] if and only if its tracial completion with respect to $T(A)$ has property Γ as defined here (see Proposition 5.20).

Definition 5.19 (cf. [23, Definition 2.1])**.** Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra. We say that $(\mathcal{M}, X)$ has *property* Γ if for any $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}$ and any $k \in \mathbb{N}$ there exist projections $p_1, \ldots, p_k \in \mathcal{M}^{\omega} \cap S'$ summing to $1_{\mathcal{M}^{\omega}}$ such that

$$\tau(ap_i) = \frac{1}{k}\tau(a), \qquad a \in S,\ \tau \in X^{\omega},\ i = 1, \ldots, k. \tag{5.56}$$

Proposition 5.20. *Let A be a separable C^*-algebra with $T(A)$ compact. Then A has uniform property Γ as in* [23, Definition 2.1] *if and only if its tracial completion with respect to $T(A)$ has property Γ in the sense of Definition 5.19.*

Remark 5.21. We have chosen to restrict our definition of property Γ to the case of factorial tracially complete C^*-algebras. Although Definition 5.19 makes sense for non-factorial tracially complete C^*-algebras, in the absence of results, we are not confident that this would be appropriate definition outside of the factorial setting.

The following simple observation provides many examples of factorial tracially complete C^*-algebras with property Γ. Combining this with Proposition 5.17 shows that the tracial completion of a separable $\mathcal{Z}$-stable C^*-algebra with respect to a compact face of traces has property Γ (cf. [23, Proposition 2.3]).

Proposition 5.22. *If $(\mathcal{M}, X)$ is a factorial McDuff tracially complete C^*-algebra, then $(\mathcal{M}, X)$ satisfies property Γ.*

Proof. Let $S \subseteq \mathcal{M}$ be a $\|\cdot\|_{2,X}$-separable set and fix $k \geq 1$. Fix a unital embedding $\phi\colon M_k \to \mathcal{M}^\omega \cap S'$. If $a \in S$, then the function $\tau(a\phi(\,\cdot\,))$ is a tracial functional on M_k. Hence $\tau(a\phi(e_{ii})) = \tau(a\phi(e_{11}))$ for all $i = 1, \ldots, k$, and since ϕ is unital and $\sum_{i=1}^n e_{ii} = 1_{M_k}$, we have $\tau(a\phi(e_{ii})) = \frac{1}{k}\tau(a)$ for all $a \in S$. Thus the projections $p_i \coloneqq \phi(e_{ii})$, $i = 1, \ldots, k$, satisfy the conditions in Definition 5.19. □

The following proposition records several equivalent formulation of property Γ.

Proposition 5.23 (cf. [22, Proposition 2.3])**.** *For a factorial tracially complete C^*-algebra $(\mathcal{M}, X)$, the following are equivalent:*

(i) *$(\mathcal{M}, X)$ satisfies property Γ.*

(ii) *For every finite set $\mathcal{F} \subseteq \mathcal{M}$ and $\epsilon > 0$, there is a self-adjoint contraction $p \in \mathcal{M}$ such that for all $a \in \mathcal{F}$ and all $\tau \in X$,*

$$\|[p, a]\|_{2,X} < \epsilon,\ \|p - p^2\|_{2,X} < \epsilon,\ \text{and}\ \left|\tau(ap) - \frac{1}{2}\tau(a)\right| < \epsilon. \tag{5.57}$$

(iii) *There is $c \in (0, 1)$ such that for every $\|\cdot\|_{2,X}$-separable set $S \subseteq \mathcal{M}^\omega$, there is a projection $p \in \mathcal{M}^\omega \cap S'$ such that $\tau(ap) = c\tau(a)$ for all $a \in S$ and $\tau \in X^\omega$.*

(iv) *There exists a faithful trace $\sigma \in T(\mathbb{C}^2)$ such that for every $\|\cdot\|_{2,X}$-separable set $S \subseteq \mathcal{M}^\omega$, there is a unital embedding $\phi : \mathbb{C}^2 \to \mathcal{M}^\omega \cap S'$ such that $\tau(a\phi(x)) = \tau(a)\sigma(x)$ for all $a \in S$, $x \in \mathbb{C}^2$, and $\tau \in X^\omega$.*

(v) *For every $\|\cdot\|_{2,X}$-separable set $S \subseteq \mathcal{M}^\omega$, there is a *-homomorphism $\phi\colon L^\infty[0, 1] \to \mathcal{M}^\omega \cap S'$ such that*

$$\tau(a\phi(f)) = \tau(a)\int_0^1 f(t)dt \tag{5.58}$$

for all $a \in S$, $f \in L^\infty[0, 1]$, and $\tau \in X^\omega$.

Proof. (i)⇒(ii): Apply the definition of property Γ with $k \coloneqq 2$ and $S \coloneqq \mathcal{F}$ and take a suitable element of a representative sequence for one of the projections witnessing property Γ to give (ii).

(ii)⇒(iii): Let $\{s^{(i)} : i \in \mathbb{N}\}$ be a countable dense set in S. For each $i \in \mathbb{N}$, let $(s_n^{(i)})_{n=1}^\infty$ be a sequence representing $s^{(i)}$. For each $n \in \mathbb{N}$, set $\epsilon_n \coloneqq 2^{-n}$ and $\mathcal{F}_n \coloneqq \{s_n^{(i)} : i \leq n\}$. Let $p_n \in \mathcal{M}$ be given as in (ii) with $(\mathcal{F}_n, \epsilon_n)$ in place of $(\mathcal{F}, \epsilon)$. Then the sequence $(p_n)_{n=1}^\infty$ induces a projection $p \in \mathcal{M}^\omega \cap S'$ such that $\tau(ap) = \frac{1}{2}\tau(a)$ for all $a \in S$ and $\tau \in X^\omega$.

(iii)⇒(iv): Let $e_1, e_2 \in \mathbb{C}^2$ be the standard basis vectors and let $p \in \mathcal{M}^\omega \cap S'$ be the projection constructed in (iii). Define $\phi : \mathbb{C}^2 \to \mathcal{M}^\omega \cap S'$ by

$\phi(e_1) \coloneqq p$ and $\phi(e_2) \coloneqq 1_{\mathcal{M}^\omega} - p$. Define $\sigma(x_1, x_2) \coloneqq cx_1 + (1-c)x_2$. Then $\tau(a\phi(x)) = \tau(a)\sigma(x)$ for all $a \in S$, $x \in \mathbb{C}^2$, and $\tau \in X^\omega$.

(iv)⇒(v): Let $\phi_1 \colon \mathbb{C}^2 \to \mathcal{M}^\omega \cap S'$ be a unital embedding such that $\tau(a\phi_1(x)) = \tau(a)\sigma(x)$ for all $a \in S$, $x \in \mathbb{C}^2$, and $\tau \in X^\omega$. Then define $S_1 \coloneqq C^*(\phi_1(\mathbb{C}^2) \cup S)$ and use (iv) again to produce a unital embedding $\phi_2 \colon \mathbb{C}^2 \to \mathcal{M}^\omega \cap S_1'$ such that $\tau(a\phi_2(x)) = \tau(a)\sigma(x)$ for all $a \in S_1$, $x \in \mathbb{C}^2$, and $\tau \in X^\omega$.

Continuing inductively, there are unital embeddings $\phi_n \colon \mathbb{C}^2 \to \mathcal{M}^\omega \cap S'$ with commuting ranges such that the induced embedding

$$\phi \colon \bigotimes_{n=1}^{\infty} \mathbb{C}^2 \to \mathcal{M}^\omega \cap S' \tag{5.59}$$

satisfies $\tau(a\phi(x)) = \tau(a)\sigma^{\otimes\infty}(x)$ for all $a \in S$, $x \in \bigotimes_{n=1}^{\infty} \mathbb{C}^2$, and $\tau \in X^\omega$.

Since $\mathcal{M}^\omega \cap S'$ is tracially complete, ϕ extends to a *-homomorphism

$$\bar{\phi} \colon \pi_{\sigma^{\otimes\infty}}\Big(\bigotimes_{n=1}^{\infty} \mathbb{C}^2\Big)'' \to \mathcal{M}^\omega \cap S'. \tag{5.60}$$

Writing, $\bar{\sigma}$ for the trace on $\pi_{\sigma^{\otimes\infty}}\Big(\bigotimes_{n=1}^{\infty} \mathbb{C}^2\Big)''$ induced by $\sigma^{\otimes\infty}$, we have $\tau(a\bar{\phi}(x)) = \tau(a)\bar{\sigma}(x)$ for all $a \in S$, $x \in \pi_{\sigma^{\otimes\infty}}\big(\bigotimes_{n=1}^{\infty} \mathbb{C}^2\big)''$, and $\tau \in X^\omega$. The uniqueness of the standard probability space then provides an isomorphism

$$\pi_{\tau_0}\Big(\bigotimes_{n=1}^{\infty} \mathbb{C}^2\Big)'' \cong L^\infty[0,1] \tag{5.61}$$

which carries $\bar{\sigma}$ to the trace on $L^\infty[0,1]$ given by integration with respect to the Lebesgue measure.

(v)⇒(i): For $k \geq 1$ and $1 \leq i \leq k$, set $p_i \coloneqq \phi(\chi_{[(i-1)/k, i/k)})$. □

Remark 5.24. Using reindexing, as in Remark 5.15, we can see that Definition 5.12 is equivalent to an *a priori* stronger definition where S is allowed to be any $\|\cdot\|_{2,X^\omega}$-separable subset of $\mathcal{M}^\omega$. Similarly, each of (iii), (iv) and (v) is equivalent to an *a priori* weaker statement with $S \subseteq \mathcal{M}$.

The local characterisation of property Γ given in Proposition 5.23(ii) provides the following permanence property. Recall that factoriality is preserved by inductive limits by Proposition 3.34.

Proposition 5.25. *Property* Γ *is preserved by inductive limits of factorial tracial complete C^*-algebras.*

Remark 5.26. It is also true that the reduced product of a sequence of factorial tracially complete C^*-algebras with property Γ is both factorial and has property Γ, but we do not yet have the machinery to prove the factorial part of this claim.[45] Once the factorial issue is sorted, it is clear that the condition in Proposition 5.23(ii) is preserved by reduced products. See Corollary 7.7(ii).

[45]Recall that in general it is not true that a reduced product of factorial tracially complete C^*-algebras is factorial (see Remark 5.10).

The definition of property Γ requires that one can tracially divide all elements of a tracially complete C^*-algebra in an approximately central fashion, whereas for a II_1 factor $\mathcal{M}$ it suffices just to divide the unit (i.e. (5.56) only needs to hold for $a = 1_{\mathcal{M}^\omega}$). In [22, Proposition 3.2], several of the present authors observed that such a result is also true for separable C^*-algebras with Bauer tracial simplices. The same holds in the tracially complete setting with a very similar proof.

Proposition 5.27 (cf. [22, Corollary 3.2])**.** *Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra such that X is a Bauer simplex. Suppose that for any $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}$ and $k \in \mathbb{N}$, there exists pairwise orthogonal projections $p_1, \dots, p_k \in \mathcal{M}^\omega \cap S'$ with $\tau(p_i) = 1/k$ for all $i = 1, \dots, k$. Then $(\mathcal{M}, X)$ has property Γ.*

Proof. First assume that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable, so we can take $\mathcal{S} \coloneqq \mathcal{M}$ in the statement. Using Remark 5.24, it will suffice to show that we automatically have

$$\tau(ap_i) = \frac{1}{k}\tau(a), \qquad a \in \mathcal{M},\ \tau \in X^\omega,\ i = 1, \dots k. \tag{5.62}$$

By [10, Proposition 3.9], $\mathcal{M}^\omega$ is a W^*-bundle over $(\partial_e X)^\omega$ (in the notation defined before [10, Proposition 3.9]),[46] so it suffices to prove (5.62) for $\tau \in (\partial_e X)^\omega$, which means it suffices to assume that τ is a limit trace coming from a sequence (τ_n) of traces in $\partial_e X$ and an ultrafilter $\omega' \supseteq \omega$. In this case, since $\partial_e X$ is compact, $\tau|_{\mathcal{M}} \in \partial_e X$ (as it is the ω'-limit of the τ_n). Define $\sigma\colon \mathcal{M} \to \mathbb{C}$ by $\sigma(x) \coloneqq \tau(xp_i)$. This is a positive tracial functional such that $\sigma(1_{\mathcal{M}}) = \frac{1}{k}$; since τ is an extreme point in $T(\mathcal{M})$, we have $\sigma = \frac{1}{k}\tau$, which amounts to (5.62).

In the non-separable case, we can use the fact that factoriality is separably inheritable (Theorem A.3(i)) to see that for any $\|\cdot\|_{2,X}$-separable subset S of $\mathcal{M}$, there exists a $\|\cdot\|_{2,X}$-separable factorial tracially complete subalgebra $\mathcal{M}_0$ of $\mathcal{M}$ which contains S and satisfies the hypotheses of the proposition and, therefore, has property Γ. Consequently, $\mathcal{M}$ has property Γ as the projections witnessing property Γ for $\mathcal{M}_0$ also witness property Γ for $\mathcal{M}$. □

In the setting of W^*-bundles, abstracting the results in [110, 97, 68], an amenable factorial type II_1 tracially complete C^*-algebra $(\mathcal{M}, X)$ with $\partial_e X$ compact and finite dimensional turns out to have property Γ. This will be contained in the forthcoming work by the third- and fifth-named authors, which will show that a factorial W^*-bundle over a finite dimensional space has property Γ when viewed as a tracially complete C^*-algebra if and only if each fibre has property Γ as a von Neumann algebra. We show here that this holds in the zero dimensional setting.

Proposition 5.28. *If $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra such that $\partial_e X$ is compact and totally disconnected and $\pi_\tau(\mathcal{M})''$ has property Γ for each $\tau \in \partial_e X$, then $(\mathcal{M}, X)$ has property Γ. In particular, if $(\mathcal{M}, X)$ is an amenable type II_1 factorial tracially complete C^*-algebra such that $\partial_e X$ is compact and totally disconnected, then $(\mathcal{M}, X)$ has property Γ.*

[46] In [10], ω is taken to be an ultrafilter, but the same considerations apply to any free filter on $\mathbb{N}$.

Proof. Note that the second statement follows from the first. Indeed, if $(\mathcal{M}, X)$ is as in the second statement, then for all $\tau \in \partial_e X$, $\pi_\tau(\mathcal{M})''$ is semidiscrete by Theorem 1.2 and is a II_1 factor by Proposition 3.14. It then follows from [29, Corollary 2.2] that $\pi_\tau(\mathcal{M})''$ has property Γ.

We now prove the first statement. As $\partial_e X$ is compact, $\mathcal{M}$ can be viewed as a W^*-bundle over $K \coloneqq \partial_e X$ by Theorem 3.37. Let $C(K) \subseteq \mathcal{M}$ be the corresponding inclusion. Fix a finite set $\mathcal{F} \subset \mathcal{M}$, $\epsilon > 0$, and $k \in \mathbb{N}$. By Proposition 5.27, it suffices to show that there are positive contractions $p_1, \ldots, p_k \in \mathcal{M}$ such that

(i) $\|[p_i, a]\|_{2,X} < \epsilon$ for all $a \in \mathcal{F}$ and $i = 1, \ldots, k$,
(ii) $\|p_i - p_i^2\|_{2,X} < \epsilon$ for all $i, \ldots, k$,
(iii) $\|p_i p_{i'}\|_{2,X} < \epsilon$ for $i, i' = 1, \ldots, k$ with $i \neq i'$, and
(iv) $|\tau(p_i) - 1/k| < \epsilon$ for all $\tau \in X$ and $i = 1, \ldots, k$.

For each $\tau \in K$, as the II_1 factor $\pi_\tau(\mathcal{M})''$ satisfies property Γ, there are mutually orthogonal projections $\bar{p}_{1,\tau}, \ldots, \bar{p}_{k,,\tau} \in \pi_\tau(\mathcal{M})''$ with trace $1/k$ such that

$$\|[\bar{p}_{i,\tau}, \pi_\tau(a)]\|_{2,\tau} < \epsilon, \qquad a \in \mathcal{F},\ i = 1, \ldots, k. \tag{5.63}$$

Using Kaplansky's density theorem to approximate each $\bar{p}_{\tau,i}$ by a positive contraction in $\mathcal{M}$ in the norm $\|\cdot\|_{2,\tau}$ together with the compactness of K, there is a finite open cover $U_1, \ldots, U_n$ of K and positive contractions $p_{i,j} \in \mathcal{M}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n$ such that

(i′) $\|[p_{i,j}, a]\|_{2,U_j} < \epsilon$ for all $a \in \mathcal{F}$, $i = 1, \ldots, k$, and $j = 1, \ldots, n$,
(ii′) $\|p_{i,j} - p_{i,j}^2\|_{2,U_j} < \epsilon$ for all $i, \ldots, k$ and $j = 1, \ldots, n$,
(iii′) $\|p_{i,j} p_{i',j}\|_{2,U_j} < \epsilon$ for $i, i' = 1, \ldots, k$ with $i \neq i'$ and $j = 1, \ldots, n$, and
(iv′) $|\tau(p_{i,j}) - 1/k| < \epsilon$ for all $\tau \in U_j$ and $i = 1, \ldots, k$.

As K is totally disconnected, after refining the open cover, we may assume that the sets $U_1, \ldots, U_n$ form a clopen partition of K. Then the elements

$$p_i \coloneqq \sum_{j=1}^{n} \chi_{U_j} p_{i,j} \in \mathcal{M}, \qquad j = 1, \ldots n, \tag{5.64}$$

are positive contractions satisfying (i)–(iv). □

It remains open whether the approximately central division of the unit as in Proposition 5.27 implies property Γ outside the setting of Bauer simplices ([22, Question 3.5]). This question is just as valid in the tracially complete framework.

Question 5.29. Does Proposition 5.27 hold for all factorial tracially complete C^*-algebras (i.e. without assuming that X is a Bauer simplex)?

Immediately following [22, Proposition 5.10], the second-, third-, and last two named authors asserted that all the Villadsen algebras of the first type – namely the examples from [114], which were used by Toms to give striking counterexamples to Elliott's original classification conjecture ([109]) – have property Γ. These algebras are so-called diagonal AH algebras (i.e. inductive limits of homogeneous algebras of a particularly nice form), and they prove a fertile testing ground for comparing regularity properties of very different

natures, with the analysis in [112] leading to a positive solution to the Toms–Winter conjecture on this class of algebras. Uniform tracial completions of diagonal AH algebras satisfy the central divisibility-of-the-unit hypothesis of Proposition 5.27 (this is [22, Proposition 5.10]) and the assertion that Villadsen algebras of the first type have property Γ was based on [112, Section 8] (where it is suggested, based on computations in [108, Theorem 4.1], that these have Bauer simplices of traces).[47] However recent results of Elliott, Li, and Niu ([36, Theorem 4.5]) show that, in fact, the trace space of Villadsen algebras of the first type is the Poulsen simplex (whenever the algebra is not $\mathcal{Z}$-stable) – maximally far from being Bauer! Accordingly, the argument in [22] is not valid, and it does not seem straightforward to determine whether the Villadsen type I algebras have property Γ.

Question 5.30. Do the uniform tracial completions of the (non-$\mathcal{Z}$-stable) Villadsen algebras of the first type have property Γ?

The connecting maps in the inductive limit construction of a Villadsen type I algebra involve a combination of coordinate projections and point evaluations, the latter of which ensure simplicity of the inductive limit. In the case of a non-$\mathcal{Z}$-stable Villadsen type I algebra A, the point evaluations are relatively sparse (see [112, Theorem 3.4 and Lemma 5.1]). As such they do not affect the structure of the tracial completion of A. For this reason, the heart of Question 5.30 lies in determining whether Γ holds for the tracially complete inductive limit of the trivial W^*-bundles

$$C([0,1]) \to C([0,1]^2, M_2) \to C([0,1]^4, M_4) \to \cdots , \tag{5.65}$$

where each stage is equipped with all its traces, and the connecting maps are given as the direct sum of the two coordinate projections.

6. Complemented partitions of unity

This section formally introduces the notation of CPoU for factorial tracially complete C^*-algebras as discussed in Section 1.4. The definition of CPoU is given in Section 6.1. Some examples of tracially complete C^*-algebras with CPoU can be found in Section 6.2, and permanence properties are discussed in Section 6.3. The proof Theorem 1.4 (property Γ implies CPoU) will be given in Section 6.4.

6.1. **Formal definition and its reformulations.**

The following definition extends the notion of CPoU introduced in [23] in the C^*-setting to factorial tracial complete C^*-algebras. More precisely, a separable C^*-algebra A with $T(A)$ compact has CPoU in the sense of [23] if and only its tracial completion with respect to $T(A)$ has CPoU in the sense of Definition 6.1 below; this is immediate from the definition and Proposition 5.7, which identifies the uniform 2-norm central sequence algebra of A with that of its uniform tracial completion.[48] Recall our convention that (unless specified otherwise)

[47] In [22], the second-, third-, and last two named authors incorrectly referenced the non-existent [109, Theorem 4.1] rather than [108, Theorem 4.1].

[48] In [23], CPoU was only defined in the separable setting, but we will not make this restriction here.

ω denotes a free filter on $\mathbb{N}$. It will follow from the local characterisation in Proposition 6.2 that Definition 6.1 does not depend on the choice of filter ω.

Definition 6.1 (cf. [23, Definition 3.1])**.** Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra. We say that $(\mathcal{M}, X)$ has *complemented partitions of unity* (CPoU) if for any $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}$, any family $a_1, \dots, a_k$ of positive elements in $\mathcal{M}$, and any scalar

$$\delta > \sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(a_i), \tag{6.1}$$

there exist orthogonal projections $p_1, \dots, p_k \in \mathcal{M}^\omega \cap S'$ summing to $1_{\mathcal{M}^\omega}$ such that

$$\tau(a_i p_i) \leq \delta \tau(p_i), \qquad \tau \in X^\omega,\ i = 1, \dots, k. \tag{6.2}$$

We note that when $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable, it suffices to take $S \coloneqq \mathcal{M}$. As with property Γ, we have have chosen to restrict the definition to the setting of factorial tracially complete C^*-algebras since it is not clear if this definition is appropriate outside of the factorial setting.

The following proposition provides an equivalent formulation of CPoU that avoids the language of reduced products. Moreover, a reindexing argument shows that Definition 6.1 is equivalent to an *a priori* stronger definition where $a_1, \dots, a_k \in \mathcal{M}^\omega$ and S can be any $\|\cdot\|_{2,X}$-separable subspace of $\mathcal{M}^\omega$.

Proposition 6.2. *Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra. Then the following are equivalent:*

(i) *$(\mathcal{M}, X)$ has CPoU;*

(ii) *for every finite set $\mathcal{F} \subseteq \mathcal{M}$, $\epsilon > 0$, $a_1, \dots, a_k \in \mathcal{M}_+$, and*

$$\delta > \sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(a_i), \tag{6.3}$$

there exist orthogonal positive contractions $e_1, \dots, e_k \in \mathcal{M}$ such that

$$\begin{aligned} &\Big\| \sum_{j=1}^{k} e_j - 1_{\mathcal{M}} \Big\|_{2,X} < \epsilon, \\ &\max_{x \in \mathcal{F}} \|[e_i, x]\|_{2,X} < \epsilon, \quad i = 1, \dots, k, \\ &\tau(a_i e_i) - \delta \tau(e_i) < \epsilon, \quad \tau \in X,\ i = 1, \dots, k; \end{aligned} \tag{6.4}$$

(iii) *for every $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}$, $a_1, \dots, a_k \in \mathcal{M}^\omega$, and*

$$\delta > \sup_{\tau \in X^\omega} \min_{1 \leq i \leq k} \tau(a_i), \tag{6.5}$$

there exist orthogonal projections $p_1, \dots, p_k \in \mathcal{M}^\omega \cap S'$ summing to $1_{\mathcal{M}^\omega}$ such that

$$\tau(a_i p_i) \leq \delta \tau(p_i), \qquad \tau \in X^\omega,\ i = 1, \dots, k. \tag{6.6}$$

Proof. (i)⇒(ii): Let $\epsilon > 0$, let $a_1, \dots, a_k \in \mathcal{M}_+$, let $\mathcal{F} \subseteq \mathcal{M}$ be a finite subset, and let $\delta > 0$ be such that (6.1) holds. Since $(\mathcal{M}, X)$ has CPoU, we obtain orthogonal projections $p_1, \dots, p_k \in \mathcal{M}^\omega \cap \mathcal{F}'$ summing to $1_{\mathcal{M}^\omega}$ such that (6.2) hold. We can lift $p_1, \dots, p_k \in \mathcal{M}^\omega$ to orthogonal positive contractions $e^{(1)}, \dots, e^{(k)} \in \ell^\infty(\mathcal{M})$ by [1, Proposition 2.6] (this is a special case of projectivity of cones over finite dimensional C^*-algebras; see [72,

Corollary 3.8]). Write $(e_j^{(i)})_{j=1}^{\infty}$ for a sequence of positive contractions representing $e^{(i)}$ for $i = 1, \ldots, k$. Note that for each $j \in \mathbb{N}$, $e_j^{(1)}, \ldots, e_j^{(k)}$ are orthogonal positive contractions. Since the p_i sum to $1_{\mathcal{M}^\omega}$ and commute with $\mathcal{F}$, we have

$$\limsup_{j\to\omega} \left\| e_j^{(1)} + \cdots + e_j^{(k)} - 1_{\mathcal{M}} \right\|_{2,X} = 0, \tag{6.7}$$

and

$$\limsup_{j\to\omega} \|[e_j^{(i)}, x]\|_{2,X} = 0, \quad x \in \mathcal{F},\ i = 1, \ldots, k. \tag{6.8}$$

Moreover, we must have

$$\limsup_{j\to\omega} \sup_{\tau\in X} \left(\tau\big(a_i e_j^{(i)}\big) - \delta\tau\big(e_j^{(i)}\big) \right) \leq 0, \quad i = 1, \ldots, k, \tag{6.9}$$

as otherwise we could choose a sequence of traces $(\tau_j)_{j=1}^{\infty}$ in X and an ultrafilter $\omega' \supseteq \omega$ such that the corresponding limit trace τ does not satisfy $\tau(a_i p_i) \leq \delta\tau(p_i)$. Therefore, taking $e_i \coloneqq e_j^{(i)}$ for suitable choice of j, the orthogonal positive contractions $e_1, \ldots, e_k$ will satisfy (6.4).

(ii)⇒(iii): Let $S \subseteq \mathcal{M}^\omega$ be a $\|\cdot\|_{2,X^\omega}$-separable subset and fix a countable dense subset $\{s^{(i)} : i \in \mathbb{N}\}$ of S. For each $i \in \mathbb{N}$, let $(s_j^{(i)})_{j=1}^{\infty}$ be a sequence representing $s^{(i)}$. Let $a_1, \ldots, a_k \in \mathcal{M}^\omega$ be positive contractions in $\mathcal{M}^\omega$. Let $(a_j^{(i)})_{j=1}^{\infty}$ be a sequence of positive contractions representing a_i for $i = 1, \ldots, k$. Let $\delta > 0$ satisfy (6.5). Then we have

$$\limsup_{j\to\omega} \sup_{\tau\in X} \min_{1\leq i\leq k} \tau\big(a_j^{(i)}\big) < \delta. \tag{6.10}$$

Therefore, there exists a set $J \in \omega$ such that for all $j \in J$, we have

$$\sup_{\tau\in X} \min_{1\leq i\leq k} \tau\big(a_j^{(i)}\big) < \delta. \tag{6.11}$$

For each $j \in \mathbb{N}$, set $\epsilon_j \coloneqq 2^{-j}$ and $\mathcal{F}_j \coloneqq \{s_j^{(i)} : i \leq j\}$. Applying (ii) for each $j \in J$, we obtain orthogonal positive contractions $e_j^{(1)}, \ldots, e_j^{(k)}$ such that

$$\begin{aligned} &\Big\| \sum_{i=1}^{k} e_j^{(i)} - 1_{\mathcal{M}} \Big\|_{2,X} < \epsilon_j, \\ &\max_{x\in\mathcal{F}_j} \big\| \big[e_j^{(i)}, x\big] \big\|_{2,X} < \epsilon_j, \quad i = 1, \ldots, k,\ x \in \mathcal{F}_j \\ &\tau\big(a_j^{(i)} e_j^{(i)}\big) - \delta\tau\big(e_j^{(i)}\big) < \epsilon_j, \quad \tau \in X,\ i = 1, \ldots, k. \end{aligned} \tag{6.12}$$

For $j \notin J$, we may choose $e_j^{(1)}, \ldots, e_j^{(k)}$ arbitrarily. Let $p_i \in \mathcal{M}^\omega$ be the element represented by $\big(e_j^{(i)}\big)_{j=1}^{\infty}$. Then (6.12) ensures that the p_i sum to $1_{\mathcal{M}^\omega}$, commute with S, and satisfy (6.6). Since $p_1, \ldots, p_k$ are orthogonal positive contractions summing to the identity, they are in fact projections.

(iii)⇒(i): If $S \subseteq \mathcal{M}$ is $\|\cdot\|_{2,X}$-separable, then viewed as a subset of $\mathcal{M}^\omega$, it is $\|\cdot\|_{2,X^\omega}$-separable. If $a_1, \ldots, a_k \in \mathcal{M}_+$, we have

$$\sup_{\tau\in X^\omega} \min_{1\leq i\leq k} \tau(a_i) = \sup_{\tau\in X} \min_{1\leq i\leq k} \tau(a_i). \tag{6.13}$$

Therefore, (i) is a special case of (iii). □

6.2. **First examples and non-examples.** The main source of example of factorial tracially complete C^*-algebras with CPoU come from Theorem 1.4, which, as a special case, implies the uniform tracial completion of a separable $\mathcal{Z}$-stable Cj-algebra with respect to a compact face of traces will have CPoU. In this subsection, we shall investigate some more elementary examples of tracially complete C^*-algebras $(\mathcal{M}, X)$ with CPoU in the case when X is a Bauer simplex. We will also prove a special case of Theorem 1.4 showing that Γ implies CPoU in the case of W^*-bundles (Proposition 6.7), which is easier and more conceptual than the general result taken up in Section 6.4). We also provide examples of type II_1 factorial tracially complete C^*-algebras without CPoU in Example 6.6.

In the setting of W^*-bundles with totally disconnected base space, a strong form a CPoU holds: one can take the partition of unity to be central as opposed to approximately central.

Proposition 6.3. *Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra such that $\partial_e X$ is compact and totally disconnected. Then for all $a_1, \dots, a_k \in \mathcal{M}_+$ and*

$$\delta > \sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(a_i), \tag{6.14}$$

there are projections $p_1, \dots, p_k \in Z(\mathcal{M})$ summing to $1_{\mathcal{M}}$ such that $\tau(a_i p_i) \leq \delta\tau(p_i)$ for all $\tau \in X$ and $i = 1, \dots, k$. In particular, $(\mathcal{M}, X)$ has CPoU.

Proof. It suffices to prove the first part since for any $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}$, we have $Z(\mathcal{M}) \subseteq \mathcal{M}^\omega \cap S'$.

For $i = 1, \dots, k$, define

$$U_i \coloneqq \{\tau \in \partial_e X : \tau(a_i) < \delta\}, \tag{6.15}$$

so that $U_1, \dots, U_k$ is an open cover of $\partial_e X$. By total disconnectedness, we may partition $\partial_e X$ into clopen sets $V_1, \dots, V_k$ such that $V_i \subseteq U_i$ for $i = 1, \dots, k$.

As $\mathcal{M}$ is factorial, X is a closed face of $T(\mathcal{M})$ and so is a Choquet simplex in its own right, which is then Bauer from the assumption that $\partial_e X$ is compact. By Theorem 3.37, there is a natural embedding $C(\partial_e X) \subseteq Z(\mathcal{M})$. Let us define

$$p_i \coloneqq \chi_{V_i} \in C(\partial_e X) \subseteq Z(\mathcal{M}). \tag{6.16}$$

For $\tau \in \partial_e X$, we have $p_i \in C(\partial_e X) \subseteq \mathcal{M}$ is in the multiplicative domain of τ, and hence $\tau(a_i p_i) = \tau(a_i)\tau(p_i)$. Since $\tau(p_i) = 0$ for $\tau \in \partial_e X \setminus V_i$ and $\tau(a_i) < \delta$ for $\tau \in V_i$, we have $\tau(a_i p_i) \leq \delta\tau(p_i)$ for all $i = 1, \dots, k$ and $\tau \in \partial_e X$. It then follows from the Krein–Milman theorem that $\tau(a_i p_i) \leq \delta\tau(p_i)$ for all $i = 1, \dots, k$ and $\tau \in X$. □

A noteworthy special case of Proposition 6.3 is when $\mathcal{M}$ is a finite dimensional C^*-algebra and $X = T(\mathcal{M})$. This is easily seen to be a tracially complete C^*-algebra as the norms $\|\cdot\|_{2,X}$ and $\|\cdot\|$ are equivalent. Although this example is fairly trivial, it played a notable role in the proof of [23, Theorem 3.8], which could be viewed as bootstrapping CPoU from the finite dimensional setting (where it is easy to verify) to the nuclear setting

with the aid of uniform property Γ. The proof of Theorem 1.4 will similarly make use of the following special case of Proposition 6.3.

Corollary 6.4. *If $\mathcal{M}$ is a finite von Neumann algebra, then the factorial tracially complete C^*-algebra $\big(\mathcal{M}, T(\mathcal{M})\big)$ has CPoU. In fact, if $a_1, \ldots, a_k \in \mathcal{M}_+$ and*

$$\delta > \sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(a_i), \tag{6.17}$$

then there are projections $p_1, \ldots, p_k \in Z(\mathcal{M})$ summing to $1_{\mathcal{M}}$ such that $\tau(a_i p_i) \leq \delta \tau(p_i)$ for all $\tau \in T(\mathcal{M})$ and $i = 1, \ldots, k$.

Proof. As all traces on $\mathcal{M}$ factor through the centre-valued trace $\mathcal{M} \to Z(\mathcal{M})$ (Proposition 2.8(i)), composition with the centre-valued trace produces a homeomorphism from the Gelfand spectrum of $Z(\mathcal{M})$ to $\partial_e T(\mathcal{M})$. In particular, $\partial_e T(\mathcal{M})$ is compact and totally disconnected (in fact hyperstonean) as $Z(\mathcal{M})$ is a von Neumann algebra, and so the result follows from Proposition 6.3. □

In the next proposition, we consider what it means for a trivial W^*-bundle to have CPoU. (See Example 3.7 for the definition of trivial W^*-bundles.) The heuristic idea is that for a W^*-bundle over a space that is not totally disconnected, the approximately central projections in Definition 6.1 cannot come from the centre of the algebra, and hence any constructions of approximately central projections must make use of central sequences in the fibres. In particular, if there are no central sequences in the fibres, then the W^*-bundle must not have CPoU.

Proposition 6.5. *Let $(\mathcal{M}, X)$ be the trivial W^*-bundle over a compact Hausdorff space K with fibre a finite factor $\mathcal{N}$. If $(\mathcal{M}, X)$ has CPoU, then either K is totally disconnected or $\mathcal{N}$ satisfies property Γ.*

Proof. Suppose $(\mathcal{M}, X)$ satisfies CPoU and K is not totally disconnected. Let $\tau_{\mathcal{N}}$ be the trace on $\mathcal{N}$ and let $S \subseteq \mathcal{N}$ be a $\|\cdot\|_{2,\tau_{\mathcal{N}}}$-separable subset. We work to show that there is a projection $q \in \mathcal{N}^\omega \cap S'$ with trace $1/2$. This will show $\mathcal{N}$ satisfies property Γ – this is a standard von Neumann algebra result; its also special case of (i)⇒(iii) of Proposition 5.23, for example.

Fix a point $x \in K$ and an open neighbourhood $U \subseteq K$ of x such that there is no clopen neighbourhood of x contained in U. Let $a_1 \in C(K) \subseteq \mathcal{M}$ be such that $0 \leq a_1 \leq 1_{\mathcal{M}}$, $a_1(x) = 1$, and $\operatorname{supp}(a_1) \subseteq U$, and set $a_2 \coloneqq 1_{\mathcal{M}} - a_1$. Then

$$\sup_{\tau \in X} \min\{\tau(a_1), \tau(a_2)\} \leq \frac{1}{2} < \frac{2}{3}. \tag{6.18}$$

Let ω be a free ultrafilter on $\mathbb{N}$ and view $S \subseteq \mathcal{M}$ as constant functions. Since $(\mathcal{M}, X)$ has CPoU, there are projections $p_1, p_2 \in \mathcal{M}^\omega \cap S'$ such that $p_1 + p_2 = 1_{\mathcal{M}^\omega}$ and

$$\tau(a_i p_i) \leq \frac{2}{3} \tau(p_i), \qquad i = 1, 2,\ \tau \in X^\omega. \tag{6.19}$$

For $i = 1, 2$, let $(p_{i,n})_{n=1}^\infty \subseteq \mathcal{M}$ be a sequence of positive contractions representing p_i. If $\tau \in X^\omega$ is the limit trace determined by the constant sequence $(\tau_{\mathcal{N}} \circ \mathrm{ev}_x)_{n=1}^\infty \subseteq X$. Then $\tau(a_1) = 1$, and hence $\tau(a_2) = 0$. Since

$a_2 \geq 0$, the Cauchy–Schwarz inequality implies $\tau(a_2 p_1) = 0$. So $\tau(a_1 p_1) = \tau(p_1)$. Now, (6.19) implies $\tau(p_1) = 0$. Therefore,

$$\lim_{n\to\omega} \tau_{\mathcal{N}}(p_{1,n}(x)) = 0. \tag{6.20}$$

Similarly, consider a sequence $(y_n)_{n=1}^{\infty} \subseteq K \setminus U$, and let $\tau \in X^\omega$ be the limit trace defined by the sequence $(\tau_{\mathcal{N}} \circ \mathrm{ev}_{y_n})_{n=1}^{\infty} \subseteq X$. Then $\tau(a_2) = 1$, and as above, this shows $\tau(p_2) = 0$. This implies $\tau(p_1) = 1$. As this holds for all sequences $(y_n)_{n=1}^{\infty} \subseteq K \setminus U$, it follows that

$$\lim_{n\to\omega} \inf_{y\in K\setminus U} \tau_{\mathcal{N}}(p_{1,n}(y)) = 1. \tag{6.21}$$

Define

$$N \coloneqq \Big\{ n \in \mathbb{N} : \tau_{\mathcal{N}}(p_{1,n}(x)) < \frac{1}{2}, \inf_{y\in K\setminus U} \tau(p_{1,n}(y)) > \frac{1}{2} \Big\}, \tag{6.22}$$

and note that $N \in \omega$ by (6.20) and (6.21). For each $n \in N$, there is a $y_n \in K$ such that $\tau_{\mathcal{N}}(p_{1,n}(y)) = 1/2$ since otherwise

$$\Big\{ y \in K : \tau_{\mathcal{N}}(p_{1,n}(y)) < \frac{1}{2} \Big\} \subseteq K \tag{6.23}$$

is a clopen neighbourhood of x contained in U. For $n \in N$, set $q_n \coloneqq p_{1,n}(y_n) \in \mathcal{N}$ and for $n \in \mathbb{N} \setminus N$, set $q_n = 0 \in \mathcal{N}$. Then the sequence $(q_n)_{n=1}^{\infty}$ defines a projection $q \in \mathcal{N}^\omega \cap S'$ with trace $1/2$. □

Proposition 6.5 provides us with explicit examples of tracially complete C^*-algebras that do not have CPoU.

Example 6.6. For any $n \geq 2$, the trivial W^*-bundle $C_\sigma([0,1], L(\mathbb{F}_n))$ does not have CPoU. Indeed, the II_1 factors $L(\mathbb{F}_n)$ do not have property Γ ([83]; see also [100, Theorem A.7.2]), and hence the result follows from Proposition 6.5.

The converse of Proposition 6.5 is also true. If $(\mathcal{M}, X)$ is a trivial W^*-bundle whose fibre is a II_1 factor with property Γ, then it is easy to show $(\mathcal{M}, X)$ has property Γ, and then Theorem 1.4 will show that $(\mathcal{M}, X)$ has CPoU. In fact, in the W*-bundle case we can give a more direct proof that property Γ implies CPoU. In light of Theorem 3.37, the following result is just the special case of Theorem 1.4 when X is a Bauer simplex. As the proof is more conceptual in this special case, we include it here.

Proposition 6.7. *Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra coming from a W^*-bundle over the space $K = \partial_e X$. Suppose $(\mathcal{M}, X)$ has property Γ. Then $(\mathcal{M}, X)$ has CPoU.*

Proof. Fix an ultrafilter ω on $\mathbb{N}$. By the Krein–Milman theorem and Proposition 3.3, we have that $\|a\|_{2,X} = \|a\|_{2,K}$ for all $a \in \mathcal{M}$. Moreover, writing K^ω for the weak*-closure of the set of limit traces coming from a sequence of traces in K, we have $\|a\|_{2,X^\omega} = \|a\|_{2,K^\omega}$ for all $a \in \mathcal{M}^\omega$.

Let $a_1, \ldots, a_k \in \mathcal{M}_+$ and $\delta > 0$ satisfy

$$\delta > \sup_{\tau\in K} \min_{1\leq i\leq k} \tau(a_i). \tag{6.24}$$

Let $S \subseteq \mathcal{M}$ be $\|\cdot\|_{2,X}$-separable subset, which we may assume contains $a_1, \ldots, a_k$. Let $U_i \coloneqq \{\tau \in K : \tau(a_i) < \delta\}$ for $i = 1, \ldots, k$. Then $\{U_1, \ldots, U_k\}$

is an open cover of K. Let $g_1, \dots, g_k \colon K \to [0,1]$ be a continuous partition of unity subordinate to this open cover. Set $h_0 \coloneqq 0$ and $h_i \coloneqq \sum_{j=1}^{i} g_i$ for $i = 1, \dots, k$.

Since $\mathcal{M}$ has property Γ, by (i)⇒(v) of Proposition 5.23, there exists a *-homomorphism $\phi \colon L^\infty[0,1] \to \mathcal{M}^\omega \cap S'$ such that

$$\tau(a\phi(f)) = \tau(a)\mathrm{tr}_{\mathrm{Leb}}(f) \tag{6.25}$$

for all $a \in S$, $f \in L^\infty[0,1]$ and $\tau \in X^\omega$, where $\mathrm{tr}_{\mathrm{Leb}}$ is integration with respect to the Lebesgue measure. Since $\mathcal{M}$ is a W^*-bundle there is a canonical copy of $C(K)$ in $Z(\mathcal{M})$. As the image of ϕ commutes with $C(K) \subseteq Z(\mathcal{M}^\omega)$, we have an induced *-homomorphism $\psi : C(K) \otimes L^\infty[0,1] \to \mathcal{M}^\omega \cap S'$. Let τ be a limit trace given by a sequence of traces $(\tau_n)_{n=1}^\infty$ in K. Then, since every $\tau_n \in K$ restricts to a point evaluation on $C(K)$, we see that

$$\begin{aligned} \tau(a\psi(g \otimes f)) &= \lim_{n\to\omega} \tau_n(a) g(\tau_n) \mathrm{tr}_{\mathrm{Leb}}(f) \\ &= \lim_{n\to\omega} \tau_n(a) \mathrm{tr}_{\mathrm{Leb}}(g(\tau_n) f) \end{aligned} \tag{6.26}$$

for all $a \in S$, $g \in C(K)$, and $f \in L^\infty[0,1]$. Identifying $C(K) \otimes L^\infty[0,1]$ with $C(K, L^\infty[0,1])$, we have

$$\tau(a\psi(F)) = \lim_{n\to\omega} \tau_n(a) \mathrm{tr}_{\mathrm{Leb}}(F(\tau_n)) \tag{6.27}$$

for all $a \in S$ and $F \in C(K, L^\infty[0,1])$.

In particular, we have $\|\psi(F)\|_{2,K^\omega} = \sup_{\tau \in K} \|F(\tau)\|_{2,\tau_{\mathrm{Leb}}}$. It follows that ψ extends to an embedding $C_\sigma(K, (L^\infty[0,1], \tau_{\mathrm{Leb}})) \to \mathcal{M}^\omega \cap S'$ (which we also denote by ψ) of the trivial W^*-bundle with fibre $L^\infty[0,1]$ such that (6.27) holds for all $a \in S$ and $F \in C_\sigma(K, (L^\infty[0,1], \tau_{\mathrm{Leb}}))$.

Using $\chi_{[h_{i-1}, h_i]} \colon K \to L^\infty([0,1])$ to denote the function $\tau \mapsto \chi_{[h_{i-1}(\tau), h_i(\tau)]}$, this function is in $C_\sigma(K, (L^\infty[0,1], \tau_{\mathrm{Leb}}))$, and so we can define

$$p_i \coloneqq \psi(\chi_{[h_{i-1}, h_i]}), \qquad i = 1, \dots, k. \tag{6.28}$$

Then $p_1, \dots, p_k$ are orthogonal projections in $\mathcal{M}^\omega \cap S'$ summing to $1_{\mathcal{M}^\omega}$. Let τ be a limit trace given by a sequence of traces $(\tau_n)_{n=1}^\infty$ in K. Then

$$\begin{aligned} \tau(a_i p_i) &= \lim_{n\to\omega} \tau_n(a_i) \mathrm{tr}_{\mathrm{Leb}}(\chi_{[h_{i-1}, h_i]}(\tau_n)) \\ &= \lim_{n\to\omega} \tau_n(a_i) g_i(\tau_n) \\ &\leq \lim_{n\to\omega} \delta g_i(\tau_n) \\ &= \delta \tau(p_i), \end{aligned} \tag{6.29}$$

where the inequality in the third line holds as for any $\tau_n \in K$, either $\tau_n(a_i) < \delta$ or $g_i(\tau_n) = 0$ because $g_1, \dots g_k$ is a partition of unity subordinate to $\{U_1, \dots, U_n\}$. An application of the Krein–Milman theorem shows that (6.29) holds for all $\tau \in X^\omega$. Therefore, $\mathcal{M}$ has CPoU. □

6.3. **Permanence properties of CPoU.** We will show CPoU is preserved by inductive limits and matrix amplifications. CPoU is also preserved under reduced products of tracially complete C^*-algebras, but the proof will be deferred to Section 7 (see Remark 6.13). We begin by showing that CPoU passes to "quotients" in the sense that restricting to a closed face of traces (and completing in the corresponding uniform 2-norm) preserves CPoU (cf.

Corollary 6.10). We isolate the following lemma, which will go into the proof of Proposition 6.9, as well as being recycled in Appendix A.

Lemma 6.8. *If $\mathcal{M}$ is a unital C^*-algebra, $Y \subseteq \mathcal{T}(\mathcal{M})$ is a closed face, and $S \subseteq \mathcal{M}$ is $\|\cdot\|$-separable, then there is a separable unital C^*-algebra $A \subseteq \mathcal{M}$ containing S such that*

$$Y_A \coloneqq \{\tau|_A : \tau \in Y\} \subseteq T(A) \tag{6.30}$$

is a closed face in $T(A)$.

Proof. We will construct a sequence $(A_n)_{n=1}^{\infty}$ of separable unital C^*-subalgebras of $\mathcal{M}$ containing S and self-adjoint elements $a_n \in A_{n+1}$ such that for each $n \geq 1$,

(i) $A_n \subseteq A_{n+1}$,
(ii) every trace on A_n extends to a trace on $\mathcal{M}$,
(iii) $\tau(a_n) \geq 0$ for all $\tau \in T(\mathcal{M})$, and
(iv) if $Y_n \coloneqq \{\tau|_{A_n} : \tau \in Y\}$, then for all $\tau \in T(\mathcal{M})$, we have $\tau(a_n) = 0$ if and only if $\tau|_{A_n} \in Y_n$.

The construction is by induction on n. Using [86, Lemma 9], there is a separable unital C^*-algebra $A_1 \subseteq \mathcal{M}$ which contains S such that every trace on A_1 extends to a trace on $\mathcal{M}$. With Y_1 as in (iv), note that Y_1 is closed[49] and $T(A_1)$ is metrisable, and hence $T(A_1) \setminus Y_1$ is an F_σ-set. By the continuity of the restriction map $T(\mathcal{M}) \to T(A_1)$, we have

$$U_1 \coloneqq \{\tau \in T(\mathcal{M}) : \tau|_{A_1} \notin Y_1\} \tag{6.31}$$

is an F_σ-set in $T(\mathcal{M})$ disjoint from Y. By Theorem 2.4, there is a continuous affine function $f\colon T(\mathcal{M}) \to [0,1]$ such that $f(\tau) = 0$ for all $\tau \in Y$ and $f(\tau) > 0$ for all $\tau \in U_1$. By Proposition 2.7, there is a self-adjoint $a_1 \in \mathcal{M}$ with $f(\tau) = \tau(a_1)$ for all $\tau \in T(\mathcal{M})$. Now repeat the argument with $A_1 \cup \{a_1\}$ in place of S to obtain A_2 and a_2, and continue in this fashion.

Let $A \subseteq \mathcal{M}$ be the $\|\cdot\|$-closure of the union of the A_n. Note that every trace $\tau \in T(A)$ extends to a trace on $\mathcal{M}$ – indeed, for each $n \geq 1$, $\tau|_{A_n}$ extends to a trace $\widetilde{\tau}_n \in T(\mathcal{M})$, and then any weak*-limit point of the traces $\widetilde{\tau}_n$ will extend τ. Now suppose $\sigma, \rho \in T(A)$ and that $\tau \coloneqq \frac{1}{2}(\sigma + \rho) \in Y_A$. Since σ and ρ extend to traces on $\mathcal{M}$, (iii) implies $\sigma(a_n), \rho(a_n) \geq 0$ for all $n \geq 1$. Since $\tau \in Y_A$, we have $\tau|_{A_n} \in Y_n$ for all $n \geq 1$, and hence $\tau(a_n) = 0$ for all $n \geq 1$. Therefore, $\sigma(a_n) = 0$ for all $n \geq 1$. Using again that σ extends to a trace on $\mathcal{M}$, it follows from (iv) that $\sigma|_{A_n} \in Y_n$ for all $n \geq 1$. Let $\widetilde{\sigma}_n \in Y$ be a trace with $\widetilde{\sigma}_n|_{A_n} = \sigma|_{A_n}$. Let $\widetilde{\sigma}$ be a weak*-limit point of the $\widetilde{\sigma}_n$; then $\widetilde{\sigma} \in Y$ and $\widetilde{\sigma}|_A = \sigma$. So $\sigma \in Y_A$. Similarly, $\rho \in Y_A$, and hence Y_A is a face in $T(A)$. The argument of Footnote 49 shows Y_A is closed in $T(A)$. □

Proposition 6.9. *If $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU and $Y \subseteq X$ is a closed face, then $\left(\overline{\mathcal{M}}^Y, Y\right)$ is factorial and satisfies CPoU.*

[49] Given a net $(\tau_i|_A)$ in Y_1 arising from a net of traces $\tau_i \in T(\mathcal{M})$ with $\tau_i|_A \to \sigma \in T(A)$, it follows that any weak*-limit point $\tau \in T(\mathcal{M})$ of the τ_i is an extension of σ, so $\sigma \in Y_1$.

Proof. Note that $\mathcal{N} \coloneqq \overline{\mathcal{M}}^Y$ is factorial by Proposition 3.23(iv). Define $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ be the canonical map. Since the range of ϕ is $\|\cdot\|_{2,Y}$-dense in $\mathcal{N}$, it suffices to show that for every $\delta > 0$, $a_1, \ldots, a_n \in \mathcal{M}$, and $S \subseteq \mathcal{M}$ such that S is $\|\cdot\|$-separable and

$$\sup_{\tau \in Y} \min_{1 \leq i \leq n} \tau(\phi(a_i)) < \delta, \tag{6.32}$$

there are projections $p_1, \ldots, p_n \in \mathcal{M}^\omega \cap S'$ such that

$$\sum_{j=1}^{n} \phi^\omega(p_j) = 1_{\mathcal{N}^\omega} \qquad \text{and} \qquad \tau(\phi^\omega(a_i p_i)) < \delta \tau(\phi^\omega(p_i)) \tag{6.33}$$

for all $i = 1, \ldots, n$ and $\tau \in Y^\omega$.

Given δ, S and $a_1, \ldots, a_n$ as above, define

$$C \coloneqq \{\tau \in T(\mathcal{M}) : \min_{1 \leq i \leq n} \tau(a_i) \geq \delta\} \subseteq T(\mathcal{M}). \tag{6.34}$$

Then C is a compact convex subset of $T(\mathcal{M})$ disjoint from the compact convex set $Y \subseteq T(\mathcal{M})$. By the Hahn–Banach theorem, there is $\alpha > 0$ and self-adjoint $b \in \mathcal{M}$ such that $\tau(b) > \alpha$ for all $\tau \in Y$ and $\tau(b) < \alpha$ for all $\tau \in C$. Replacing b with $b + (\|b\| + 1)1_{\mathcal{M}}$ and α with $\alpha + \|b\| + 1$, we may assume that $b \in \mathcal{M}_+$. Finally, by scaling b, there exists $a_0 \in \mathcal{M}_+$ such that $\tau(a_0) > \delta$ for all $\tau \in Y$ and $\tau(a_0) < \delta$ for all $\tau \in C$.

If $\tau \in X$ such that $\tau(a_i) \geq \delta$ for all $i = 1, \ldots, n$, then $\tau \in C$, and hence $\tau(a_0) = f(\tau) < \delta$. Therefore,

$$\sup_{\tau \in X} \min_{0 \leq i \leq n} \tau(a_i) < \delta. \tag{6.35}$$

By Lemma 6.8, there is a separable unital C^*-algebra $A \subseteq \mathcal{M}$ containing S and a_0 such that the set $Y_A \subseteq T(A)$ defined in (6.30) is a face in $T(A)$. Now, since $(\mathcal{M}, X)$ has CPoU, there are projections $p_0, p_1, \ldots, p_n \in \mathcal{M}^\omega \cap A'$ such that

$$\sum_{j=0}^{n} p_j = 1_{\mathcal{M}^\omega} \qquad \text{and} \qquad \tau(a_i p_i) \leq \delta \tau(p_i) \tag{6.36}$$

for all $i = 0, \ldots, n$ and $\tau \in X^\omega$.

To verify (6.33), it suffices to show $\phi^\omega(p_0) = 0$. Suppose this fails. Then there is a trace $\tau \in Y^\omega$ with $\tau(\phi^\omega(p_0)) \neq 0$. Define $\sigma_0\colon A \to \mathbb{C}$ by

$$\sigma_0(a) \coloneqq \frac{\tau(\phi^\omega(ap_0))}{\tau(\phi^\omega(p_0))}, \qquad a \in A. \tag{6.37}$$

Then σ_0 is a trace on A that is dominated by a multiple of $\tau \circ \phi|_A \in Y_A$, and since Y_A is a face in $T(A)$, we have $\sigma_0 \in Y_A$. By the definition of Y_A, there is a trace $\sigma \in Y$ such that $\sigma|_A = \sigma_0$. Then $\sigma(a_0) > \delta$ by the choice of a_0. But this contradicts (6.36) with $i = 0$ and with $\tau \circ \phi^\omega$ in place of τ. Therefore, $\phi^\omega(p_0) = 0$, as required. □

As recalled at the beginning of Section 6.1, a C^*-algebra A with $T(A)$ has compact has CPoU in the sense of [23] if and only if, in our notation, the uniform tracial completion of A with respect to $T(A)$ has CPoU as a tracially complete C^*-algebra. As a consequence of Proposition 6.9, CPoU passes to quotients of C^*-algebras.

Corollary 6.10. *Let A be a C^*-algebra with $T(A)$ compact and let $I \trianglelefteq A$ be an ideal. Then $T(A/I)$ is compact. Further, if A satisfies CPoU in the sense of* [23]*, then so does A/I.*

Proof. We may identify $T(A/I)$ with the closed face Y of $T(A)$ consisting of traces on A vanishing on I so that $T(A/I)$ is certainly compact. Apply Proposition 6.9 after making this identification – note that we are implicitly using Proposition 3.23(v) to identify the uniform tracial completion of $\overline{A}^{T(A)}$ with respect to $T(A/I)$ with the uniform tracial completion of A/I with respect to $T(A/I)$. □

Now we show that CPoU is preserved under direct limits. While the corresponding result for the McDuff property and property Γ followed easily from the approximate characterisations of these properties, this permanence property for CPoU will take more work. The new difficulty arises from the supremum over traces in (6.1) since traces on the finite terms of an inductive system will not generally extend to traces on the limit. We overcome this challenge with the following lemma.

Lemma 6.11. *Let $(\mathcal{M}, X) \coloneqq \varinjlim \big((\mathcal{M}_n, X_n), \phi_n^{n+1}\big)$ be an inductive limit of tracially complete C^*-algebras and write the inductive limit morphisms as $\phi_n^\infty \colon (\mathcal{M}_n, X_n) \to (\mathcal{M}, X)$. Suppose $k, m_0 \geq 1$, $\delta > 0$, and $b_1, \ldots, b_k \in \mathcal{M}_{m_0}$ are positive with*

$$\sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(\phi_{m_0}^\infty(b_i)) < \delta. \tag{6.38}$$

Then there exists $m \geq m_0$ such that

$$\sup_{\tau \in X_m} \min_{1 \leq i \leq k} \tau(\phi_{m_0}^m(b_i)) < \delta, \tag{6.39}$$

where $\phi_{m_0}^m \coloneqq \phi_{m-1}^m \circ \cdots \circ \phi_{m_0}^{m_0+1}$.

Proof. Suppose the conclusion of the lemma does not hold. For each $m \geq m_0$, since X_m is compact, the failure of (6.39) implies there exists $\tau_m \in X_m$ such that

$$\min_{1 \leq i \leq k} \tau_m(\phi_{m_0}^m(b_i)) \geq \delta. \tag{6.40}$$

Fix a free ultrafilter ω on $\mathbb{N}$, and for every $m \geq m_0$, define

$$\sigma_m \coloneqq \lim_{n \to \omega} \tau_n \circ \phi_m^n \in X_m, \tag{6.41}$$

which exists as X_m is compact and Hausdorff. If $n \geq m_2 \geq m_1 \geq m_0$, then $\tau_n \circ \phi_{m_1}^n = \tau_n \circ \phi_{m_2}^n \circ \phi_{m_1}^{m_2}$ for all $n \geq m_2$. Since ω is a free ultrafilter, we obtain $\sigma_{m_1} = \sigma_{m_2} \circ \phi_{m_1}^{m_2}$. Therefore, the sequence $(\sigma_m)_{m=m_0}^\infty$ induces a trace $\sigma \in X$ such that $\sigma_m = \sigma \circ \phi_m^\infty$ for all $m \geq m_0$.

By (6.40), $\min_{1 \leq i \leq k} \sigma_{m_0}(b_i) \geq \delta$. Hence, $\min_{1 \leq i \leq k} \sigma(\phi_{m_0}^\infty(b_i)) \geq \delta$, contrary to (6.38). This completes the proof. □

With the above technical lemma in hand, we now proceed with the proof that CPoU passes to inductive limits.

Proposition 6.12. *Let $(\mathcal{M}, X) \coloneqq \varinjlim \big((\mathcal{M}_n, X_n), \phi_n^{n+1}\big)$ be an inductive limit of factorial tracially complete C^*-algebras. If $(\mathcal{M}, X)$ has CPoU for all $n \geq 1$, then $(\mathcal{M}, X)$ has CPoU as well.*[50]

Proof. Write the inductive limit morphisms as $\phi_n^\infty \colon (\mathcal{M}_n, X_n) \to (\mathcal{M}, X)$. We will use (ii)⇒(i) of Proposition 6.2. As $\bigcup_{m=1}^\infty \phi_m^\infty(\mathcal{M}_m)$ is $\|\cdot\|_{2,X}$-dense in $\mathcal{M}$, we may assume the finite set $\mathcal{F} \coloneqq \{x_1, \dots, x_l\}$ in the statement of Proposition 6.2(ii) is contained in $\bigcup_{m=1}^\infty \phi_m^\infty(\mathcal{M}_m)$. By rescaling, we may assume that each x_j is a contraction. Fix $\epsilon > 0$ and consider a family $a_1, \dots, a_k$ of positive elements in $\mathcal{M}$ and a scalar

$$\delta > \sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(a_i). \tag{6.42}$$

By the $\|\cdot\|_{2,X}$-density of $\bigcup_{m=1}^\infty \phi_m^\infty(\mathcal{M}_m)$ in $\mathcal{M}$, there are an integer $m_0 \geq 1$ and positive $b_1, \dots, b_k \in \mathcal{M}_{m_0}$ such that

$$\|a_i - \phi_{m_0}^\infty(b_i)\|_{2,X} < \epsilon/3. \tag{6.43}$$

Since $\tau \circ \phi_{m_0}^\infty \in X_{m_0}$ for all $\tau \in X$, this implies

$$\sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(\phi_{m_0}^\infty(b_i)) < \delta + \epsilon/3. \tag{6.44}$$

Accordingly, by Lemma 6.11, we can find $m \geq m_0$ large enough so that

$$\sup_{\tau \in X_m} \min_{1 \leq i \leq k} \tau(\phi_{m_0}^m(b_i)) < \delta + \epsilon/3, \tag{6.45}$$

and enlarging m if necessary, we may find $y_1, \dots, y_l \in \mathcal{M}_m$ such that $\phi_m^\infty(y_j) = x_j$ for $1 \leq j \leq l$.

Now we apply CPoU in $\mathcal{M}_m$ using Proposition 6.2(ii) to obtain pairwise orthogonal positive contractions $f_1, \dots, f_k \in \mathcal{M}_m$ such that

$$\begin{aligned} \|[f_i, y_j]\|_{2,X_m} &< \epsilon, && i = 1, \dots, k,\ j = 1 \dots, l, \\ \tau(f_1 + \cdots + f_k) &> 1 - \epsilon, && \tau \in X_m, \text{ and} \\ \tau(\phi_{m_0}^m(b_i) f_i) &< (\delta + \epsilon/3)\tau(f_i) + \epsilon/3 \\ &\leq \delta\tau(f_i) + 2\epsilon/3, && \tau \in X_m,\ i = 1, \dots, k. \end{aligned} \tag{6.46}$$

Set $e_i \coloneqq \phi_m^\infty(f_i) \in \mathcal{N}$. These are pairwise orthogonal positive contractions. By the $\|\cdot\|_{2,X_m}$-$\|\cdot\|_{2,X}$-contractivity of ϕ_m^∞, we have

$$\|[e_i, x_j]\|_{2,X} < \epsilon, \qquad i = 1, \dots, k,\ j = 1, \dots, l. \tag{6.47}$$

As $(\phi_m^\infty)^*(X) \subseteq X_m$,

$$\tau(e_1 + \cdots + e_k) > 1 - \epsilon, \qquad \tau \in X, \tag{6.48}$$

and

$$\begin{aligned} \tau(a_i e_i) &\overset{(6.43)}{\leq} \tau(\phi_{m_0}^\infty(b_i) e_i) + \epsilon/3 \\ &= (\phi_m^\infty)^*(\tau)(\phi_{m_0}^m(b_i) f_i) + \epsilon/3 \\ &\overset{(6.46)}{<} \delta(\phi_m^\infty)^*(\tau)(f_i) + \epsilon = \delta\tau(e_i) + \epsilon, \qquad \tau \in X. \end{aligned} \tag{6.49}$$

Therefore, $(\mathcal{M}, X)$ has CPoU. □

[50] Note that $(\mathcal{M}, X)$ is factorial by Proposition 3.34.

Remark 6.13 (cf. Remark 5.26)**.** We will later show that a reduced products of factorial tracially complete C^*-algebras with CPoU is factorial and has CPoU (Corollary 7.7(i)). Once the reduced product is shown to be factorial, the reduced product will have CPoU since the local characterisation of CPoU in Proposition 6.2(ii) is easily seen to be preserved by reduced products.

The property CPoU also passes to matrix algebras.

Proposition 6.14. *If $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebras with CPoU, then so is $\big(\mathcal{M} \otimes M_d, X \otimes \{\mathrm{tr}_d\}\big)$ for all $d \in \mathbb{N}$.*

Proof. First note that $(\mathcal{M} \otimes M_d, X \otimes \{\mathrm{tr_d}\})$ is a factorial tracially complete C^*-algebra by Proposition 3.18. Now fix a $\|\cdot\|_{2,X\otimes\{\mathrm{tr}_d\}}$-separable set $S \subseteq \mathcal{M} \otimes M_d$ and let $a_1, \dots, a_k \in (\mathcal{M} \otimes M_d)_+$ and $\delta > 0$ be such that

$$\sup_{\tau\in X} \min_{1\le i\le k} (\tau \otimes \mathrm{tr}_d)(a_i) < \delta. \tag{6.50}$$

Let $T \subseteq \mathcal{M}$ be the set of entries of elements of S and note that T is $\|\cdot\|_{2,X}$-separable. Applying CPoU to the elements $(\mathrm{id}_{\mathcal{M}} \otimes \mathrm{tr}_d)(a_i) \in \mathcal{M}_+$, there are projections $q_1, \dots, q_k \in \mathcal{M}^\omega \cap T'$ such that

$$\sum_{j=1}^{k} q_j = 1_{\mathcal{M}} \quad \text{and} \quad \tau((\mathrm{id}_{\mathcal{M}} \otimes \mathrm{tr}_d)(a_i)q_i) \le \delta\tau(q_i) \tag{6.51}$$

for all $i = 1, \dots, k$ and $\tau \in X$. Then $p_i \coloneqq q_i \otimes 1_{M_d}$, $1 \le i \le k$, are projections in $(\mathcal{M} \otimes M_d)^\omega \cap T'$ which verify CPoU.[51] □

6.4. **Property Γ implies CPoU.** We now have the machinery to prove Theorem 1.4, giving a large supply of examples of tracially complete factorial C^*-algebras which satisfy CPoU. In particular, all (uniform tracial completions of) unital $\mathcal{Z}$-stable C^*-algebras satisfy CPoU, removing the nuclearity condition from [23, Theorem 3.8].

In the separable setting, the proof reduces to the strategy outlined towards the end of Section 1.4, which is modelled on the proof of [23, Theorem 3.8]. The following result is the weak form of CPoU discussed in the first step of the outline. This was shown to hold for uniform tracial completions of nuclear C^*-algebras with compact trace simplex in [23, Lemma 3.6] – we extend the result to all factorial tracially complete C^*-algebras.

Theorem 6.15. *Let $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra such that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. If $a_1, \dots, a_k \in \mathcal{M}_+$,*

$$\delta > \sup_{\tau\in X} \min_{1\le i\le k} \tau(a_i), \tag{6.52}$$

and $q \in \mathcal{M}^\omega \cap \mathcal{M}'$ is a projection with $\tau(q) > 0$ for all $q \in X^\omega$, then for every $\|\cdot\|_{2,X^\omega}$-separable set $S \subseteq \mathcal{M}^\omega$, there are positive contractions $e_1, \dots, e_k \in M^\omega \cap S'$ summing to q with $\tau(a_i e_i) \le \delta\tau(e_i)$ for all $\tau \in X^\omega$ and $i = 1, \dots, k$.

Proof. Let $\gamma \coloneqq \inf_{\tau\in X^\omega} \tau(q)$, and note that $\gamma > 0$ as X^ω is compact. By Kirchberg's ϵ-test (Lemma 5.1), it suffices to show that for any finite set

[51] Note that we are using Proposition 5.8 to identify $\mathcal{M}^\omega \otimes M_d$ and $(\mathcal{M} \otimes M_d)^\omega$.

$\mathcal{F} \subseteq \mathcal{M}$ and $\epsilon > 0$, there are positive contractions $e_1, \dots, e_k \in \mathcal{M}^\omega$ such that

$$\begin{aligned} &\Big\| \sum_{j=1}^{k} e_j - q \Big\|_{2,X^\omega} < \epsilon, \\ &\max_{x \in \mathcal{F}} \|[x, e_i]\|_{2,X} < \epsilon, \qquad i = 1, \dots, k, \text{ and} \\ &\tau(a_i e_i) - \delta\tau(e_i) < \gamma^{-1}\epsilon, \qquad \tau \in X^\omega,\ i = 1, \dots, k. \end{aligned} \tag{6.53}$$

Fix the finite set $\mathcal{F}$ and the tolerance $\epsilon > 0$. We will first show that for any $\tau \in X$, there are

$$e_1', \dots, e_k', b_1, \dots, b_k, c_1, \dots c_k \in \mathcal{M} \tag{6.54}$$

with each e_i' and b_i positive, $\|e_i'\| \leq 1$, and c_i in the span of the commutators in $\mathcal{M}$ such that

$$\begin{aligned} &\Big\| \sum_{j=1}^{k} e_j' - 1_{\mathcal{M}} \Big\|_{2,\tau} < \epsilon, \\ &\max_{x \in \mathcal{F}} \|[x, e_i']\|_{2,\tau} < \epsilon, \quad i = 1, \dots, k, \text{ and} \\ &\|(a_i - \delta)e_i' - b_i - c_i\|_{2,\tau} < \epsilon, \quad i = 1, \dots, k. \end{aligned} \tag{6.55}$$

Fix $\tau \in X$. For each $\sigma \in T(\pi_\tau(\mathcal{M})'')$, we have $\sigma \circ \pi_\tau \in X$ by Lemma 2.10. Therefore,

$$\delta > \sup_{\sigma \in T(\pi_\tau(\mathcal{M})'')} \min_{1 \leq i \leq k} \sigma(\pi_\tau(a_i)). \tag{6.56}$$

By Corollary 6.4, there are projections $\bar{e}_1, \dots, \bar{e}_k \in Z(\pi_\tau(\mathcal{M})'')$ summing to $1_{\pi_\tau(\mathcal{M})''}$ such that $\sigma(a_i\bar{e}_i) \leq \delta\sigma(\bar{e}_i)$ for all $i = 1, \dots, k$ and $\sigma \in T(\pi_\tau(\mathcal{M})'')$. For $i = 1, \dots, k$, define

$$\bar{b}_i \coloneqq \mathrm{tr}_{\pi_\tau(\mathcal{M})''}(a_i\bar{e}_i - \delta\bar{e}_i) \in Z(\pi_\tau(\mathcal{M})''). \tag{6.57}$$

Then $\bar{b}_i$ is positive and

$$\sigma(a_i\bar{e}_i) + \sigma(\bar{b}_i) = \delta\sigma(\bar{e}_i), \qquad \sigma \in T(\pi_\tau(\mathcal{M})''). \tag{6.58}$$

Let $\bar{c}_i = (a_i - \delta)\bar{e}_i - \bar{b}_i$. Then $\bar{c}_i$ vanishes on all traces, so $\bar{c}_i$ is a sum of commutators (see [43, Théorème 2.3]).[52] The existence of the required e_i', b_i, and c_i then follows from Kaplansky's density theorem.

The pointwise result above, holding for all $\epsilon > 0$, and Lemma 4.7[53] imply that for any $\epsilon > 0$ there are positive contractions $e_1', \dots e_k' \in \mathcal{M}$, positive

[52] A standard Hahn–Banach argument implies that since $\bar{c}_i$ vanishes on all traces, $\bar{c}_i$ is in the $\|\cdot\|$-closed span of the commutators. After adjusting the bounds, this weaker result is sufficient to run the proof without quoting [43, Théorème 2.3]

[53] We apply Lemma 4.7 with $\mathcal{C}$ equal to the set of all tuples $(e_1, \dots, e_k, b_1, \dots, b_k, c_1, \dots, c_k) \in \mathcal{M}^{3k}$ such that $e_1, \dots, e_k$ are positive contractions, $b_1, \dots, b_k$ are positive, and $c_1, \dots, c_k$ are sums of commutators (this set is evidently convex); the functions f_i are given by $(e_1, \dots, c_k) \mapsto \sum_{i=1}^k e_i - 1_{\mathcal{M}}$, $(e_1, \dots, c_k) \mapsto [x, e_i]$ for $x \in \mathcal{F}$ and $i = 1, \dots, k$, and $(e_1, \dots, c_k) \mapsto (a_i - \delta)e_i - b_i - c_i$ (these are evidently affine).

elements $b_1, \dots, b_k \in \mathcal{M}$, and elements $c_1, \dots, c_k \in \mathcal{M}$ in the span of the commutators in $\mathcal{M}$ such that

$$\begin{aligned} \Big\| \sum_{j=1}^{k} e'_j - 1_{\mathcal{M}} \Big\|_{2,X} &< \epsilon, \\ \max_{x \in \mathcal{F}} \|[x, e'_i]\|_{2,X} &< \epsilon, \quad i = 1, \dots, k, \text{ and} \\ \|(a_i - \delta)e'_i - b_i - c_i\|_{2,X} &< \epsilon, \quad i = 1, \dots, k. \end{aligned} \tag{6.59}$$

The final equation in (6.59) implies that for all $\tau \in X$, we have

$$\tau\big((a_i - \delta)e'_i - b_i - c_i\big) < \epsilon. \tag{6.60}$$

Rearranging, this implies

$$\tau(a_i e'_i) < \delta\tau(e'_i) + \tau(b_i) + \tau(c_i) + \epsilon \leq \delta\tau(e'_i) + \epsilon, \tag{6.61}$$

where the last inequality holds since b_i is positive and c_i in the closed span on commutators.

We now have positive contractions $e'_1, \dots, e'_k \in \mathcal{M}$ such that

$$\begin{aligned} \Big\| \sum_{j=1}^{k} e'_j - 1_{\mathcal{M}} \Big\|_{2,X^\omega} &< \epsilon, \\ \max_{x \in \mathcal{F}} \|[x, e'_i]\|_{2,X} &< \epsilon, \quad i = 1, \dots, k, \text{ and} \\ \tau(a_i e'_i) - \delta\tau(e'_i) &< \epsilon, \quad \tau \in X,\ i = 1, \dots, k. \end{aligned} \tag{6.62}$$

For $i = 1, \dots, k$, define a positive contraction $e_i \coloneqq e'_i q \in \mathcal{M}$. The first two conditions in (6.53) follow from the corresponding conditions in (6.62). We work to verify the third condition in (6.53).

Fix $\tau \in X^\omega$. By hypothesis, $\tau(q) > 0$, so we may define $\tau_q \in \mathcal{M}^*$ by $\tau_q(a) \coloneqq \tau(q)^{-1}\tau(a)$. As q commutes with $\mathcal{M}$, $\tau_q \in T(\mathcal{M})$. Further, since $\tau|_{\mathcal{M}} \in X$, $\tau_q \leq \tau(q)^{-1}\tau|_{\mathcal{M}}$, and X is face, we have $\tau_q \in X$. By the third condition in (6.62), we have

$$\tau_q(a_i e'_i) < \delta\tau_q(e'_i) + \epsilon \qquad i = 1, \dots, k, \tag{6.63}$$

or equivalently,

$$\tau(a_i e'_i q) < \delta\tau(e_i q) + \epsilon \qquad i = 1, \dots, k. \tag{6.64}$$

Dividing by $\tau(q)$ yields the final condition in (6.53). □

The rest of the proof that property Γ implies CPoU proceeds as in [23]. The following proposition is a tracially complete version of the "tracial projectionisation" result in [23, Lemma 2.4] and the proof is essentially identical. For the sake of completeness, we provide a sketch.

Proposition 6.16 (Projectionisation)**.** *Suppose $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with property Γ. If $S \subseteq \mathcal{M}^\omega$ is a $\|\cdot\|_{2,X^\omega}$-separable subset and $e \in \mathcal{M}^\omega \cap S'$ is a positive contraction, then there is a projection $p \in \mathcal{M}^\omega \cap S'$ commuting with e such that $\tau(ae) = \tau(ap)$ for all $a \in S$ and $\tau \in X^\omega$.*

Proof. Let $k \geq 1$. Using property Γ, we may fix projections $r_1, \ldots, r_k \in \mathcal{M}^\omega \cap S' \cap \{e\}'$ partitioning the unit with

$$\tau(ar_i) = \frac{1}{k}\tau(a), \qquad \tau \in X^\omega,\ a \in C^*(S \cup \{e\}),\ i = 1, \ldots, k. \tag{6.65}$$

For $i = 1, \ldots, k$, consider the continuous function $f_i \colon [0,1] \to \mathbb{R}$ given by

$$f_i(t) \coloneqq \begin{cases} 0, & 0 \leq t \leq (i-1)/k, \\ kt - i + 1, & (i-1)/k \leq t \leq i/n, \\ 1, & i/k \leq t \leq 1. \end{cases} \tag{6.66}$$

Then set $q \coloneqq \sum_{i=1}^k f_i(e) r_i \in \mathcal{M}^\omega \cap S' \cap \{e\}'$, which is a positive contraction. A computation (as in [23, Equations (2.9) and (2.10)])[54] shows

$$\tau(ae) = \tau(aq) \quad \text{and} \quad \tau(q - q^2) < \frac{1}{4k}, \quad \tau \in X^\omega,\ a \in S. \tag{6.67}$$

The result follows from Kirchberg's ϵ-test (Lemma 5.1). □

Applying the above projectionisation to each of the e_i from the conclusion of Theorem 6.15 will produce projections which have the correct behaviour on traces, but fail to be orthogonal. As in [23], this will be addressed via another application of property Γ with a construction deemed "orthogonalisation" in [23]. The orthogonalised projections will no longer sum to the unit, but they will sum to a projection of constant trace $\frac{1}{k}$. From here, CPoU follows from a maximality argument by repeating the construction in the complementary corner – this is the reason for the projection q in Theorem 6.15. (A geometric series argument could be used instead of the maximality argument for this last step.)

Theorem 1.4. *Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra with property Γ. Then $(\mathcal{M}, X)$ has CPoU. In particular, unital C^*-algebras with uniform property Γ (e.g. unital $\mathcal{Z}$-stable C^*-algebras) have CPoU.*

Proof. The results of Appendix A reduce the theorem to the case when $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. Namely, Theorem A.3 shows that both factoriality plus property Γ and factoriality plus CPoU are strongly separably inheritable properties; therefore, to show one implies the other, it suffices to do so in the $\|\cdot\|_{2,X}$-separable case.

For the rest of the proof, we assume $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. Suppose $a_1 \ldots, a_k \in \mathcal{M}_+$ and $\delta > 0$ are given as in the definition of CPoU, i.e. they satisfy

$$\delta > \sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(a_i). \tag{6.68}$$

Let $I \subseteq [0,1]$ denote the set of $\alpha \in [0,1]$ such that for all $\|\cdot\|_{2,X^\omega}$-separable sets $S \subseteq \mathcal{M}^\omega$, there are orthogonal projections $p_1, \ldots, p_k \in \mathcal{M}^\omega \cap S'$ such that

$$\sum_{j=1}^k \tau(p_j) = \alpha \qquad \text{and} \qquad \tau(a_i p_i) \leq \delta \tau(p_i) \tag{6.69}$$

[54] [23, Equation (2.7)] is still correct if the right side is changed to $\frac{1}{4} 1_{C([0,1])}$, and using this, the right side of [23, Equation (2.10)] can be improved to $1/4n$.

for all $\tau \in X^\omega$ and $i = 1, \dots, k$. Clearly $I \neq \emptyset$ as $0 \in I$, and I is closed by Kirchberg's ϵ-test. Hence I contains a maximal element α. It suffices to show $\alpha = 1$ since this forces $\sum_{j=1}^k p_j = 1_{\mathcal{M}^\omega}$. We will assume $\alpha < 1$ and show $\alpha + \frac{1}{k}(1-\alpha) \in I$, which contradicts the maximality of α.

Assume $\alpha < 1$ and let $S \subseteq \mathcal{M}^\omega$ be a $\|\cdot\|_{2,X^\omega}$-separable set with $\mathcal{M} \subseteq S$. By the assumption on α, there are mutually orthogonal projections $p_1' \dots, p_k' \in \mathcal{M}^\omega \cap S'$ such that

$$\sum_{j=1}^k \tau(p_j') = \alpha \qquad \text{and} \qquad \tau(a_i p_i') \leq \delta \tau(p_i') \tag{6.70}$$

for all $\tau \in X^\omega$ and $i = 1, \dots, k$. Define

$$q \coloneqq 1_{\mathcal{M}^\omega} - \sum_{j=1}^k p_j' \in \mathcal{M}^\omega \cap S' \subseteq \mathcal{M}^\omega \cap \mathcal{M}' \tag{6.71}$$

Note that $\tau(q) = 1 - \alpha > 0$ for all $\tau \in X^\omega$.

Theorem 6.15 provides positive contractions $e_1, \dots, e_k \in \mathcal{M}^\omega \cap S'$ summing to q with $\tau(a_i e_i) \leq \delta \tau(e_i)$ for all $\tau \in X^\omega$ and $i = 1, \dots, k$. Let $q_1, \dots, q_k \in \mathcal{M}^\omega \cap (S \cup \{q, e_1, \dots, e_k\})'$ be projections as in Proposition 6.16 satisfying $\tau(ae_i) = \tau(aq_i)$ for all $a \in C^*(S \cup \{q\})$ and $i = 1, \dots, k$. Let $A \subseteq \mathcal{M}^\omega$ denote the C^*-algebra generated by S, the e_i, and the q_i, and using property Γ, let $r_1, \dots, r_k \in \mathcal{M}^\omega \cap A'$ be projections summing to $1_{\mathcal{M}^\omega}$ such that $\tau(ar_i) = \frac{1}{k}\tau(a)$ for all $\tau \in X^\omega$, $i = 1, \dots, k$, and $a \in A$

Define $p_i'' \coloneqq q_i r_i$ for $i = 1, \dots k$. These are orthogonal projections since $r_1, \dots, r_k$ are orthogonal projections and each q_i is a projection commuting with q_i. We have

$$\sum_{j=1}^k \tau(p_i'') = \frac{1}{k}\sum_{j=1}^k \tau(q_i) = \frac{1}{k}\sum_{j=1}^k \tau(e_i) = \frac{1}{k}\tau(q) = \frac{1}{k}(1-\alpha) \tag{6.72}$$

for all $\tau \in X^\omega$. Further, for all $\tau \in X^\omega$ and $i = 1, \dots, k$, we have

$$\tau(a_i p_i'') = \frac{1}{k}\tau(a_i q_i) = \frac{1}{k}\tau(a_i e_i) \leq \frac{\delta}{k}\tau(e_i) = \frac{\delta}{k}\tau(q_i) = \delta\tau(p_i''). \tag{6.73}$$

Note also that $e_i q = e_i$, and hence $\tau(q_i q) = \tau(e_i q) = \tau(e_i) = \tau(q_i)$ for all $\tau \in X^\omega$. As q_i and q are commuting positive elements, we have $q_i q \leq q_i$, and hence the equality on traces implies $q_i q = q_i$. Therefore, $p_i'' \leq q_i' \leq q$ for all $i = 1, \dots, k$. Set $p_i \coloneqq p_i' + p_i'' \in \mathcal{M}^\omega \cap S'$. Since $p_1', \dots, p_k', p_1'', \dots, p_k''$ are mutually orthogonal projections, the same is true of $p_1, \dots, p_k$. Further, the projections p_i witness $\alpha + \frac{1}{k}(1-\alpha) \in I$, which contradicts the maximality of α. □

7. Applications of CPoU

The power of CPoU lies in the fact that we can prove many useful structural properties for a reduced product of a sequence of factorial tracially complete C^*-algebras with CPoU. That is the topic of this subsection.

Our first goal is show in Section 7.1 that a reduced product of factorial tracially complete C^*-algebras with CPoU is factorial. In fact, we will show all traces on the reduced product are in the closed convex hull of the limit traces

– solving the trace problem for such reduced products. Along the way, we show that such reduced products have real rank zero and comparison of projections with respect to limit traces. We then analyse the unitary group of factorial tracially complete C^*-algebras with CPoU in Section 7.2 and show that a uniform 2-norm dense set of unitaries are exponentials. This also provides a stability result for unitaries showing that in the uniform 2-norm, every approximate unitary is close to a unitary. Finally, Section 7.3 strengthens the comparison property obtained in Section 7.1 mentioned above and classifies projections in factorial tracially complete C^*-algebras with CPoU via traces.

7.1. **Traces on reduced products.** The following lemma gives an approximate version of the existence of spectral projections in factorial tracially complete C^*-algebras with CPoU, and the proof is a typical application of CPoU. The idea is the following: if $\mathcal{M}$ is a von Neumann algebra and $x \in \mathcal{M}$ is self-adjoint, then there is a projection $q \in \mathcal{M}$ such that $qx = x_+$; indeed, one may take q to be the spectral projection of x corresponding to the interval $[0, \infty)$. Then CPoU provides a method for transferring this result from tracial von Neumann algebra completions of a tracially complete C^*-algebra $(\mathcal{M}, X)$ to produce an analogous result in $\mathcal{M}$ up to a small $\|\cdot\|_{2,X}$-error.

Lemma 7.1. *Suppose $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU. If $x \in \mathcal{M}$ is self-adjoint and $\epsilon > 0$, then there is a positive contraction $q \in \mathcal{M}$ such that*

$$\|q - q^2\|_{2,X} < \epsilon \qquad \text{and} \qquad \|qx - x_+\|_{2,X} < \epsilon. \tag{7.1}$$

Proof. For each $\tau \in X$, let $\bar{q}_\tau \in \pi_\tau(\mathcal{M})''$ be the spectral projection of $\pi_\tau(a)$ corresponding to the interval $[0, \infty)$ so that $\bar{q}_\tau \pi_\tau(x) = \pi_\tau(x_+)$. By Kaplansky's density theorem, there is a positive contraction $q_\tau \in \mathcal{M}$ such that

$$\|q_\tau - q_\tau^2\|_{2,\tau} < \frac{\epsilon}{\sqrt{3}} \qquad \text{and} \qquad \|q_\tau x - x_+\|_{2,\tau} < \frac{\epsilon}{\sqrt{3}}. \tag{7.2}$$

For all $\tau \in X$, define

$$a_\tau \coloneqq |q_\tau - q_\tau^2|^2 + |q_\tau x - x_+|^2 \in \mathcal{M}_+ \tag{7.3}$$

and note that $\tau(a_\tau) < 2\epsilon^2/3$. For $\tau \in X$, let

$$U_\tau \coloneqq \Big\{\sigma \in X : \sigma(a_\tau) < \frac{2\epsilon^2}{3}\Big\}, \tag{7.4}$$

which is an open neighbourhood of τ. As X is compact, there are traces $\tau_1, \ldots, \tau_k \in X$ such that $(U_{\tau_i})_{i=1}^k$ is an open cover of X. Therefore,

$$\sup_{\tau \in X} \min_{1 \le i \le k} \tau(a_{\tau_i}) < \frac{2\epsilon^2}{3}. \tag{7.5}$$

Set $S \coloneqq \{q_{\tau_i} : i = 1, \ldots, k\} \cup \{x\}$. As $(\mathcal{M}, X)$ has CPoU, there are pairwise orthogonal projections $p_1, \ldots, p_k \in \mathcal{M}^\omega \cap S'$ such that

$$\sum_{i=1}^k p_i = 1_{\mathcal{M}^\omega} \qquad \text{and} \qquad \tau(a_{\tau_i} p_i) \le \frac{2\epsilon^2}{3} \tau(p_i) \tag{7.6}$$

for all $1 \leq i \leq k$ and $\tau \in X^\omega$. Let $q \coloneqq \sum_{i=1}^k q_{\tau_i} p_i \in \mathcal{M}_+^\omega$. Since $p_1, \ldots, p_k$ are mutually orthogonal projections summing to the identity which commute with $\{q_{\tau_i} : i = 1, \ldots, k\} \cup \{x\}$, we have

$$\begin{aligned} |q - q^2|^2 + |qx - x_+|^2 &= \sum_{i=1}^k |q_{\tau_i}^2 - q_{\tau_i}|^2 p_i + \sum_{i=1}^k |q_{\tau_i} x - x_+|^2 p_i \\ &= \sum_{i=1}^k a_{\tau_i} p_i. \end{aligned} \tag{7.7}$$

Now, we compute

$$\|q - q^2\|_{2,\tau}^2 + \|qx - x_+\|_{2,\tau}^2 = \sum_{i=1}^k \tau(a_{\tau_i} p_i) \leq \frac{2\epsilon^2}{3} \sum_{i=1}^k \tau(p_i) < \epsilon^2 \tag{7.8}$$

for all $\tau \in X^\omega$. If $(q_n)_{n=1}^\infty$ is a sequence of positive contractions in $\mathcal{M}$ representing q, then for ω-many n,

$$\|q_n - q_n^2\|_{2,X} < \epsilon \qquad \text{and} \qquad \|q_n x - x_+\|_{2,X} < \epsilon. \tag{7.9}$$

□

We will prove a stronger version of the following result in Corollary 7.15 once we show reduced products of factorial tracially complete C^*-algebras with CPoU are factorial and have CPoU.

Proposition 7.2. *Let $\big((\mathcal{M}_n, X_n)\big)_{n=1}^\infty$ be a sequence of factorial tracially complete C^*-algebras with CPoU. Then $\prod^\omega \mathcal{M}_n$ has real rank zero.*

Proof. Define $(\mathcal{M}, X) \coloneqq \big(\prod^\omega \mathcal{M}_n, \sum^\omega X_n\big)$ and let $x \in \mathcal{M}$ be self-adjoint. Let $(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty \mathcal{M}_n$ be a self-adjoint element lifting x and then use Lemma 7.1 to produce a positive contraction $(q_n)_{n=1}^\infty \in \prod_{n=1}^\infty \mathcal{M}_n$ such that for all $n \geq 1$,

$$\|q_n - q_n^2\|_{2,X_n} < \frac{1}{n} \qquad \text{and} \qquad \|q_n x_n - (x_n)_+\|_{2,X_n} < \frac{1}{n}. \tag{7.10}$$

Let $q \in \mathcal{M}^\omega$ denote the image of $(q_n)_{n=1}^\infty$. Then we have

$$\|q - q^2\|_{2,X} = 0 \qquad \text{and} \qquad \|qx - x_+\|_{2,X} = 0. \tag{7.11}$$

Hence, q is a projection with $qx = x_+$, which also implies that x and q commute.

Fix $\epsilon > 0$ and note that

$$y \coloneqq x + \epsilon(2q - 1_{\mathcal{M}}) = (x_+ + \epsilon)q - (x_- + \epsilon)q^\perp \in \mathcal{M} \tag{7.12}$$

is self-adjoint and invertible with $\|x - y\| \leq \epsilon$. Hence, the invertible self-adjoint elements in $\mathcal{M}$ form a dense subset of the self-adjoint elements of $\mathcal{M}$. Therefore, $\mathcal{M}$ has real rank zero. □

Now we turn to comparison of projections in tracially complete C^*-algebras. Just as Murray–von Neumann subequivalence of projections in finite von Neumann algebras is determined by traces, we will show in Theorem 7.18 that a factorial tracial complete C^*-algebra $(\mathcal{M}, X)$ with CPoU has comparison of projections with respect to X, i.e. that a projection $p \in \mathcal{M}$ is Murray–von Neumann subequivalent to a projection $q \in \mathcal{M}$ if and only if

$\tau(p) \leq \tau(q)$ for all $\tau \in X$. Before doing this, we need the following approximate comparison result, which will lead to comparison of projections in reduced products.

Lemma 7.3. *Suppose $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU, $\epsilon > 0$, and $p, q \in \mathcal{M}$ are positive contractions such that*

$$\tau(p - p^2) < \epsilon, \quad \tau(q - q^2) < \epsilon, \quad \text{and} \quad \tau(p) < \tau(q) + \epsilon \tag{7.13}$$

for all $\tau \in X$. Then there is a contraction $v \in \mathcal{M}$ such that

$$\|v^* q v - p\|_{2,X} < \big(2\sqrt{2} + \sqrt{5}\big)\sqrt{\epsilon}. \tag{7.14}$$

Proof. Using that X is compact, there is an $\epsilon' \in (0, \epsilon)$ with

$$\tau(p - p^2) < \epsilon', \quad \tau(q - q^2) < \epsilon', \quad \text{and} \quad \tau(p) < \tau(q) + \epsilon' \tag{7.15}$$

for all $\tau \in X$. We first show that for each $\tau \in X$, there is a partial isometry $\bar{v}_\tau \in \pi_\tau(\mathcal{M})''$ such that

$$\|\bar{v}_\tau^* \pi_\tau(q) \bar{v}_\tau - \pi_\tau(p)\|_{2,\tau} < \delta \coloneqq \big(2\sqrt{2} + \sqrt{5}\big)\sqrt{\epsilon'}. \tag{7.16}$$

Fix $\tau \in X$ and let $p_\tau, q_\tau \in \pi_\tau(\mathcal{M})''$ denote the spectral projections of $\pi_\tau(p)$ and $\pi_\tau(q)$ corresponding to the interval $[1/2, 1]$. Then

$$\begin{aligned} |\pi_\tau(p) - p_\tau|^2 \leq |\pi_\tau(p) - p_\tau| \leq 2(\pi_\tau(p) - \pi_\tau(p)^2) \qquad \text{and} \\ |\pi_\tau(q) - q_\tau|^2 \leq |\pi_\tau(q) - q_\tau| \leq 2(\pi_\tau(q) - \pi_\tau(q)^2), \end{aligned} \tag{7.17}$$

so that (7.15) gives

$$\|\pi_\tau(p) - p_\tau\|_{2,\tau} \leq \sqrt{2\epsilon'} \text{ and } \|\pi_\tau(q) - q_\tau\|_{2,\tau} \leq \sqrt{2\epsilon'}. \tag{7.18}$$

By Lemma 2.10, we have $\sigma \circ \pi_\tau \in X$ for all $\sigma \in T(\pi_\tau(\mathcal{M})'')$, and so (7.15) and (7.17) imply that

$$\sigma(p_\tau) < \sigma(q_\tau) + 5\epsilon', \qquad \sigma \in T(\pi_\tau(\mathcal{M})''). \tag{7.19}$$

By the general comparison theorem for projections in von Neumann algebras (see for example [104, Theorem V.1.8]), there exists a central projection $z_\tau \in \pi_\tau(\mathcal{M})''$ with $z_\tau p_\tau \precsim z_\tau q_\tau$ and $z_\tau^\perp q_\tau \precsim z_\tau^\perp p_\tau$. Fix a partial isometry $\bar{v}_\tau \in \pi_\tau(\mathcal{M})''$ such that

$$z_\tau \bar{v}_\tau^* \bar{v}_\tau = z_\tau p_\tau \quad \text{and} \quad z_\tau \bar{v}_\tau \bar{v}_\tau^* \leq z_\tau q_\tau, \tag{7.20}$$

$$z_\tau^\perp \bar{v}_\tau^* \bar{v}_\tau \leq z_\tau^\perp p_\tau \quad \text{and} \quad z_\tau^\perp \bar{v}_\tau \bar{v}_\tau^* = z_\tau^\perp q_\tau. \tag{7.21}$$

In the special case where $z_\tau = 1_{\mathcal{M}}$, then (7.20) implies that $p_\tau = \bar{v}_\tau^* q_\tau \bar{v}_\tau$; hence, the required inequality (7.16) is now a consequence of (7.17) and (7.15). Suppose now that $z_\tau \neq 1_{\mathcal{M}}$. Then $\tau(z_\tau^\perp) \neq 0$, so we may define $\sigma \in T(\pi_\tau(\mathcal{M})'')$ by $\sigma(a) \coloneqq \tau(z_\tau^\perp)^{-1} \tau(z_\tau^\perp a)$. Then (7.20) implies that $z_\tau p_\tau = z_\tau \bar{v}_\tau^* q_\tau \bar{v}_\tau$, and (7.21) implies that $z_\tau^\perp p_\tau - z_\tau^\perp \bar{v}_\tau^* q_\tau \bar{v}_\tau$ is a projection. Therefore,

we have

$$\begin{aligned}\|p_\tau - \bar{v}_\tau^* q_\tau \bar{v}_\tau\|_{2,\tau}^2 &= \|z_\tau(p_\tau - \bar{v}_\tau^* q_\tau \bar{v}_\tau) + z_\tau^\perp(p_\tau - \bar{v}_\tau^* q_\tau \bar{v}_\tau)\|_{2,\tau}^2 \\ &= \|z_\tau^\perp p_\tau - z_\tau^\perp \bar{v}_\tau^* q_\tau \bar{v}_\tau\|_{2,\tau}^2 \\ &= \tau(z_\tau^\perp p_\tau - \bar{v}_\tau^* z_\tau^\perp q_\tau \bar{v}_\tau) \\ &= \tau(z_\tau^\perp p_\tau) - \tau(z_\tau^\perp \bar{q}_\tau \bar{v}_\tau \bar{v}_\tau^*) \\ &= \tau(z_\tau^\perp)\big(\sigma(p_\tau) - \sigma(q_\tau)\big) \\ &< 5\epsilon',\end{aligned} \tag{7.22}$$

where the final estimate uses (7.19). Combining (7.22) with (7.18) proves (7.16).

Now we use CPoU to obtain (7.14) from (7.16). By Kaplansky's density theorem, there is a contraction $v_\tau \in \mathcal{M}$ such that

$$\|v_\tau^* q v_\tau - p\|_{2,\tau} < \delta. \tag{7.23}$$

Define $a_\tau \coloneqq |v_\tau^* q v_\tau - p|^2$, and use this to define the open neighbourhood

$$U_\tau \coloneqq \{\sigma \in X : \sigma(a_\tau) < \delta^2\} \tag{7.24}$$

of τ in X. As X is compact, there are $\tau_1, \dots, \tau_k \in X$ such that $(U_{\tau_i})_{i=1}^k$ covers X. Therefore,

$$\sup_{\tau \in X} \min_{1 \le i \le k} \tau(a_{\tau_i}) < \delta^2. \tag{7.25}$$

Set $S := \{v_{\tau_i} : i = 1, \dots, k\} \cup \{p, q\}$. As $(\mathcal{M}, X)$ has CPoU, there are mutually orthogonal projections $e_1, \dots, e_k \in \mathcal{M}^\omega \cap S'$ such that

$$\sum_{i=1}^k e_i = 1_{\mathcal{M}^\omega} \qquad \text{and} \qquad \tau(a_{\tau_i} e_i) \le \delta^2 \tau(e_i) \tag{7.26}$$

for all $\tau \in X^\omega$ and $i = 1, \dots, k$. Define $v \coloneqq \sum_{i=1}^k e_i v_{\tau_i} \in \mathcal{M}^\omega$. Since $e_1, \dots, e_k$ are mutually orthogonal projections summing to the identity which commute with $\{v_{\tau_i} : i = 1, \dots, k\} \cup \{p, q\}$, we have

$$|v^* q v - p|^2 = \sum_{i=1}^k |v_{\tau_i}^* q v_{\tau_i} - p|^2 e_i = \sum_{i=1}^k a_{\tau_i} e_i. \tag{7.27}$$

Hence, for each $\tau \in X^\omega$,

$$\|v^* q v - p\|_{2,\tau}^2 = \tau(|v^* q v - p|^2) = \sum_{i=1}^k \tau(a_{\tau_i} e_i) \le \delta^2 \sum_{j=1}^k \tau(e_i) = \delta^2. \tag{7.28}$$

Finally, if $(v_n)_{n=1}^\infty \subseteq \mathcal{M}$ is a sequence representing v, then

$$\lim_{n \to \omega} \|v_n^* q v_n - p\|_{2,X} = \|v^* q v - p\|_{2,X^\omega} \le \delta < (2\sqrt{2} + \sqrt{5})\sqrt{\epsilon}, \tag{7.29}$$

and the result follows. □

Proposition 7.4. *Suppose $\big((\mathcal{M}_n, X_n)\big)_{n=1}^\infty$ is a sequence of factorial tracially complete C^*-algebras with CPoU and $p, q \in \prod^\omega \mathcal{M}_n$ are projections.*

(i) *If $\tau(p) \le \tau(q)$ for all $\tau \in \sum^\omega X_n$, then p is Murray–von Neumann subequivalent to q.*

(ii) *If $\tau(p) = \tau(q)$ for all $\tau \in \sum^\omega X_n$, then p and q are unitarily equivalent.*

Proof. Lemma 7.3 immediately implies (i). For (ii), use (i) to obtain $v \in \prod^\omega \mathcal{M}_n$ with $v^*v = p$ and $vv^* \leq q$. Since $q - vv^* \geq 0$ and

$$\tau(q - vv^*) = \tau(q) - \tau(p) = 0 \tag{7.30}$$

for all $\tau \in \sum^\omega X_n$, we have $vv^* = q$. Repeating this argument with $p^\perp$ and $q^\perp$ in place of p and q produces a partial isometry $w \in \prod^\omega \mathcal{M}_n$ such that $w^*w = p^\perp$ and $ww^* = q^\perp$. Then $u \coloneqq v + w$ is a unitary with $upu^* = q$. □

Combining Propositions 7.2 and 7.4 resolves the trace problem (Question 1.1) for reduced products of tracially complete C^*-algebras with CPoU. This generalises the case of McDuff W^*-bundles from [10, Proposition 3.32] (taking $A \coloneqq \mathbb{C}$ in that proposition).

Theorem 7.5. *If $\big((\mathcal{M}_n, X_n)\big)_{n=1}^\infty$ is a sequence of factorial tracially complete C^*-algebras with CPoU, then $T\big(\prod^\omega \mathcal{M}_n\big) = \sum^\omega X_n$.*

Proof. Let $(\mathcal{M}, X) \coloneqq \big(\prod^\omega \mathcal{M}_n, \sum^\omega X_n\big)$. Then $\mathcal{M}$ has real rank zero by Proposition 7.2 and comparison of projections with respect to X by Proposition 7.4. Further, for all $d \geq 1$, $\mathcal{M} \otimes M_d$ has comparison of projections with respect to $X \otimes \{\mathrm{tr_d}\}$ since Proposition 5.8 identifies $(\mathcal{M} \otimes M_d, X \otimes \{\mathrm{tr}_d\})$ with $\prod^\omega (\mathcal{M}_n \otimes M_d, X_n \otimes \{\mathrm{tr}_d\})$ and each of the algebras $(\mathcal{M}_n \otimes M_d, X_n \otimes \{\mathrm{tr}_d\})$ are factorial and have CPoU by Proposition 6.14.

Let $V(\mathcal{M})$ denote the Murray–von Neumann semigroup of $\mathcal{M}$ and let $[p] \in V(\mathcal{M})$ denote the class of a projection $p \in \mathcal{M} \otimes M_d$ for an integer $d \geq 1$. For $\tau \in T(\mathcal{M})$, let $\hat{\tau}$ denote the induced state on $V(\mathcal{M})$ given by

$$\hat{\tau}([p]) \coloneqq (\tau \otimes \mathrm{Tr}_d)(p) \tag{7.31}$$

for a projection p in $\mathcal{M} \otimes M_d$, where Tr_d is the unnormalised trace on M_d so that $\mathrm{Tr}_d(1_{M_d}) = d$. By comparison of projections in matrices over $\mathcal{M}$, the natural map

$$V(\mathcal{M}) \to \mathrm{Aff}(X) \colon x \mapsto (\tau \mapsto \hat{\tau}(x)) \tag{7.32}$$

is an order embedding.

Now suppose $\sigma \in T(\mathcal{M})$. By [9, Corollary 2.7], the state $\hat{\sigma}$ on $V(\mathcal{M})$ extends to a state ϕ on $\mathrm{Aff}(X)$. Since all states on $\mathrm{Aff}(X)$ are point evaluations, (see Proposition 2.1(i)), there is a trace $\tau \in X$ such that $\phi(f) = f(\tau)$ for all $f \in \mathrm{Aff}(X)$. Then, by construction, $\hat{\sigma} = \hat{\tau}$, and hence $\sigma(p) = \tau(p)$ for all projections $p \in \mathcal{M}$. As $\mathcal{M}$ has real rank zero by Proposition 7.2, $\mathcal{M}$ is the $\|\cdot\|$-closed span of its projections, and hence $\sigma = \tau \in X$. □

Remark 7.6. Versions of Theorem 7.5 have appeared previously in [10, Proposition 3.22] and [23, Proposition 4.6] in the settings of ultrapowers of factorial McDuff W^*-bundles and uniform tracial ultrapowers of C^*-algebras with CPoU, respectively. The proofs in these references are based on a result of Fack and de la Harpe ([43, Théorème 2.3]). This style of argument could also be adapted to the setting of tracially complete C^*-algebras, providing an alternative proof of Theorem 7.5.

Theorem 7.5 also proves that factoriality passes to reduced products in the presence of CPoU. This allows us to prove both property Γ and CPoU pass to reduced products as promised in Remarks 5.26 and 6.13.

Corollary 7.7. *Suppose $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of factorial tracially complete C^*-algebras with reduced product $(\mathcal{M}, X)$.*

(i) *If each $(\mathcal{M}_n, X_n)$ has CPoU, then $(\mathcal{M}, X)$ is factorial and has CPoU.*
(ii) *If each $(\mathcal{M}_n, X_n)$ satisfies property Γ, then $(\mathcal{M}, X)$ is factorial and satisfies property Γ.*

Proof. To see (i), note that $(\mathcal{M}, X)$ is factorial by Theorem 7.5. The approximation property in Proposition 6.2(ii) characterising CPoU passes to reduced products, so $(\mathcal{M}, X)$ satisfies CPoU. For (ii), Theorem 1.4 implies each $(\mathcal{M}_n, X_n)$ satisfies CPoU, so $(\mathcal{M}, X)$ is factorial by (i). The approximation property in Proposition 5.23(ii) characterising Γ passes to reduced products, and hence $(\mathcal{M}, X)$ satisfies property Γ. □

Corollary 7.7 and Kirchberg's ϵ-test (Lemma 5.1) allow for the following variation of CPoU for reduced products. The point is that the projections p_i in the definition of CPoU can be chosen in the reduced product instead of the reduced power of the reduced product.

Corollary 7.8. *Suppose $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of factorial tracially complete C^*-algebras with CPoU. If $a_1, \dots, a_k \in (\prod^{\omega} \mathcal{M}_n)_+$, $S \subseteq \prod^{\omega} \mathcal{M}_n$ is $\|\cdot\|_{2,X^\omega}$-separable, and*

$$\sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(a_i) < \delta, \tag{7.33}$$

then there are projections $p_1, \dots, p_k \in \prod^{\omega} \mathcal{M}_n \cap S'$ such that

$$\sum_{i=1}^{k} p_i = 1_{\mathcal{M}^\omega} \qquad \text{and} \qquad \tau(a_i p_i) \leq \delta \tau(p_i) \tag{7.34}$$

for all $1 \leq i \leq k$ and $\tau \in X^\omega$.

7.2. **Structure of unitaries.** With Corollary 7.7 in hand, we derive some properties unitaries in tracially complete C^*-algebras with CPoU. The following result and its proof are analogous to [21, Proposition 2.1].

Proposition 7.9. *Suppose $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of factorial tracially complete C^*-algebras with CPoU and $S \subseteq \prod^{\omega} \mathcal{M}_n$ is a $\|\cdot\|_{2,X^\omega}$-separable subset. If $u \in \prod^{\omega} \mathcal{M}_n \cap S'$ is a unitary, then there is a self-adjoint $h \in \prod^{\omega} \mathcal{M}_n \cap S'$ such that $u = e^{ih}$ and $\|h\| \leq \pi$.*

Proof. Since this theorem involves the numbers π and $i = \sqrt{-1}$, we shall write σ_τ to denote the GNS representation corresponding to a trace τ and use the letter j for our summation index. For the sake of brevity, we write $(\mathcal{M}, X) \coloneqq \big(\prod^{\omega} \mathcal{M}_n, \sum^{\omega} X_n\big)$.

Fix $\epsilon > 0$ and a finite set $\mathcal{F} \subseteq \mathcal{M}$. By Kirchberg's ϵ-test (Lemma 5.1), it suffices to show that there is a self-adjoint $h \in \mathcal{M}$ with $\|h\| \leq \pi$ such that

$$\|u - e^{ih}\|_{2,X} \leq \epsilon \qquad \text{and} \qquad \max_{x \in \mathcal{F}} \|[h, x]\|_{2,X} \leq \epsilon. \tag{7.35}$$

For each $\tau \in X$, there is a self-adjoint $\bar{h}_\tau \in \sigma_\tau(\mathcal{M})'' \cap \sigma_\tau(\mathcal{F})'$ with $\|\bar{h}_\tau\| \leq \pi$ such that $\sigma_\tau(u) = e^{i\bar{h}_\tau}$. By Kaplansky's density theorem, there is a self-adjoint $h_\tau \in \mathcal{M}$ with $\|h_\tau\| \leq \pi$ such that

$$\|u - e^{ih_\tau}\|_{2,\tau} < (|\mathcal{F}| + 1)^{-1/2}\epsilon \qquad \text{and}$$
$$\max_{x \in \mathcal{F}} \|[h_\tau, x]\|_{2,\tau} < (|\mathcal{F}| + 1)^{-1/2}\epsilon.$$

Then define

$$a_\tau \coloneqq |u - e^{ih_\tau}|^2 + \sum_{x \in \mathcal{F}} |[h_\tau, x]|^2 \in \mathcal{M}_+, \tag{7.36}$$

so that $\tau(a_\tau) < \epsilon^2$. By the compactness of X, there are $\tau_1, \ldots, \tau_k \in X$ such that

$$\sup_{\tau \in X} \min_{1 \leq j \leq k} \tau(a_{\tau_j}) < \epsilon^2. \tag{7.37}$$

By CPoU (in the form of Corollary 7.8), there are mutually orthogonal projections

$$p_1, \ldots, p_k \in \mathcal{M} \cap \mathcal{F}' \cap \{u\}' \cap \{h_{\tau_1}, \ldots, h_{\tau_k}\}' \tag{7.38}$$

such that

$$\sum_{j=1}^{k} p_j = 1_{\mathcal{M}} \qquad \text{and} \qquad \tau(a_{\tau_j} p_j) \leq \epsilon^2 \tau(p_j) \tag{7.39}$$

for all $1 \leq j \leq k$ and $\tau \in X$.

Define $h \coloneqq \sum_{j=1}^{k} h_{\tau_j} p_j \in \mathcal{M}$ and note that $\|h\| \leq \pi$ as $\|h_{\tau_j}\| \leq \pi$ for all $1 \leq j \leq k$ as the p_j are mutually orthogonal projections commuting with the h_{τ_j}. Also, for each $\tau \in X$, we have

$$\|u - e^{ih}\|_{2,\tau}^2 \leq \sum_{j=1}^{k} \tau(a_{\tau_j} p_j) \leq \epsilon^2 \sum_{j=1}^{k} \tau(p_j) = \epsilon^2, \tag{7.40}$$

and for each $\tau \in X$ and $x \in \mathcal{F}$,

$$\|[h, x]\|_{2,\tau}^2 \leq \sum_{j=1}^{k} \tau(a_{\tau_j} p_j) \leq \epsilon^2 \sum_{j=1}^{k} \tau(p_j) = \epsilon^2. \tag{7.41}$$

This verifies (7.35). □

Applying Proposition 7.9 to matrix algebras (with $S = \emptyset$) yields the following K-theoretic computation.

Corollary 7.10. *If $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of factorial tracially complete C*-algebras with CPoU, then $K_1\big(\prod^{\omega} \mathcal{M}_n\big) = 0$.*

Proof. Proposition 7.9 implies that the unitary group of $\prod^{\omega} \mathcal{M}_n$ is path-connected, and Propositions 6.14 and 5.8 imply the same result for matrices over $\prod^{\omega} \mathcal{M}_n$. □

The following result facilitates the use of Theorem 5.11 in the presence of CPoU.

Corollary 7.11. *If $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of factorial tracially complete C^*-algebras with CPoU and $u \in \prod^{\omega} \mathcal{M}_n$ is a unitary, then there is a sequence of unitaries $(u_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{M}_n$ lifting u.*

Using a sequence of counterexamples argument (see for example [71]), Corollary 7.11 also provides a uniform 2-norm stability result for unitaries.

Corollary 7.12. *For all $\epsilon, c > 0$, there is a $\delta > 0$ such that if $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU and $v \in \mathcal{M}$ with $\|v\| \leq c$ and $\|v^*v - 1\|_{2,X} < \delta$, there is a unitary $u \in \mathcal{M}$ such that $\|u - v\|_{2,X} < \epsilon$.*

Proof. Suppose the result is false and that $\epsilon, c > 0$ provide a counterexample. For each $n \in \mathbb{N}$, let $(\mathcal{M}_n, X_n)$ be a tracially complete C^*-algebra with CPoU and let $v_n \in \mathcal{M}_n$ be such that $\|v_n\| \leq c$, $\|v_n^* v_n - 1\|_{2,X_n} < \frac{1}{n}$, but for all unitaries $u \in \mathcal{M}_n$, we have

$$\|u - v_n\|_{2,X_n} \geq \epsilon. \tag{7.42}$$

The sequence $(v_n)_{n=1}^{\infty}$ induces an element $v \in \prod^{\omega} \mathcal{M}_n$. We have $\|v^*v - 1\|_{2,\sum^{\omega} X_n} = 0$, and hence v is an isometry. Since $1 - vv^* \geq 0$ and $\tau(1 - vv^*) = 0$ for all $\tau \in X$, we further have that v is unitary. By Corollary 7.11, there is a sequence of unitaries $(u_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{M}_n$ lifting v. Now,

$$\lim_{n \to \omega} \|u_n - v_n\|_{2,X_n} = 0, \tag{7.43}$$

which contradicts (7.42). □

Proposition 7.9 also implies that if $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU, then there is a $\|\cdot\|_{2,X}$-dense set of unitaries in $\mathcal{M}$ that have the form e^{ih} for a self-adjoint $h \in \mathcal{M}$ with $\|h\| \leq \pi$. We do not know if every unitary in $\mathcal{M}$ has this form. Without CPoU, there are certainly commutative counterexamples such as $\big(C(\mathbb{T}), T(C(\mathbb{T}))\big)$, but we do not know of any counterexamples among type II_1 tracially complete C^*-algebras $(\mathcal{M}, X)$ – with or without CPoU –, or even among tracially complete C^*-algebras $(\mathcal{M}, X)$ for which $\pi_\tau(\mathcal{M})''$ is diffuse for each $\tau \in X$.

Question 7.13. If $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU, is every unitary in $\mathcal{M}$ an exponential? Slightly less ambitiously, is the unitary group of $\mathcal{M}$ path connected in the C^*-norm topology?

In every finite von Neumann algebra, every element has a unitary polar decomposition. The same holds in reduced products of factorial tracially complete C^*-algebras with CPoU. In fact, this can be done in an approximately central way.

Proposition 7.14. *Suppose $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of factorial tracially complete C^*-algebras with CPoU and $S \subseteq \prod^{\omega} \mathcal{M}_n$ is a $\|\cdot\|_{2,X^{\omega}}$-separable subset. If $a \in \mathcal{M} \cap S'$, then there is a unitary $u \in \prod^{\omega} \mathcal{M}_n \cap S'$ such that $a = u|a|$. Further, if $a = a^*$, we may arrange that $u = u^*$.*

Proof. For the sake of brevity, define $(\mathcal{M}, X) := \big(\prod^{\omega} \mathcal{M}_n, \sum^{\omega} X_n\big)$. We may assume S is a set of contractions. Fix $\epsilon > 0$ and a finite set $\mathcal{F} \subseteq S$. For the first part, by Kirchberg's ϵ-test, it suffices to show that if $a \in \mathcal{M}$, then there is a unitary $u \in \mathcal{M}$ such that

$$\|a - u|a|\|_{2,X} \leq \epsilon \qquad \text{and} \qquad \max_{b \in \mathcal{F}} \|[u, b]\|_{2,X} \leq \epsilon. \tag{7.44}$$

By the existence of unitary polar decompositions in finite von Neumann algebras, for each $\tau \in X$, there is a unitary $\bar{u}_\tau \in \pi_\tau(\mathcal{M})'' \cap \pi_\tau(\mathcal{F})'$ such that

$$\pi_\tau(a) = \bar{u}_\tau \pi_\tau(|a|). \tag{7.45}$$

Since $\pi_\tau(\mathcal{M})'' \cap \pi_\tau(\mathcal{F})'$ is a von Neumann algebra, $\bar{u}_\tau = e^{i\bar{h}_\tau}$ for some self-adjoint $\bar{h}_\tau \in \pi_\tau(\mathcal{M})'' \cap \pi_\tau(\mathcal{F})'$.

By Kaplansky's density theorem, we can approximate $\bar{h}_\tau$ by a self-adjoint in $\pi_\tau(\mathcal{M})$ and lift to a self-adjoint element $h_\tau \in \mathcal{M}$ such that unitary $u_\tau \coloneqq e^{ih_\tau} \in \mathcal{M}$ satisfies

$$\begin{aligned} \|a - u_\tau|a|\|_{2,\tau} &< (2 + |\mathcal{F}|)^{-1/2}\epsilon, \\ \max_{b\in\mathcal{F}} \|[u_\tau, b]\|_{2,\tau} &< (2 + |\mathcal{F}|)^{-1/2}\epsilon. \end{aligned} \tag{7.46}$$

For $\tau \in X$, define

$$c_\tau \coloneqq \left|a - u_\tau|a|\right|^2 + \sum_{b\in\mathcal{F}} \left|[u_\tau, b]\right|^2 \in \mathcal{M}_+, \tag{7.47}$$

and note that $\tau(c_\tau) < \epsilon^2$ for all $\tau \in X$. By the compactness of X, there are $\tau_1, \ldots, \tau_k \in X$ such that

$$\sup_{\tau\in X} \min_{1\le j\le k} \tau(c_{\tau_j}) < \epsilon^2. \tag{7.48}$$

By CPoU (in the form of Corollary 7.8), there are mutually orthogonal projections

$$p_1, \ldots, p_k \in \mathcal{M} \cap \mathcal{F}' \cap \{a\}' \cap \{u_{\tau_1}, \ldots, u_{\tau_k}\}' \tag{7.49}$$

such that

$$\sum_{j=1}^{k} p_j = 1_{\mathcal{M}} \qquad \text{and} \qquad \tau(c_{\tau_j} p_j) \le \epsilon^2 \tau(p_j), \tag{7.50}$$

for all $1 \le j \le k$ and $\tau \in X$. Define $u \coloneqq \sum_{i=j}^{n} u_{\tau_j} p_j$. As the u_{τ_j} are unitaries commuting with the p_j, we have that u is a unitary. We compute that for all $\tau \in X$ and $b \in \mathcal{F}$,

$$\|a - u|a|\|_{2,\tau}^2 \le \sum_{j=1}^{k} \tau(c_{\tau_j} p_j) \le \epsilon^2 \sum_{j=1}^{k} \tau(p_j) = \epsilon^2 \tag{7.51}$$

and

$$\|[u, b]\|_{2,\tau}^2 \le \sum_{j=1}^{k} \tau(c_{\tau_j} p_j) \le \epsilon^2 \sum_{j=1}^{k} \tau(p_j) = \epsilon^2 \tag{7.52}$$

which implies (7.44), as required.

In the case that $a^* = a$, the unitaries $\bar{u}_\tau$ can be taken to be self-adjoint. For this, by ensuring the unitaries u_τ are chosen with $\|u_\tau - u_\tau^*\|_{2,\tau}$ sufficiently small, and replacing c_τ with $c_\tau + |u_\tau - u_\tau^*|^2$ in the above proof, one may also arrange $\|u - u^*\|_{2,X^\omega} \le \epsilon$ in (7.44). □

As promised, we have the following strengthened version of Proposition 7.2.

Corollary 7.15. *If $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of factorial tracially complete C^*-algebras with CPoU and $S \subseteq \prod^{\omega} \mathcal{M}_n$ is a $\|\cdot\|_{2,X^\omega}$-separable subset, then $\prod^{\omega} \mathcal{M}_n \cap S'$ has real rank zero and stable rank one.*

Proof. Suppose $a \in \prod^{\omega} \mathcal{M}_n \cap S'$ and write $a = u|a|$ for a unitary $u \in \prod^{\omega} \mathcal{M}_n \cap S'$ as in Proposition 7.14. Then for each $\epsilon > 0$, the element $b \coloneqq u(|a| + \epsilon)$ is invertible and $\|a - b\| \leq \epsilon$. Therefore, $\prod^{\omega} \mathcal{M}_n \cap S'$ has stable rank one. If a is self-adjoint and we take $u = u^*$, then u and $|a|$ commute. It follows that b is self-adjoint, and this shows $\prod^{\omega} \mathcal{M}_n \cap S'$ has real rank zero. □

In the same spirit as the comments after Corollary 7.11, an approximate version of stable rank one can be obtained for all factorial tracially complete C^*-algebras $(\mathcal{M}, X)$ with CPoU. Indeed, combining Proposition 7.14 (with $S = \emptyset$) and Corollary 7.11 shows that a $\|\cdot\|_{2,X}$-dense set of elements $a \in \mathcal{M}$ have the form $a = u|a|$ for some unitary $u \in \mathcal{M}$. Then the proof of Corollary 7.15 shows that a $\|\cdot\|_{2,X}$-dense set of elements in $\mathcal{M}$ are invertible. The analogous statement for self-adjoint elements is less clear – this would require a version of Corollary 7.11 for self-adjoint unitaries (or, equivalently, for projections).

Question 7.16. If $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU, does $\mathcal{M}$ have real rank zero or stable rank one?

7.3. **Classification of projections.** Questions 7.13 and 7.16 expose a common drawback of the CPoU technique – even when exact results or $\|\cdot\|$-approximate results are possible in finite von Neumann algebras, a direct application of CPoU will always introduce a uniform 2-norm error. In the case of classification of projections, we can overcome this defect by controlling the 2-norm distance from the unitary implementing the equivalence to the unit. The following lemma and its proof is analogous to [105, Lemma XIV.2.1] where the result is shown for finite von Neumann algebras. It is possible to prove this result by combining the von Neumann algebra result with a CPoU argument similar to those above, but with the structural results of reduced products already obtained, a more direct proof is possible.

Lemma 7.17. *Suppose $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU. If $p, q \in \mathcal{M}$ are projections with $\tau(p) = \tau(q)$ for all $\tau \in X$ and $\epsilon > 0$, then there is a unitary $u \in \mathcal{M}$ such that*

$$\|upu^* - q\|_{2,X} < \epsilon \qquad \text{and} \qquad \|u - 1_{\mathcal{M}}\|_{2,X} < 2\sqrt{2}\|p - q\|_{2,X} + \epsilon. \tag{7.53}$$

Proof. By the liftability of unitaries in reduced products (Corollary 7.11), it suffices to show there is a unitary $u \in \mathcal{M}^\omega$ such that

$$upu^* = q \qquad \text{and} \qquad \|u - 1_{\mathcal{M}^\omega}\|_{2,X^\omega} \leq 2\sqrt{2}\|p - q\|_{2,X}. \tag{7.54}$$

By Proposition 7.4, there is a unitary $v \in \mathcal{M}^\omega$ such that $vpv^* = q$. Set $a \coloneqq pvp + p^\perp vp^\perp$. As $a \in \mathcal{M}^\omega \cap \{p\}'$, Proposition 7.14 implies there is a unitary $w \in \mathcal{M}^\omega \cap \{p\}'$ such that $a = w|a|$. Then $u \coloneqq vw^*$ is a unitary with $upu^* = q$.

Note that $(pvp - pv)^*(p^\perp vp^\perp - p^\perp v) = 0$. Therefore, by the Pythagorean identity,

$$\begin{aligned} \|a - v\|_{2,X^\omega}^2 &= \|pvp - pv\|_{2,X^\omega}^2 + \|p^\perp vp^\perp - p^\perp v\|_{2,X^\omega}^2 \\ &= \|p(q-p)v\|_{2,X^\omega}^2 + \|p^\perp(q^\perp - p^\perp)v\|_{2,X^\omega}^2 \\ &\leq 2\|p - q\|_{2,X}^2. \end{aligned} \tag{7.55}$$

Further, since $\|a\| \leq 1$, we have $(1_{\mathcal{M}^\omega} - |a|)^2 \leq (1_{\mathcal{M}^\omega} - |a|^2)^2$ in $\mathcal{M}^\omega$. Therefore,

$$\begin{aligned} \|w - a\|_{2,X^\omega}^2 &= \|1 - |a|\|_{2,X^\omega}^2 \\ &\leq \|1 - |a|^2\|_{2,X^\omega}^2 \\ &= \|p - pv^*pvp\|_{2,X^\omega}^2 + \|p^\perp - p^\perp v^* p^\perp v p^\perp\|_{2,X^\omega}^2 \\ &= \|pv^*(q-p)vp\|_{2,X^\omega}^2 + \|p^\perp v^*(q^\perp - p^\perp)vp\|_{2,X^\omega}^2 \\ &\leq 2\|p - q\|_{2,X}^2. \end{aligned} \tag{7.56}$$

Combining (7.55) and (7.56) shows

$$\|u - 1_{\mathcal{M}^\omega}\|_{2,X^\omega} = \|v - w\|_{2,X^\omega} \leq 2\sqrt{2}\|p - q\|_{2,X}. \tag{7.57}$$ □

The second part of the following result is Theorem 1.3(ii) from the introduction. Note also that taking $(\mathcal{M}, X)$ to be a trivial W^*-bundle over a compact Hausdorff space K with fibre being a II_1 factor with property Γ in the following result proves Theorem E from the overview, using Theorem 1.4 to verify the CPoU hypothesis.

Theorem 7.18. *Suppose $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra with CPoU and $p, q \in \mathcal{M}$ are projections.*

(i) *If $\tau(p) \leq \tau(q)$ for all $\tau \in X$, then p is Murray–von Neumann subequivalent to q.*
(ii) *If $\tau(p) = \tau(q)$ for all $\tau \in X$, then p and q are unitarily equivalent.*

Proof. First we show (ii). Let $(\epsilon_n)_{n=1}^\infty \subseteq (0, \infty)$ be a decreasing sequence with $\sum_{n=1}^\infty \epsilon_n < \infty$. We will construct a sequence of unitaries $(u_n)_{n=1}^\infty \subseteq \mathcal{M}$ such that

$$\begin{aligned} &\|u_n p u_n^* - q\|_{2,X} < \epsilon_n, && n \geq 1, \\ &\|u_{n+1} - u_n\|_{2,X} < 4\epsilon_{n-1}, && n \geq 2. \end{aligned} \tag{7.58}$$

Then $(u_n)_{n=1}^\infty \subseteq \mathcal{M}$ is a $\|\cdot\|$-bounded, $\|\cdot\|_{2,X}$-Cauchy sequence, and hence converges to some $u \in \mathcal{M}$. Since multiplication is $\|\cdot\|_{2,X}$-continuous on $\|\cdot\|$-bounded sets, it follows that u is a unitary and $upu^* = q$.

The sequence $(u_n)_{n=1}^\infty$ will be constructed inductively. Lemma 7.17 implies there is a unitary $u_1 \in \mathcal{M}$ such that

$$\|u_1 p u_1^* - q\|_{2,X} < \epsilon_1. \tag{7.59}$$

Assuming u_n has been constructed, applying Lemma 7.17 to the projections $u_n p u_n^*$ and q, there is a unitary $v_n \in \mathcal{M}$ such that

$$\begin{aligned} \|v_n u_n p u_n^* v_n^* - q\|_{2,X} &< \epsilon_{n+1}, \\ \|v_n - 1_{\mathcal{M}}\|_{2,X} &< 2\sqrt{2}\epsilon_n + \epsilon_{n+1}. \end{aligned} \tag{7.60}$$

Now set $u_{n+1} \coloneqq v_n u_n$. Then $\|u_{n+1} p u_{n+1} - q\|_{2,X} < \epsilon_{n+1}$ and

$$\begin{aligned} \|u_{n+1} - u_n\|_{2,X} &\leq \|u_n\| \|v_n - 1\|_{2,X} \\ &< 2\sqrt{2}\epsilon_n + \epsilon_{n+1} \\ &\leq 3\epsilon_n + \epsilon_n \\ &= 4\epsilon_n, \end{aligned} \tag{7.61}$$

as required.

Now we show (i). By Proposition 7.4, there is a partial isometry $v \in \mathcal{M}^\infty$ such that $v^*v = p$ and $vv^* \leq q$. Suppose $r\colon \mathbb{N} \to \mathbb{N}$ is a function with $\lim_{n\to\infty} r(n) = \infty$ and consider the reparameterisation morphism $r^*\colon \mathcal{M}^\infty \to \mathcal{M}^\infty$ as defined just before Theorem 5.11. Since $p, q \in \mathcal{M}$, we have $r^*(p) = p$ and $r^*(q) = q$. For all $\tau \in X^\infty$,

$$\tau(r^*(q - vv^*)) = \tau(r^*(q)) - \tau(r^*(p)) = \tau(q) - \tau(p) = \tau(q - vv^*). \tag{7.62}$$

By Proposition 7.4, $q - vv^*$ and $r^*(q - vv^*)$ are unitarily equivalent in $\mathcal{M}^\infty$. By Corollary 7.11, unitaries in $\mathcal{M}^\infty$ lift to unitaries in $\ell^\infty(\mathcal{M})$, so we may apply intertwining through reparameterisation (Theorem 5.11) in the case where S is a one point space, and obtain a projection $p' \in \mathcal{M}$ that is unitarily equivalent to $q - vv^*$ in $\mathcal{M}^\infty$. The projections $p \oplus p'$ and $q \oplus 0$ in $M_2(\mathcal{M})$ agree on traces. By Proposition 6.14, $M_2(\mathcal{M})$ has CPoU. Hence, $p \oplus p'$ and $q \oplus 0$ are unitarily equivalent by (ii). Therefore, $p \precsim q$ in $M_2(\mathcal{M})$ and so also in $\mathcal{M}$. □

The uniqueness theorem for projections is complemented by a corresponding existence theorem building projections with prescribed behaviour on traces. This is obtained by applying CPoU to Proposition 2.8(iv). Note that the following result is precisely Theorem 1.3(ii) from the introduction.

Theorem 7.19. *Suppose $(\mathcal{M}, X)$ is a type II_1 factorial tracially complete C^*-algebra with CPoU. If $f \in \mathrm{Aff}(X)$ with $0 \leq f \leq 1$, then there is a projection $p \in \mathcal{M}$ such that $\tau(p) = f(\tau)$ for all $\tau \in T(\mathcal{M})$.*

Proof. It suffices to construct a projection $p \in \mathcal{M}^\infty$ such that

$$\tau(p) = f(\tau|_{\mathcal{M}}), \qquad \tau \in X^\infty. \tag{7.63}$$

Indeed, assuming this has been done, for every function $r\colon \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} r(n) = \infty$, we have

$$\tau(r^*(p)) = f((\tau \circ r^*)|_{\mathcal{M}}) = f(\tau|_{\mathcal{M}}) = \tau(p) \tag{7.64}$$

for all $\tau \in X^\infty$, and hence by Lemma 7.3, $r^*(p)$ and p are unitarily equivalent. Since unitaries in $\mathcal{M}^\infty$ lift to unitaries in $\ell^\infty(\mathcal{M})$ (Corollary 7.11), intertwining through reparameterisation (Theorem 5.11) implies that there is a projection $p_0 \in \mathcal{M}$ that is unitarily equivalent to p. Then $\tau(p_0) = f(\tau)$ for all $\tau \in X$.

We work to build a projection $p \in \mathcal{M}^\infty$ satisfying (7.63). First, we extend f to all of $T(\mathcal{M})$ using Theorem 2.3. Then, by Proposition 2.7, there is a self-adjoint $c \in \mathcal{M}$ such that $\tau(c) = f(\tau)$ for all $\tau \in T(\mathcal{M})$.

Fix $\tau \in X$ and $\epsilon > 0$. Since $\pi_\tau(\mathcal{M})''$ is type II_1, Proposition 2.8(iv) implies there is a projection $\bar{p}_\tau \in \pi_\tau(\mathcal{M})''$ such that $\sigma(\bar{p}_\tau) = \sigma(\pi_\tau(c))$ for all $\sigma \in T(\pi_\tau(\mathcal{M})'')$. Therefore, by [43, Théorème 3.2] and Kaplansky's

density theorem, there are a positive contraction $p_\tau \in \mathcal{M}$ and elements $x_{j,\tau}, y_{j,\tau} \in \mathcal{M}$, for $j = 1, \dots, 10$, of norm at most $12(\|c\| + 1)$ such that

$$\left\| p_\tau - c - \sum_{j=1}^{10} [x_{j,\tau}, y_{j,\tau}] \right\|_{2,\tau}^2 < \epsilon \quad \text{and} \quad \tau(p_\tau - p_\tau^2) < \epsilon \tag{7.65}$$

for all $\tau \in X$. Define

$$a_\tau \coloneqq p_\tau - p_\tau^2 + \left| p_\tau - c - \sum_{j=1}^{10} [x_{j,\tau}, y_{j,\tau}] \right|^2 \in \mathcal{M}_+ \tag{7.66}$$

and note that $\tau(a_\tau) < 2\epsilon$.

As X is compact, there are $\tau_1, \dots, \tau_k \in X$ such that

$$\sup_{\tau \in X} \min_{1 \le i \le k} \tau(a_{\tau_i}) < 2\epsilon.$$

By CPoU, there are projections $q_1, \dots, q_k \in \mathcal{M}^\infty$ summing to $1_{\mathcal{M}^\infty}$ and commuting with c, p_{τ_i}, x_{j,τ_i} and y_{j,τ_i} for all $i = 1, \dots, k$ and $j = 1, \dots, 10$, such that

$$\tau(q_i a_{\tau_i}) \le 2\epsilon \tau(q_i), \qquad \tau \in X^\infty, \quad i = 1, \dots, k. \tag{7.67}$$

Define

$$p \coloneqq \sum_{i=1}^{k} q_i p_{\tau_i}, \quad x_j \coloneqq \sum_{i=1}^{k} q_i x_{j,\tau_i}, \quad \text{and} \quad y_j \coloneqq \sum_{i=1}^{k} q_i y_{j,\tau_i} \tag{7.68}$$

for $j = 1, \dots, 10$. Then we have

$$\left\| p - c - \sum_{i=1}^{10} [x_j, y_j] \right\|_{2,X^\infty} \le \sqrt{2\epsilon} \quad \text{and} \quad \sup_{\tau \in X^\infty} \tau(p - p^2) \le \sqrt{2}\epsilon \tag{7.69}$$

The result of the previous paragraph and Kirchberg's ϵ-test implies there are a projection $p \in \mathcal{M}^\infty$ and elements $x_j, y_j \in \mathcal{M}^\infty$ such that

$$p = c + \sum_{j=1}^{10} [x_j, y_j]. \tag{7.70}$$

Then for all $\tau \in X^\infty$, $\tau(p) = \tau(c) = f(\tau|_{\mathcal{M}})$ as required in (7.63). □

The classification of projections given in Theorems 7.18 and 7.19 allows us to compute the Murray–von Neumann semigroup and the K_0-group for factorial tracially complete C^*-algebra with CPoU. This is immediate from the previous two results and Propositions 5.8 and 6.14, which allow us to apply the previous two results to projections in matrix amplifications.

Corollary 7.20. *Suppose $(\mathcal{M}, X)$ is a type II_1 factorial tracially complete C^*-algebra with CPoU. The natural maps*

$$V(\mathcal{M}) \to \mathrm{Aff}(X)_+ \qquad \text{and} \qquad K_0(\mathcal{M}) \to \mathrm{Aff}(X) \tag{7.71}$$

are isomorphisms of ordered monoids and ordered groups, respectively.

8. Hyperfiniteness

There is a natural notion of hyperfiniteness for tracially complete C^*-algebras which is modelled on the Murray–von Neumann notion of hyperfiniteness for II_1 factors from [83]. It is defined by a local approximation condition, asking that every finite set is approximately contained in a finite dimensional C^*-subalgebra and the existence of an inductive limit decomposition.

Section 8.1 contains the definition of hyperfiniteness along with the statements of all the results needed to prove the regularity theorem: namely, hyperfiniteness implies amenability and CPoU. The proof of the latter implication occupies most of this section. En route to proving CPoU, we prove an inductive limit decomposition for hyperfinite tracially complete C^*-algebras in the separable setting (Theorem 8.4). Section 8.2 requires the finite dimensional perturbation lemmas required to obtain the inductive limit decomposition. The proofs of Theorems 8.3 and 8.4 are given in Section 8.3.

8.1. Definition and main properties.

We define hyperfiniteness analogously to Murray and von Neumann's notion for II_1 factors in [83].

Definition 8.1. We say that a tracially complete C^*-algebra $(\mathcal{M}, X)$ is *hyperfinite* if for every finite set $\mathcal{F} \subseteq \mathcal{M}$ and $\epsilon > 0$, there is a finite dimensional C^*-algebra $F \subseteq \mathcal{M}$ such that for every $a \in \mathcal{F}$, there is a $b \in F$ with $\|a - b\|_{2,X} < \epsilon$.

The passage from hyperfiniteness to amenability is a little subtle. The obvious approach is to verify the completely positive approximation property directly by taking the "downward" maps to be conditional expectations onto finite dimensional subalgebras and the "upward" maps to be the inclusions. For the estimates to match up correctly, one must arrange for the conditional expectations to be continuous in the uniform 2-norms, and it is not clear if this can be done. This issue does not arise in the setting of tracial von Neumann algebras (as the trace-preserving conditional expectation onto a subalgebra is necessarily 2-norm contractive), so we can obtain amenability by working locally.

Theorem 8.2. *Every hyperfinite tracially complete C^*-algebra is amenable.*

Proof. Suppose $(\mathcal{M}, X)$ is a hyperfinite tracially complete C^*-algebras. For all $\tau \in X$, the tracial von Neumann algebra $\pi_\tau(\mathcal{M})''$ is hyperfinite and hence is semidiscrete. The result follows from Theorem 1.2 (which was proved in Theorem 4.9). □

The following result obtaining CPoU from hyperfiniteness is more subtle yet.

Theorem 8.3. *Every hyperfinite factorial tracially complete C^*-algebra satisfies CPoU.*

It is easy to see that finite dimensional C^*-algebras satisfy CPoU in the sense that if F is a finite dimensional C^*-algebra, then $\big(F, T(F)\big)$ is a factorial tracially complete C^*-algebras with CPoU (this is a very special case

of Corollary 6.4). If one strengthens the definition of hyperfiniteness (Definition 8.1) to require that the finite dimensional subalgebras $F \subseteq \mathcal{M}$ is factorial when viewed as a tracially complete C^*-algebra with the traces inherited from $(\mathcal{M}, X)$ (or equivalently, that every trace on F extends to a trace in X), then CPoU for $(\mathcal{M}, X)$ follows directly from CPoU from $\big(F, T(F)\big)$.

For a general finite dimensional C^*-subalgebra $F \subseteq \mathcal{M}$, we do not have a way of relating general traces in $T(F)$ to traces of the form $\tau|_F$ for $\tau \in X$, so the above argument does not work. A somewhat related issue was addressed by Murray and von Neumann in the setting of II_1 factors in [83]. They defined a II_1 factor $\mathcal{M}$ to be "approximately finite (B)" if it satisfies for all finite set $\mathcal{F} \subseteq \mathcal{M}$ and $\epsilon > 0$, there is a finite dimensional C^*-algebra $F \subseteq \mathcal{M}$ such that for all $a \in \mathcal{F}$, there is a $b \in F$ such that $\|a - b\|_2 < \epsilon$, and they define $\mathcal{M}$ to be "approximately finite (A)" if one can further arrange for F to be a factor. It is a non-trivial result that these two notions coincide (see [83, Lemma 4.6.2]).[55]

We circumvent the need to establish the equivalence discussed in the previous paragraph by showing that (in the separable setting) hyperfiniteness is equivalent to the a-priori stronger property of being a sequential inductive limit of finite dimensional algebras. Murray and von Neumann established such a result for separably acting hyperfinite II_1 factors (using the terminology "approximately finite (C)" for sequential inductive limits), and Bratteli proved a C^*-version of this result in his seminal paper on AF C^*-algebras, ([11]).

Theorem 8.4. *If $(\mathcal{M}, X)$ is a hyperfinite tracially complete C^*-algebra such that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable, then $(\mathcal{M}, X)$ is an inductive limit of tracially complete C^*-algebras $(\mathcal{M}_n, X_n)$, $n \geq 1$, such that each $\mathcal{M}_n$ is finite dimensional.*

The proof has the same flavour as the corresponding results for C^*-algebras and tracial von Neumann algebras. We will prove Theorem 8.4 in Section 8.3 after we set out the relevant uniform 2-norm perturbation and near-containment lemmas in the next section.

Once, Theorem 8.4 is in place, it will allow us to realise a separable factorial hyperfinite C^*-algebras as a tracial completion of the tracial completion of an AF algebra with respect to a closed face of its traces. Such tracial completions have CPoU by the permanence properties of Section 6.3. This will prove Theorem 8.3 in the separable case, and we will deduce the non-separable case from there.

8.2. **Perturbation lemmas.** The goal of this subsection is to prove a uniform 2-norm 'near-containment' result (Proposition 8.8) for finite dimensional subalgebras. Similar results were used by Murray and von Neumann in the von Neumann algebra setting, and Glimm and Bratteli for C^*-algebras (cf. [83, 51, 11]). The main observation is that although the Borel functional

[55] In the end, for type II_1 factorial tracially complete C^*-algebras, the two analogous notions of hyperfiniteness are equivalent as a consequence of Theorem 9.15 using that the tracially complete C^*-algebras $(\mathcal{R}_X, X)$ of Example 3.35 satisfy the stronger notion of hyperfiniteness. However, this is circular since Theorem 9.15 depends on Theorem 8.3.

calculus used in the von Neumann algebraic results does not exist in a tracially complete C^*-algebra, it is defined in all finite dimensional subalgebras.

We start with a perturbation result for almost orthogonal projections.

Lemma 8.5. *For all $\epsilon > 0$ and $n \in \mathbb{N}$, there is a $\delta > 0$ such that for all finite dimensional tracially complete C^*-algebras (F, X) and all self-adjoint contractions $q_1, \ldots, q_n \in F$ with*

$$\|q_i - q_i^2\|_{2,X} < \delta \qquad \text{and} \qquad \|q_i q_j\|_{2,X} < \delta, \tag{8.1}$$

for all $i, j = 1, \ldots, n$ with $i \neq j$, there are mutually orthogonal projections $p_1, \ldots, p_n \in F$ such that $\|p_i - q_i\|_{2,X} < \epsilon$.

Proof. We will prove the result by induction on n. When $n = 1$, set $\delta = \epsilon/2$, let $f \colon \mathbb{R} \to \mathbb{R}$ denote the characteristic function of $[1/2, \infty)$, and define $p_1 \coloneqq f(q_1)$. Then

$$|f(t) - t| \leq 2|t - t^2|, \qquad t \in \mathbb{R}, \tag{8.2}$$

and so $\|p_1 - q_1\|_{2,X} \leq 2\|q_1 - q_1^2\|_{2,X} < \epsilon$.

Assuming the result has been proven for $n \in \mathbb{N}$, let $\delta' > 0$ be given by applying the lemma with this n and with $\epsilon/(8n)$ in place of ϵ. Define

$$\delta \coloneqq \min\Big\{\delta', \frac{\epsilon}{8n+4}\Big\}. \tag{8.3}$$

Let $q_1, \ldots, q_{n+1} \in F$ be given as in the statement. As $\delta \leq \delta'$, the choice of δ' implies there are mutually orthogonal projections $p_1, \ldots, p_n \in F$ such that $\|p_i - q_i\|_{2,X} < \epsilon/(10n+2)$ for $i = 1, \ldots, n$.

Define $p \coloneqq \sum_{i=1}^n p_i$, $q \coloneqq q_{n+1}$, and $p_{n+1} \coloneqq f(p^\perp q p^\perp)$. Then p_{n+1} is a projection and is orthogonal to each of $p_1, \ldots, p_n$. Using the bimodule property of (3.2), we can estimate

$$\begin{aligned}
\|p_{n+1} - q_{n+1}\|_{2,X} &\leq \|f(p^\perp q p^\perp) - p^\perp q p^\perp\|_{2,X} + \|p^\perp q p^\perp - q\|_{2,X} \\
&\leq 2\|p^\perp q p^\perp - (p^\perp q p^\perp)^2\|_{2,X} + 2\|pq\|_{2,X} \\
&\leq 2\|q - qp^\perp q\|_{2,X} + 2\|pq\|_{2,X} \\
&\leq 2\|q - q^2\|_{2,X} + 4\|pq\|_{2,X} \\
&< 2\delta + 4\sum_{i=1}^n \|p_i q\|_{2,X} \\
&\leq 2\delta + 4\sum_{i=1}^n \big(\|p_i - q_i\|_{2,X} + \|q_i q\|_{2,X}\big) \\
&< 2\delta + 4n\Big(\frac{\epsilon}{8n} + \delta\Big) \leq \epsilon
\end{aligned}$$

□

The following perturbation result for partial isometries will be used to perturb the off diagonal matrix units of a near inclusion of finite dimensional C^*-algebras.

Lemma 8.6. *Suppose (F, X) is a finite dimensional tracially complete C^*-algebra and $p, q \in F$ are projections. If $w \in F$ is a contraction, then there is a partial isometry $v \in F$ such that $vp = v = qv$ and*

$$\|v - w\|_{2,X} \leq 6 \max\big\{\|w^*w - p\|_{2,X}^{1/2}, \|ww^* - q\|_{2,X}^{1/2}\big\}. \tag{8.4}$$

Proof. Define

$$\delta \coloneqq \max\big\{\|w^*w - p\|_{2,X}, \|ww^* - q\|_{2,X}\big\}. \tag{8.5}$$

Let $u \coloneqq qwp$ and note that u is a contraction and

$$\max\big\{\|u^*u - p\|_{2,X}, \|uu^* - q\|_{2,X}\big\} \leq 3\delta. \tag{8.6}$$

Indeed,

$$\begin{aligned} \|u^*u - p\|_{2,X} &= \|pw^*qwp - p\|_{2,X} \\ &\leq \|pw^*(q - ww^*)wp\|_{2,X} + \|pw^*w(w^*w - p)p\|_{2,X} \\ &\quad + \|p(w^*w - p)p\|_{2,X} \\ &\leq 3\delta, \end{aligned} \tag{8.7}$$

where the last inequality uses (8.5) and the bimodule property of (3.2). The other inequality in (8.6) follows similarly.

Define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) \coloneqq \begin{cases} 0 & 0 \leq t \leq \frac{1}{4} \\ t^{-1/2} & \frac{1}{4} < t \leq 1 \end{cases} \tag{8.8}$$

and define $v \coloneqq ug(u^*u)$. Then v^*v is a projection, and hence v is a partial isometry. As $up = u = qu$, we also have $vp = v = qv$. After solving some polynomial inequalities, we get that for $t \in [0,1]$, we have

$$0 \leq |t - tg(t)| \leq \frac{4}{3}(t - t^2) \qquad \text{and} \qquad 0 \leq g(t) \leq 2. \tag{8.9}$$

Therefore, for each $\tau \in X$,

$$\begin{aligned} \|u - v\|_{2,\tau}^2 &= \tau\big((u - ug(u^*u))^*(u - ug(u^*u))\big) \\ &= \tau\big(u^*u - 2u^*ug(u^*u) + u^*ug(u^*u)^2\big) \\ &= \tau\big(u^*u - u^*ug(u^*u)\big) + \tau\big((u^*ug(u^*u) - u^*u)g(u^*u)\big) \\ &\leq \tau\big(|u^*u - u^*ug(u^*u)|\big) + 2\tau\big(|u^*u - u^*ug(u^*u)|\big) \\ &\overset{(8.9)}{\leq} 4\tau(u^*u - (u^*u)^2) \\ &\leq 4(\|u^*u - p\|_{2,\tau} + \|p - (u^*u)^2\|_{2,\tau}) \\ &\overset{(8.6)}{\leq} 36\delta, \end{aligned} \tag{8.10}$$

using Cauchy–Schwarz, and the fact that p is a projection. □

The perturbations of the previous two lemmas may change the trace of the diagonal matrix units. The following is used to account for this.

Lemma 8.7. *If (F, X) is a finite dimensional tracially complete C^*-algebra and $p_1, \ldots, p_n, p \in F$ are projections with $p_i \leq p$ for all $i = 1, \ldots, n$, then*

$$\Big\|p - \bigwedge_{i=1}^n p_i\Big\|_{2,X} \leq \Big(\sum_{i=1}^n \|p - p_i\|_{2,X}^2\Big)^{1/2} \leq \sum_{i=1}^n \|p - p_i\|_{2,X}. \tag{8.11}$$

Proof. The second inequality amounts to $\sum_{i=1}^n \lambda_i^2 \leq (\sum_{i=1}^n \lambda)^2$ for positive λ_i, so is immediate. We prove the first inequality by induction on n with the case $n = 1$ being trivial. Let $q_1 \coloneqq \bigwedge_{i=1}^n p_i$ and $q_2 \coloneqq p_{n+1}$. By [7, Proposition III.1.1.3],

$$(q_1 - (q_1 \wedge q_2)) \text{ and } ((q_1 \vee q_2) - q_2), \tag{8.12}$$

are Murray–von Neumann equivalent. Therefore, for each $\tau \in X$,

$$\begin{aligned} \|p - (q_1 \wedge q_2)\|_{2,\tau}^2 &= \tau(p - (q_1 \wedge q_2)) \\ &= \tau(p - q_1) + \tau(q_1 - (q_1 \wedge q_2)) \\ &= \tau(p - q_1) + \tau((q_1 \vee q_2) - q_2) \\ &\leq \tau(p - q_1) + \tau(p - q_2) \\ &= \|p - q_1\|_{2,\tau}^2 + \|p - q_2\|_{2,\tau}^2. \end{aligned} \tag{8.13}$$

Hence, we have

$$\Big\|p - \bigwedge_{i=1}^{n+1} p_i\Big\|_{2,X}^2 \leq \Big\|p - \bigwedge_{i=1}^{n} p_i\Big\|_{2,X}^2 + \|p - p_{n+1}\|_{2,X}^2 \tag{8.14}$$

The first inequality in (8.11) now follows by induction. □

The last three lemmas combine to prove the following stability result for near inclusions of finite dimensional tracially complete C^*-algebras.

Proposition 8.8. *Suppose $(\mathcal{M}, X)$ is a tracially complete C^*-algebra, let F be a finite dimensional C^*-subalgebra of $\mathcal{M}$, and let $\mathcal{F} \subseteq F$ be a system of matrix units. For all $\epsilon > 0$, there exists $\delta > 0$ such that if $G \subseteq \mathcal{M}$ is a finite dimensional C^*-algebra and for all $a \in \mathcal{F}$, there exists $b \in G$ with $\|a - b\|_{2,X} < \delta$, then there is a (not necessarily unital) *-homomorphism $\phi\colon F \to G$ such that*

$$\|\phi(a) - a\|_{2,X} \leq \epsilon\|a\|, \qquad a \in F. \tag{8.15}$$

Proof. Let

$$\big(e_{i,j}^{(k)}\big)_{i,j=1,\dots,d_k}^{k=1,\dots,m} \subseteq F \tag{8.16}$$

be a system of matrix units for F. After replacing ϵ with a scalar multiple (depending only on the dimension of F), it suffices to construct a corresponding system of matrix units

$$\big(f_{i,j}^{(k)}\big)_{i,j=1,\dots,d_k}^{k=1,\dots,m} \subseteq G \tag{8.17}$$

such that $\big\|e_{i,j}^{(k)} - f_{i,j}^{(k)}\big\|_{2,X} < \epsilon$ for all i, j, k.

Let $d = \max\{d_1, \dots, d_m\}$ and choose $\epsilon' > 0$ with

$$(4\epsilon' + 24(\epsilon')^{1/2})(d+1) < \epsilon. \tag{8.18}$$

Apply Lemma 8.5 with ϵ' in place of ϵ and with $\sum_{k=1}^m d_k$ in place of n to obtain $\delta' > 0$, and define $\delta := \min\{\epsilon'/3, \delta'/9\} > 0$. By assumption and Lemma 3.27, for each $i = 1, \dots, d_k$ and $k = 1, \dots, m$, there are contractions $q_i^{(k)}, w_i^{(k)} \in G$ such that

$$\big\|q_i^{(k)} - e_{i,i}^{(k)}\big\|_{2,X} < 3\delta \leq \epsilon' \quad \text{and} \quad \big\|w_i^{(k)} - e_{i,1}^{(k)}\big\|_{2,X} < 3\delta \leq \epsilon'. \tag{8.19}$$

For all $k, k' = 1, \dots, m$, $i = 1, \dots, d_k$, and $i' = 1, \dots, d_{k'}$ such that $(i,k) \neq (i',k')$, we have

$$\|(q_i(k))^2 - q_i^{(k)}\|_{2,X} < 9\delta \leq \delta', \quad \text{and} \quad \|q_i^{(k)} q_{i'}^{(k')}\|_{2,X} < 6\delta < \delta'. \tag{8.20}$$

By the choice of δ', we may invoke the conclusion of Lemma 8.5 to produce mutually orthogonal projections $p_i^{(k)} \in G$ such that for all i and k,

$$\|p_i^{(k)} - q_i^{(k)}\|_{2,X} < \epsilon'. \tag{8.21}$$

Now for each i and k, we have

$$w_i^{(k)*} w_i^{(k)} \overset{(8.19)}{\approx}_{2\epsilon'} e_{1,i}^{(k)} e_{i,1}^{(k)} = e_{1,1}^{(k)} \overset{(8.19)}{\approx}_{\epsilon'} q_1^{(k)} \overset{(8.21)}{\approx}_{\epsilon'} p_1^{(k)}, \tag{8.22}$$

and similarly $w_i^{(k)} w_i^{(k)*} \approx_{4\epsilon'} p_i^{(k)}$, where the approximations are in $\|\cdot\|_{2,X}$. By Lemma 8.6, for each i and k, there is a partial isometry $v_i^{(k)} \in G$ with source contained in $p_1^{(k)}$ and range contained in $p_i^{(k)}$ and such that

$$\|v_i^{(k)} - w_i^{(k)}\|_{2,X} \leq 12(\epsilon')^{1/2}. \tag{8.23}$$

For $k = 1, \ldots, m$ and $i, j = 1, \ldots, d_k$, define

$$r^{(k)} \coloneqq \bigwedge_{i=1}^{d_k} v_i^{(k)*} v_i^{(k)} \leq p_1^{(k)} \quad \text{and} \quad f_{i,j}^{(k)} \coloneqq v_i^{(k)} r^{(k)} v_j^{(k)*}. \tag{8.24}$$

Then the $f_{i,j}^{(k)} \in G$ satisfy the matrix unit relations, so it suffices to show that $\left\|e_{i,j}^{(k)} - f_{i,j}^{(k)}\right\|_{2,X} < \epsilon$ for all i, j, k. For each $k = 1, \ldots, m$, we have

$$v_i^{(k)*} v_i^{(k)} \overset{(8.23)}{\approx}_{24(\epsilon')^{1/2}} w_i^{(k)*} w_i^{(k)} \overset{(8.22)}{\approx}_{4\epsilon'} p_1^{(k)} \tag{8.25}$$

where the approximations are in the $\|\cdot\|_{2,X}$-norm. Therefore, Lemma 8.7 implies

$$\|p_1^{(k)} - r^{(k)}\|_{2,X} < (4\epsilon' + 24(\epsilon')^{1/2}) d_k. \tag{8.26}$$

For all $k = 1, \ldots, m$ and $i, j = 1, \ldots, d_k$, we have

$$e_{i,j}^{(k)} = e_{i,1}^{(k)} e_{1,j}^{(k)} \overset{(8.19)}{\approx}_{2\epsilon'} w_i^{(k)} w_i^{(k)*} \overset{(8.23)}{\approx}_{24(\epsilon')^{1/2}} v_i^{(k)} v_i^{(k)*} = v_i^{(k)} p_1^{(k)} v_i^{(k)*}. \tag{8.27}$$

Combining (8.26) and (8.27), we have

$$\left\|e_{i,j}^{(k)} - f_{i,j}^{(k)}\right\|_{2,X} \leq (4\epsilon' + 24(\epsilon')^{1/2})(d_k + 1) \overset{(8.18)}{<} \epsilon. \tag{8.28}$$

□

8.3. **Hyperfinite implies CPoU.** The perturbation results of the previous subsection allow us to prove Theorem 8.4, from which we will deduce Theorem 8.3.

Proof of Theorem 8.4. Suppose $(\mathcal{M}, X)$ is hyperfinite and $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. Let $\mathcal{G}_n$ be an increasing sequence of finite subsets of $\mathcal{M}$ with $\|\cdot\|_{2,X}$-dense union and let $\epsilon_n > 0$ be such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. We will inductively construct sequences of finite dimensional subalgebras $F_n \subseteq \mathcal{M}$ and *-homomorphisms $\phi_n \colon F_n \to F_{n+1}$ such that for all $n \geq 1$,

(i) for all $a \in \mathcal{G}_n$, there is a $b \in F_n$ such that $\|a - b\|_{2,X} < \epsilon_n$, and
(ii) for all $b \in F_n$, $\|\phi_n(b) - b\|_{2,X} < \epsilon_n \|b\|$.

We can find F_1 satisfying (i) by the definition of hyperfiniteness. Assuming that F_n has been constructed, let $\mathcal{F}_n$ be a system of matrix units for F_n, and let $\delta_n > 0$ be given by applying Proposition 8.8 with F_n and ϵ_n in place of F and ϵ. By the definition of hyperfiniteness, there is a finite dimensional C^*-algebra $F_{n+1} \subseteq \mathcal{M}$ such that (i) holds and for all $a \in \mathcal{F}_n$, there is a $b \in F_{n+1}$ such that $\|a - b\|_{2,X} < \delta_n$. Then Proposition 8.8 yields the *-homomorphism $\phi_n^{n+1} \colon F_n \to F_{n+1}$ required in (ii), which completes the construction.

For $n, m \in \mathbb{N}$ with $m < n$, let $\phi_m^n \colon F_m \to F_n$ be given by composing the maps ϕ_k^{k+1} for $m \leq k < n$. For $m \in \mathbb{N}$ and $b \in F_m$, condition (ii) above together with the choice of ϵ_n imply that the bounded sequence $(\phi_m^n(b))_{n=m}^\infty$ is $\|\cdot\|_{2,X}$-Cauchy. Define $\psi_m \colon F_m \to \mathcal{M}$ by

$$\psi_m(b) \coloneqq \lim_{n\to\infty} \phi_m^n(b), \qquad b \in F_m. \tag{8.29}$$

Note that $\psi_n \circ \phi_m^n = \psi_m$ for all $m, n \in N$ with $m < n$. If $A \coloneqq \varinjlim (F_n, \phi_n)$ is the C^*-algebraic inductive limit of the algebras F_n and $\phi_{\infty,n} \colon F_n \to A$ are the natural maps, then there is an induced *-homomorphism $\psi \colon A \to \mathcal{M}$ such that $\psi \circ \phi_{\infty,n} = \psi_n$ for all $n \in \mathbb{N}$.

Since A is an AF algebra and quotients of AF algebras are also AF algebras (see [31, Theorem III.4.4], for example), we have that $\psi(A)$ is an AF subalgebra of $\mathcal{M}$. It suffices to show that $\psi(A) \subseteq \mathcal{M}$ is $\|\cdot\|_{2,X}$-dense. For each $n \in \mathbb{N}$, condition (ii) and the definition of ψ_n implies

$$\|\phi_n(b) - b\|_{2,X} \leq \sum_{m=n}^{\infty} \epsilon_m \|b\|, \qquad b \in F_n. \tag{8.30}$$

As $\sum_{n=1}^\infty \epsilon_n < \infty$ and the sets $\mathcal{G}_n$ are increasing, the above inequality and condition (i) imply that $\mathcal{G}_n$ is contained in the $\|\cdot\|_{2,X}$-closure of $\psi(A)$ for all $n \geq 1$. As the sets $\mathcal{G}_n$ have $\|\cdot\|_{2,X}$-dense union in $\mathcal{M}$, this completes the proof that $(\mathcal{M}, X)$ is an inductive limit of finite-dimensional tracially complete C^*-algebras. The converse is straightforward. □

We now have the pieces to show hyperfinite implies CPoU in the factorial setting.

Proof of Theorem 8.3. Let $(\mathcal{M}, X)$ be a hyperfinite factorial tracially complete C^*-algebra. Suppose first that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. Then, by Theorem 8.4, there is an increasing sequence $(F_n)_{n=1}^\infty$ of finite dimensional unital C^*-subalgebras of $\mathcal{M}$ with $\|\cdot\|_{2,X}$-dense union. Let $A \subseteq \mathcal{M}$ denote the norm closure of the union of the F_n so that A is an AF C^*-algebra. If $X_A \coloneqq \{\tau|_A : \tau \in X\}$, then Corollary 3.29 implies that X_A is a closed face in $T(A)$ and $\big(\overline{A}^{X_A}, X_A\big) \cong (\mathcal{M}, X)$.

Note that $(\overline{A}^{T(A)}, T(A))$ is the inductive limit of the factorial tracially complete C^*-algebras $\big(F_n, T(F_n)\big)$ and each of these has CPoU by Corollary 6.4. Therefore $(\overline{A}^{T(A)}, T(A))$ has CPoU by Proposition 6.12. Further, note that the tracial completion of $\overline{A}^{T(A)}$ with respect to X_A is $(\overline{A}^{X_A}, X_A) \cong (\mathcal{M}, X)$. This has CPoU by Proposition 6.9.

The general case follows from the $\|\cdot\|_{2,X}$-separable case as hyperfiniteness is separably inheritable and factorial CPoU tracially complete C^*-algebras

are strongly separably inheritable (see the discussion immediately following Definition A.1 in Appendix A). The proofs of the (strong) separable inheritability are given in Theorem A.3.

□

9. Classification and Regularity

In this final section, we combine the pieces together to obtain our main results. The classification theorem (Theorem C) will follow from the existence and uniqueness results for morphisms obtained in Sections 9.2 and 9.1, respectively, via standard intertwining arguments.

In more detail, we will show that for all suitable tracially complete C^*-algebras $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$, such as those covered in the classification theorem, any continuous affine map $Y \to X$ is induced by a morphism $(\mathcal{M}, X) \to (\mathcal{N}, Y)$, and this map is unique up to approximate unitary equivalence. From here, if $X \cong Y$, then the existence theorem can be applied twice to produce morphisms $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ and $\psi\colon (\mathcal{N}, Y) \to (\mathcal{M}, X)$ which are inverses of each other on traces. Then, two applications of the uniqueness theorem allow us to show that $\psi \circ \phi$ and $\phi \circ \psi$ are approximately unitary equivalent to the identity maps on $\mathcal{M}$ and $\mathcal{N}$, respectively. An application of (a tracially complete version of) Elliott's intertwining argument will then imply $(\mathcal{M}, X) \cong (\mathcal{N}, Y)$.

The uniqueness result is covered in Section 9.1 and follows from a mostly standard uniqueness theorem for weakly nuclear *-homomorphisms into finite von Neumann algebras (Proposition 9.2) and a CPoU argument. The existence result in Section 9.2 is more subtle. On the first pass, we will only be able to obtain an "approximate existence" result showing the existence of approximately multiplicative maps $\mathcal{M} \to \mathcal{N}$ approximately implementing a given continuous affine map $Y \to X$; a discussion of the strategy behind this can be found at the beginning of Section 9.2. As is standard in the C^*-algebra classification literature, this approximate existence result can be paired with an analogous uniqueness result for approximately multiplicative maps via an approximate intertwining argument to strengthen our existence result and produce a genuine morphism $(\mathcal{M}, X) \to (\mathcal{N}, Y)$ with prescribed behaviour on traces (Theorem 9.12).

In the proof of Theorem 9.12, we will avoid explicitly using an approximate uniqueness theorem and this final intertwining argument by making use of the intertwining via reparameterisation technique given in Theorem 5.11. More precisely, the approximate existence result will provide a *-homomorphism $(\mathcal{M}, X) \to (\mathcal{N}^\infty, Y^\infty)$. Our uniqueness result will apply to morphisms into $(\mathcal{N}^\infty, Y^\infty)$ since the hypotheses on the codomain of the existence theorem are preserved by reduced powers. This then allows us to use Theorem 5.11 to obtain the final existence result.

With the classification theorem in hand, the structure theorem for separable type II_1 factorial amenable tracially complete C^*-algebras with property Γ (Theorem B) will follow.

9.1. **Uniqueness results for morphisms.**

This subsection gives a uniqueness result for tracially nuclear *-homomorphisms into factorial tracially

complete C^*-algebras with CPoU. Uniqueness will be up to approximate unitary equivalence in uniform 2-norm, as follows.

Definition 9.1. If $(\mathcal{M}, X)$ is a tracially complete C^*-algebra and $\phi, \psi\colon A \to \mathcal{M}$ are functions, we say that ϕ and ψ are *approximately unitarily equivalent* if there is a net of unitaries $(u_\lambda) \subseteq \mathcal{M}$ such that

$$\|u_\lambda \phi(a) u_\lambda^* - \psi(a)\|_{2,X} \to 0, \qquad a \in A. \tag{9.1}$$

Note that when A is a separable C^*-algebra and ϕ and ψ are $\|\cdot\|$-continuous we may arrange the net (u_λ) to be a sequence.

We begin by recording following the uniqueness result for von Neumann algebras which is a consequence of Connes' theorem. Several variations and special cases of this can be found in the literature (see [28, Proposition 2.1] or [99, Proposition 1.1] for example); the result most often appears when A is nuclear and $\mathcal{M}$ is a II_1 factor. Most, if not all, variations of this argument (including the one below) follow the same strategy:

(i) use Connes' theorem to replace the domain with a hyperfinite von Neumann algebra;
(ii) use an $\epsilon/3$-argument to replace the domain with a finite dimensional algebra;
(iii) use the Murray–von Neumann classification of projections by traces to compare matrix units in the codomain.

Proposition 9.2. *Suppose A is a C^*-algebra, $\mathcal{N}$ is a finite von Neumann algebra, and $\phi, \psi\colon A \to \mathcal{N}$ are weakly nuclear *-homomorphisms. Then there is a net of unitaries $(u_\lambda) \subseteq \mathcal{N}$ such that*

$$\sigma\text{-strong}^*\text{-}\lim_\lambda u_\lambda \phi(a) u_\lambda^* = \psi(a), \qquad a \in A, \tag{9.2}$$

if and only if $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in T(\mathbb{N})$.

Proof. Suppose that $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in T(\mathcal{N})$. Let $\mathcal{M} \subseteq \mathcal{N}$ denote the von Neumann subalgebra generated by $\phi(A)$. Recall that since $\mathcal{N}$ is finite, $T(\mathcal{N})$ forms a faithful set of traces, i.e. if $x \in \mathcal{N}$ is non-zero, then there exists $\tau \in T(\mathcal{N})$ with $\tau(x^*x) \neq 0$. Accordingly the hypothesis ensures $\ker(\phi) = \ker(\psi)$, and so ψ factorises through $\phi(A)$. Let $\bar{\phi}\colon \mathcal{M} \to \mathcal{N}$ denote the inclusion map and let $\bar{\psi}\colon \mathcal{M} \to \mathcal{N}$ denote the unique normal *-homomorphism with $\bar{\psi} \circ \phi = \psi$. Then $\tau \circ \bar{\phi} = \tau \circ \bar{\psi}$ for all normal traces $\tau \in T(\mathcal{N})$ (and hence also for all $\tau \in T(\mathcal{N})$).

As $\mathcal{N}$ is finite, there is a normal conditional expectation $\mathcal{N} \to \mathcal{M}$ (see [15, Lemma 1.5.10]; for example), and so, the corestriction of ϕ to a *-homomorphism $A \to \mathcal{M}$ is weakly nuclear. Then $\mathcal{M}$ is hyperfinite by the proof of (6) implies (5) of [12, Theorem 3.2.2].[56]

Let $F \subseteq \mathcal{M}$ be a finite dimensional C^*-algebra, so that $\tau \circ \bar{\phi}|_F = \tau \circ \bar{\psi}|_F$. It is a standard consequence of the classification of projections in the finite von Neumann algebra $\mathcal{N}$ by traces that $\bar{\phi}|_F$ and $\bar{\psi}|_F$ are unitarily equivalent.[57]

[56] This reference handles the case that $\mathcal{M} = \pi_\tau(A)''$ for a trace $\tau \in T(A)$ and $\phi = \pi_\tau$. To prove the claim in our setting, fix a faithful normal *-homomorphism $\pi\colon \mathcal{M} \to \mathcal{B}(H)$ so that $\mathcal{M} \cong \pi(\mathcal{M}) = \pi(\phi(A))''$, and in the proof in [12], replace π_τ with $\pi \circ \phi$.

[57] Let $e_{i,j}^{(k)}$ be a system of matrix units for F. For each $k = 1, \ldots, m$, we have

$$\tau\big(\bar{\phi}(e_{1,1}^{(k)})\big) = \tau\big(\bar{\psi}(e_{1,1}^{(k)})\big), \qquad \tau \in T(\mathcal{N}),$$

Therefore $\bar{\phi}$ and $\bar{\psi}$ are strong*-approximate unitarily equivalent, and hence so too are ϕ and ψ. The converse is immediate. □

Our main uniqueness result is obtained by a CPoU argument to pass from local uniqueness at the fibres to a global statement. This works in essentially the same way as [21, Theorem 2.2] (which handles the case when A is nuclear).[58]

Theorem 9.3. *Suppose A is a C^*-algebra, $(\mathcal{N}, Y)$ is a factorial tracially complete C^*-algebra with CPoU, and $\phi, \psi\colon A \to \mathcal{N}$ are tracially nuclear *-homomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in Y$.*

Proof. Fix a finite set $\mathcal{F} \subseteq A$ and $\epsilon > 0$. For each pair of traces $\tau \in Y$ and $\sigma \in T(\pi_\tau(\mathcal{N})'')$, we have $\sigma \circ \pi_\tau \in Y$ by Lemma 2.10, and hence

$$\sigma \circ \pi_\tau \circ \phi = \sigma \circ \pi_\tau \circ \psi. \tag{9.3}$$

As $\pi_\tau \circ \phi$ and $\pi_\tau \circ \psi$ are weakly nuclear, Proposition 9.2 implies there is a unitary $\bar{u}_\tau \in \pi_\tau(\mathcal{N})''$ such that

$$\max_{a \in \mathcal{F}} \|\bar{u}_\tau \pi_\tau(\phi(a)) \bar{u}_\tau^* - \pi_\tau(\psi(a))\|_{2,\tau} < \epsilon, \qquad \tau \in Y. \tag{9.4}$$

As $\pi_\tau(\mathcal{N})''$ is a von Neumann algebra, we know that $\bar{u}_\tau = e^{i\bar{h}_\tau}$ for some self-adjoint $\bar{h}_\tau \in \pi_\tau(\mathcal{N})''$. Applying Kaplansky's density theorem to $\bar{h}_\tau \in \pi_\tau(\mathcal{N})''$ and making use of the existence of self-adjoint lifts, we deduce that for each $\tau \in Y$, each unitary $\bar{u}_\tau$ is a $\|\cdot\|_{2,\tau}$-limit of unitaries of the form $\pi_\tau(u_\tau)$ for a unitary $u_\tau = e^{ih_\tau} \in \mathcal{N}$, and in particular, we may find unitaries $u_\tau \in \mathcal{N}$ so that

$$\max_{a \in \mathcal{F}} \|u_\tau \phi(a) u_\tau^* - \psi(a)\|_{2,\tau} < \epsilon, \qquad \tau \in Y. \tag{9.5}$$

For $\tau \in Y$, define

$$a_\tau \coloneqq \sum_{a \in \mathcal{F}} \left|u_\tau \phi(a) u_\tau^* - \psi(a)\right|^2 \in \mathcal{N}_+ \tag{9.6}$$

and note that $\tau(a_\tau) < |\mathcal{F}|\epsilon^2$ for all $\tau \in Y$. As Y is compact, there are traces $\tau_1, \ldots, \tau_n \in Y$ such that

$$\sup_{\tau \in Y} \min_{1 \le i \le n} \tau(a_{\tau_i}) < |\mathcal{F}|\epsilon^2. \tag{9.7}$$

Define $S \coloneqq \phi(\mathcal{F}) \cup \psi(\mathcal{F}) \cup \{u_{\tau_1}, \ldots, u_{\tau_n}\} \subseteq \mathcal{N}$. As $(\mathcal{N}, Y)$ has CPoU, there are projections $p_1, \ldots, p_n \in \mathcal{N}^\omega \cap S'$ such that

$$\sum_{j=1}^n p_j = 1_{\mathcal{N}^\omega} \qquad \text{and} \qquad \tau(a_{\tau_i} p_i) \le |\mathcal{F}|\epsilon^2 \tag{9.8}$$

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and hence there is a partial isometry $v_k \in \mathcal{N}$ with source $\bar{\phi}(e_{1,1}^{(k)})$ and range $\bar{\psi}(e_{1,1}^{(k)})$. Define

$$u \coloneqq \sum_{k=1}^m \sum_{i=1}^{d_k} \bar{\psi}(e_{i,1}^{(k)}) v_k \bar{\phi}(e_{1,i}^{(k)}) \in \mathcal{N}.$$

Then $u \in \mathcal{N}$ is a unitary with $u\bar{\phi}(x)u^* = \bar{\psi}(x)$ for all $x \in F$.

[58]The setup in [21] considers uniform tracial sequence algebras B^∞ as codomains when B is a separable C^*-algebra with CPoU, but there are no additional difficulties in working in the abstract framework of tracially complete C^*-algebras.

for all $i = 1, \dots, n$ and $\tau \in Y^\omega$. Then, using the fact that the p_i commute with S and are orthogonal, $u \coloneqq \sum_{i=1}^n p_i u_{\tau_i} \in \mathcal{N}^\omega$ is a unitary and

$$\max_{a \in \mathcal{F}} \|u\phi(a)u^* - \psi(a)\|_{2,Y^\omega} \leq |\mathcal{F}|^{1/2}\epsilon. \tag{9.9}$$

By Corollary 7.11, there is a sequence of unitaries $(u_n)_{n=1}^\infty \subseteq \mathcal{N}$ representing u. Then

$$\limsup_{n \to \omega} \max_{a \in \mathcal{F}} \|u_n\phi(a)u_n^* - \psi(a)\|_{2,Y} \leq |\mathcal{F}|^{1/2}\epsilon, \tag{9.10}$$

which proves the theorem. □

In the special case of reduced products of factorial tracially complete C^*-algebras with CPoU, the uniqueness theorem gives on-the-nose unitary equivalence.

Corollary 9.4 (cf. [21, Theorem 2.2])**.** *Let A be a separable C^*-algebra and $(\mathcal{N}, Y)$ is a factorial tracially complete C^*-algebra with CPoU. If $\phi, \psi \colon A \to \mathcal{N}^\omega$ are tracially nuclear *-homomorphisms, then ϕ and ψ are unitarily equivalent if and only if $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in Y^\omega$.*

Proof. As $(\mathcal{N}, Y)$ is a factorial tracially complete C^*-algebra with CPoU, the same is true for $(\mathcal{N}^\omega, Y^\omega)$ by Corollary 7.7. Hence, applying Theorem 9.3, we get that ϕ and ψ are $\|\cdot\|_{2,Y^\omega}$-approximately unitary equivalent if and only if $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in Y^\omega$. By the separability of A and a standard application of Kirchberg's ϵ-test (Lemma 5.1), $\phi, \psi \colon A \to \mathcal{N}^\omega$ are $\|\cdot\|_{2,Y^\omega}$-approximately unitary equivalent if and only if they are unitary equivalent. □

9.2. **Existence results for morphisms.** We will give a general existence result showing that morphisms can be constructed from amenable tracially complete C^*-algebras to factorial tracially complete C^*-algebras with CPoU with prescribed tracial information (see Corollary 9.9 below).

By construction, the approximate morphisms we produce will approximately factor through finite dimensional C^*-algebras. In fact, given a separable C^*-algebra A, a factorial tracially complete C^*-algebra $(\mathcal{N}, Y)$, and a continuous affine map $\gamma \colon Y \to T(A)$, we will produce approximate factorisations of γ through the trace simplices $T(F_n)$ of finite dimensional C^*-algebras F_n. If $\gamma(\tau)$ satisfies a suitable approximation property for all $\tau \in Y$ (e.g. amenability), the maps $T(F_n) \to T(A)$ may be approximately implemented by approximate morphisms $A \to F_n$. Further, when $(\mathcal{N}, Y)$ has CPoU, the classification of projections in $\mathcal{N}$ (Theorems 7.18 and 7.19) allows us to show the maps $Y \to T(F_n)$ are implemented by morphisms $F_n \to \mathcal{N}$. The compositions $A \to F_n \to \mathcal{N}$ provide the desired approximate morphism.

The following lemma gives the required existence result for morphisms out of finite dimensional C^*-algebras. When $\mathcal{N}$ is a II_1 factor, this is a standard result, and the proof here is essentially identical using Theorems 7.18 and 7.19 in place of the classification of projections in II_1 factors.

Lemma 9.5. *If F is a finite dimensional C^*-algebra, $(\mathcal{N}, Y)$ is a type II_1 factorial tracially complete C^*-algebra with CPoU, and $\gamma \colon Y \to T(F)$ is a continuous affine map, then there is a unital *-homomorphism $\phi \colon F \to \mathcal{N}$ with $\tau \circ \phi = \gamma(\tau)$ for all $\tau \in Y$.*

Proof. We first assume that F is commutative and let $e_1, \dots, e_m$ denote the minimal projections of F. It suffices to construct pairwise orthogonal projections $p_1, \dots, p_n \in F$ such that $\tau(p_i) = \gamma(\tau)(e_i)$ for all $\tau \in Y$ and $i = 1, \dots, m$ since we may then define a *-homomorphism $\phi\colon F \to \mathcal{N}$ by $\phi(e_i) \coloneqq p_i$. Further, since $\tau(\phi(1_F)) = 1$ for all $\tau \in Y$, we will have $\phi(1_F) = 1_{\mathcal{N}}$.

The existence of p_1 follows immediately from Theorem 7.19. Assuming $p_1, \dots, p_k$ have been constructed some some $k < m$, Theorem 7.19 provides a projection $p'_{k+1} \in \mathcal{N}$ such that $\tau(p'_{k+1}) = \gamma(\tau)(e_{k+1})$ for all $\tau \in Y$. Note that

$$\tau(p'_{k+1}) = \gamma(\tau)(e_{k+1}) \leq 1 - \sum_{i=1}^{k} \gamma(\tau)(e_i) = \tau\Big(1_{\mathcal{N}} - \sum_{i=1}^{k} p_i\Big) \tag{9.11}$$

for all $\tau \in Y$. Then Theorem 7.18 implies there is a partial isometry v with $p'_{k+1} = v^*v$ and such that $p_{k+1} \coloneqq vv^*$ is orthogonal to each of $p_1, \dots p_k$. This completes the proof when F is commutative.

Next consider a general finite dimensional C^*-algebra F with a system of matrix units

$$\big(e_{i,j}^{(k)}\big)_{1 \leq i,j \leq d_k}^{1 \leq k \leq m} \subseteq F. \tag{9.12}$$

By the first part of the proof, there are mutually orthogonal projections $p_i^{(k)} \in \mathcal{N}$ for $k = 1, \dots, m$ and $i = 1 \dots, d_k$ such that

$$\tau(p_i^{(k)}) = \gamma(\tau)\big(e_{i,i}^{(k)}\big) \qquad \text{for all } \tau \in Y. \tag{9.13}$$

By Theorem 7.18, for each $k = 1, \dots, m$ and $i = 1 \dots, d_k$, there is a partial isometry $v_i^{(k)} \in \mathcal{N}$ such that

$$v_i^{(k)*} v_i^{(k)} = p_i^{(k)} \qquad \text{and} \qquad v_i^{(k)} v_i^{(k)*} = p_1^{(k)}. \tag{9.14}$$

Then we may define a *-homomorphism $\phi\colon F \to \mathcal{N}$ by

$$\phi\big(e_{i,j}^{(k)}\big) \coloneqq v_i^{(k)*} v_j^{(k)} \tag{9.15}$$

for all $k = 1, \dots, m$ and $i, j = 1, \dots, d_k$ Then $\tau \circ \phi = \gamma(\tau)$ for all $\tau \in Y$, and since $\tau(\phi(1_F)) = 1$ for all $\tau \in Y$, we have $\phi(1_F) = 1_{\mathcal{N}}$. □

We now work towards an existence result for constructing morphisms $A \to \mathcal{N}^\omega$ with prescribed tracial data. One of the most influential embedding results is Connes' theorem from [29] that every separably acting injective II_1 factor embeds into $\mathcal{R}^\omega$.[59] This is a key ingredient in showing that such factors are isomorphic to $\mathcal{R}$. In the same paper Connes poses his eponymous embedding problem asking if every separably acting II_1 factor embeds into $\mathcal{R}^\omega$ (see the paragraph above [29, Notation 5.6]). Note that since every II_1 factor contains $\mathcal{R}$, we have that every II_1 factor which embeds into $\mathcal{R}^\omega$ also embeds into $\mathcal{N}^\omega$ for each II_1 factor $\mathcal{N}$.

Our embedding result will be along these lines showing that if A is separable C^*-algebra, $(\mathcal{N}, Y)$ is a type II_1 factorial tracially complete C^*-algebra

[59]This statement does not appear explicitly in [29], but follows by Lemma 5.2 and $7 \Rightarrow 6$ of Theorem 5.1. Condition 7 of Theorem 5.1 is verified by composing the trace on $\mathcal{N}$ with a conditional expectation onto $\mathcal{N}$.

with CPoU, then there is a *-homomorphism $A \to \mathcal{N}^\omega$ with prescribed tracial data in the case that all the relevant traces on A factorise through $\mathcal{R}^\omega$. We first provide the following local characterisation of traces factorising through $\mathcal{R}^\omega$.

Definition 9.6. A trace τ on a C^*-algebra A is *hyperlinear* if there is a net of self-adjoint linear maps $(\psi_\lambda \colon A \to M_{d(\lambda)})$ such that for all $a, b \in A$,

(i) $\|\psi_\lambda(ab) - \psi_\lambda(a)\psi_\lambda(b)\|_2 \to 0$,
(ii) $\mathrm{tr}_{d(\lambda)}(\psi_\lambda(a)) \to \tau(a)$, and
(iii) $\limsup_\lambda \|\psi_\lambda(a)\| < \infty$.

For applications to classification, the hyperlinear traces of interest will be amenable, so that the maps ψ_i can be taken to be c.p.c. However, we will prove the existence theorem in terms of hyperlinear traces since it takes little extra effort.

Remark 9.7. We collect here a few general observations about hyperlinear traces.

(i) As usual, when A is unital, we may arrange for each ψ_λ to be unital, and when A is separable, we can arrange for the net (ψ_λ) to be a sequence.
(ii) If A is separable, we choose a sequence $(\psi_n)_{n=1}^\infty$ as above and view $M_{d(n)}$ as a unital subalgebra of $\mathcal{R}$. In this way, the ψ_n induce a *-homomorphism $\psi \colon A \to \mathcal{R}^\omega$ with $\mathrm{tr}_{\mathcal{R}^\omega} \circ \psi = \tau$. Note that part (iii) of Definition 9.6 is needed to guarantee that ψ_n is well-defined.
(iii) If G is a discrete group, then G is hyperlinear if and only if the canonical trace on the reduced group C^*-algebra $C^*_\lambda(G)$ is hyperlinear. This is the reason for the terminology.
(iv) Definition 9.6(iii) is equivalent to

$$\limsup_\lambda \|\psi_\lambda(a)\| \leq \|a\|, \qquad a \in A. \tag{9.16}$$

Indeed, if Λ is the index set of the net and we view each $M_{d(\lambda)}$ as a subalgebra of $\mathcal{R}$, , then the ψ_λ induce a *-homomorphism $\psi \colon A \to \ell^\infty(\Lambda, \mathcal{R})/c_0(\Lambda, \mathcal{R})$, where $c_0(\Lambda, \mathcal{R})$ consists of all bounded strongly null nets. As *-homomorphisms are contractive, (9.16) follows.

The following is our main existence result for *-homomorphisms, although in the classification theorem, it will be accessed through Corollary 9.9 which gives a restatement in terms of reduced products when the domain is separable. We keep track of the additional details regarding factorisations through finite dimensional C^*-algebras as it is conceivable that these will play a role, for example, in subsequent nuclear dimension calculations.

Theorem 9.8. *Suppose A is a C^*-algebra, $(\mathcal{N}, Y)$ is a type II_1 factorial tracially complete C^*-algebra with CPoU, and $\gamma \colon Y \to T(A)$ is a continuous affine map such that $\gamma(\tau)$ is hyperlinear for all $\tau \in Y$. For every finite set $\mathcal{F} \subseteq A$ and $\epsilon > 0$, there are a finite dimensional C^*-algebra F, a self-adjoint linear map $\psi \colon A \to F$, and a unital *-homomorphism $\phi \colon F \to \mathcal{N}$ such that for all $a, b \in \mathcal{F}$:*

(i) $\|\psi(ab) - \psi(a)\psi(b)\|_{2,T(F)} < \epsilon$,

(ii) $|\tau(\phi(\psi(a))) - \gamma(\tau)(a)| < \epsilon$ *for all* $\tau \in Y$, *and*
(iii) $\|\psi(a)\| < \|a\| + \epsilon$.

Moreover, if A *is unital, we may arrange for* ψ *to be unital, if* $\gamma(\tau)$ *is amenable for all* $\tau \in Y$, *we may arrange for* ψ *to be c.p.c., and if* $\gamma(\tau)$ *is quasidiagonal for all* $\tau \in T(A)$, *we may arrange for*

(i′) $\|\psi(ab) - \psi(a)\psi(b)\| < \epsilon$

for all $a, b \in \mathcal{F}$. *Finally, the last three statements can be performed simultaneously.*

Proof. Since Y is a closed face in $T(\mathcal{N})$, we have that Y is a Choquet simplex (Theorem 2.6). Applying Theorem 2.2 to the continuous affine maps $\mathrm{ev}_a \circ \gamma$ for $a \in \mathcal{F}$, there is a finite dimensional Choquet simplex Z together with continuous affine maps $\alpha' \colon Z \to Y$ and $\beta \colon Y \to Z$ so that

$$|\gamma(\alpha'(\beta(\tau)))(a) - \gamma(\tau)(a)| < \epsilon, \qquad a \in \mathcal{F},\ \tau \in Y. \tag{9.17}$$

Define $\alpha \coloneqq \gamma \circ \alpha' \colon Z \to T(A)$ so that $\alpha(\tau)$ is hyperlinear for all $\tau \in Z$ and

$$|\alpha(\beta(\tau))(a) - \gamma(\tau)(a)| < \epsilon, \qquad a \in \mathcal{F},\ \tau \in Y. \tag{9.18}$$

Let $\rho_1, \dots, \rho_n \in Z$ denote the extreme points of Z. For $i = 1, \dots, n$, since $\alpha(\rho_i) \in T(A)$ is hyperlinear (by the assumption on γ), there are $d_i \in \mathbb{N}$ and *-linear maps $\psi_i \colon A \to M_{d_i}$ such that for all $a, b \in \mathcal{F}$,

$$\begin{aligned} \|\psi_i(ab) - \psi_i(a)\psi_i(b)\|_{2,\mathrm{tr}_{d_i}} &< \epsilon, \\ |\mathrm{tr}_{d_i}(\psi_i(a)) - \alpha(\rho_i)(a)| &< \epsilon, \quad \text{and} \\ \|\psi_i(a)\| - \|a\| &< \epsilon. \end{aligned} \tag{9.19}$$

Define $F \coloneqq \bigoplus_{i=1}^n M_{d_i}$ and let $\psi \coloneqq \bigoplus_{i=1}^n \phi_i \colon A \to F$. We identify Z with $T(F)$ via the affine map given by

$$Z \to T(F) \colon \rho_i \mapsto \mathrm{tr}_{d_i} \circ \pi_i, \tag{9.20}$$

where $\pi_i \colon F \to M_{d_i}$ denotes the projection map for $i = 1, \dots, n$. By Lemma 9.5, there is a unital *-homomorphism $\phi \colon F \to \mathcal{N}$ such that $\tau \circ \phi = \beta(\tau)$ for all $\tau \in Y$. Then all three conditions in the theorem follow from the corresponding conditions in (9.19).

The additional claims in the theorem follow by choosing the ψ_i with the appropriate properties in the second paragraph of the proof. □

Corollary 9.9. *Suppose* A *is a separable* C^**-algebra,* $(\mathcal{N}, Y)$ *is a type* II_1 *factorial tracially complete* C^**-algebra with CPoU. Given a continuous affine map* $\gamma \colon Y \to T(A)$ *such that* $\gamma(\tau)$ *is hyperlinear for all* $\tau \in Y$, *there is a unital* **-homomorphism* $\theta \colon A \to \mathcal{N}^\omega$ *such that* $\tau \circ \theta = \gamma(\tau|_{\mathcal{N}})$ *for all* $\tau \in Y^\omega$.

Proof. Since A is separable, we may choose a countable dense $\mathbb{Q}[i]$-subalgebra $A_0 \subseteq A$. By Theorem 9.8, there are sequences $(F_n)_{n=1}^\infty$ of finite dimensional C^*-algebras, $(\psi_n \colon A_0 \to F_n)_{n=1}^\infty$ of self-adjoint linear maps, and $(\phi_n \colon F_n \to \mathcal{N})_{n=1}^\infty$ of unital *-homomorphisms such that for all $a, b \in A_0$ and $\tau \in Y$,

(i) $\|\psi_n(ab) - \psi_n(a)\psi_n(b)\|_{2,T(F_n)} \to 0$,
(ii) $|\tau(\phi(\psi(a))) - \gamma(\tau)(a)| \to 0$, and
(iii) $\limsup_{n\to\infty} \|\psi(a_n)\| \leq \|a\|$.

The sequences $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ induce *-homomorphisms

$$A_0 \xrightarrow{\psi} \prod^{\omega} \big(F_n, T(F_n)\big) \xrightarrow{\phi} \mathcal{N}^{\omega}. \tag{9.21}$$

Define $\theta_0 \coloneqq \phi \circ \psi$. By (iii), θ_0 is contractive, and hence extends to a *-homomorphism $\theta\colon A \to \mathcal{N}^{\omega}$ with the required property. □

If the range of γ is contained in the uniformly amenable traces on A, then θ will necessarily be tracially nuclear by Theorem 4.9. In the next subsection, we will use this observation together with the uniqueness result in Corollary 9.4 to strengthen this existence theorem – namely, θ can be chosen to take values in $\mathcal{N}$ (Theorem 9.12).

As a first application of Theorem 9.8, we establish a W^*-bundle version of the Connes' embedding problem assuming a positive solution in each fibre. In the following result, we assume ω is a free ultrafilter on $\mathbb{N}$ (as opposed to a free filter on $\mathbb{N}$ as above).

Corollary 9.10. *Suppose $\mathcal{M}$ is a W^*-bundle over a compact metrisable space K with separably acting fibres. If each fibre of $\mathcal{M}$ admits a trace-preserving embedding into $\mathcal{R}^{\omega}$, then there is an embedding $\mathcal{M} \hookrightarrow C_\sigma(K, \mathcal{R})^{\omega}$ which restricts to the identity on $C(K)$.*

Proof. As in Proposition 3.6, we may view $\mathcal{N} \coloneqq C_\sigma(K, \mathcal{R})$ as a factorial tracially complete C^*-algebra $(\mathcal{N}, Y)$ where $Y \cong \mathrm{Prob}(K)$ is the set of traces given by integrating the trace on $\mathcal{R}$ over K. It is easy to see that $\mathcal{N}$ is McDuff, and hence $(\mathcal{N}, Y)$ has CPoU by Theorem 1.4. Similarly, we may view $\mathcal{M}$ as a tracially complete C^*-algebra $(\mathcal{M}, X)$ where $X \cong \mathrm{Prob}(K)$ where again, a Radon probability measure on K induces on trace on $\mathcal{M}$ by integrating the traces on the fibres of $\mathcal{M}$.

Let $\gamma\colon Y \to X$ be the continuous affine map induced by the identity map on $\mathrm{Prob}(K)$. By hypothesis, $\gamma(\tau) \in X$ is hyperlinear for all $\tau \in K$. It is easy to show that the hyperlinear traces on a unital C^*-algebra form a closed convex set, and hence $\gamma(\tau) \in X$ is hyperlinear for all $\tau \in Y$. Since K is metrisable and each fibre of $\mathcal{M}$ is separably acting, we have that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. Let $A \subseteq \mathcal{M}$ be a $\|\cdot\|$-separable unital C^*-subalgebra.

By Corollary 9.9, there is a *-homomorphism $\theta_0\colon A \to \mathcal{N}^{\omega}$ such that $\gamma(\tau)|_A = \tau \circ \theta_0$ for all $\tau \in Y^{\omega}$. Then θ_0 extends by continuity to a morphism $(\mathcal{M}, X) \to (\mathcal{N}^{\omega}, Y^{\omega})$. By construction, the inclusion map $C(K) \hookrightarrow \mathcal{N}^{\omega}$ and $\theta|_{C(K)}\colon C(K) \hookrightarrow \mathcal{N}^{\omega}$ agree on traces, and hence are unitarily equivalent by Corollary 9.4. If $u \in \mathcal{N}^{\omega}$ is a unitary conjugating $\theta|_{C(K)}$ to the inclusion map, then $\mathrm{ad}(u) \circ \theta$ is the desired embedding. □

Question 9.11. Can Corollary 9.10 be improved to provide an embedding $\mathcal{M} \to C_\sigma(K, \mathcal{R}^{\omega})$ (which restricts to the identity on $C(K)$)?

9.3. **Classification and consequences.** The existence and uniqueness results for morphisms in the last sections together can be combined with tracially complete versions of standard intertwining arguments to produce our main classification theorems. First, we use the intertwining via reparameterisation technique (Theorem 5.11) to obtain a classification result for tracially nuclear morphisms.

Theorem 9.12. *Let $(\mathcal{N}, Y)$ be a a type II_1 factorial tracially complete C^*-algebra with CPoU.*

(i) *Let A be a separable C^*-algebra. If $\gamma\colon Y \to T(A)$ is a continuous affine map such that $\gamma(\tau)$ is uniformly amenable for all $\tau \in Y$, then there is a tracially nuclear *-homomorphism $\phi\colon A \to \mathcal{N}$ such that $\tau \circ \phi = \gamma(\tau)$ for all $\tau \in Y$, and this ϕ is unique up to approximate unitary equivalence.*

(ii) *Let $(\mathcal{M}, X)$ be a $\|\cdot\|_{2,X}$-separable tracially complete C^*-algebra. If $\gamma\colon Y \to X$ is a continuous affine function such that $\gamma(Y)$ is contained in the uniformly amenable traces on $\mathcal{M}$ then there is a tracially nuclear *-homomorphism $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ such that $\tau \circ \phi = \gamma(\tau)$ for all $\tau \in Y$, and this ϕ is unique up to approximate unitary equivalence.*

(iii) *Let $(\mathcal{M}, X)$ be a $\|\cdot\|_{2,X}$-separable amenable tracially complete C^*-algebra. If $\gamma\colon Y \to X$ is a continuous affine function, then there is a tracially nuclear *-homomorphism $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ such that $\tau \circ \phi = \gamma(\tau)$ for all $\tau \in Y$, and this ϕ is unique up to approximate unitary equivalence.*

Proof. (i). The uniqueness part of (i) is Theorem 9.3. For the existence part of (i), since uniformly amenable traces are hyperlinear, Corollary 9.9 implies there is a *-homomorphism $\theta_\infty\colon A \to \mathcal{N}^\infty$ with $\tau \circ \theta_\infty = \gamma(\tau|_{\mathcal{N}})$ for all $\tau \in Y^\infty$. By Theorem 4.9, θ is tracially nuclear. Note that if $r\colon \mathbb{N} \to \mathbb{N}$ is a function with $\lim_{n\to\infty} r(n) = \infty$ and $r^*\colon \mathcal{N}^\infty \to \mathcal{N}^\infty$ is the reparameterisation map as in Theorem 5.11, then

$$\tau \circ \theta_\infty = \gamma(\tau|_{\mathcal{N}}) = \gamma((\tau \circ r^*)|_{\mathcal{N}}) = \tau \circ r^* \circ \theta_\infty, \qquad \tau \in Y^\infty. \tag{9.22}$$

By Corollary 9.4, $r^* \circ \theta$ and θ are unitarily equivalent. By Corollary 7.11, unitaries in $\mathcal{N}^\infty$ lift to unitaries in $\ell^\infty(\mathcal{N})$, and hence Theorem 5.11 provides a *-homomorphism $\theta\colon A \to \mathcal{N}$ such that $\iota_{\mathcal{N}} \circ \theta$ and θ_∞ are unitarily equivalent. Then θ satisfies $\tau \circ \theta = \gamma(\tau)$ for all $\tau \in \mathcal{N}$, and Theorem 4.9 implies that θ is tracially nuclear.

(ii). Let $A \subseteq \mathcal{M}$ be a separable $\|\cdot\|_{2,X}$-dense C^*-subalgebra. Then uniqueness follows from uniqueness in (i) as all morphisms between tracially complete C^*-algebras are automatically contractive between the uniform 2-norms. Part (i) gives a (tracially nuclear) *-homomorphism $\tilde{\phi}\colon A \to \mathcal{N}$ with $\tau \circ \tilde{\phi} = \tilde{\gamma}(\tau)$ for $\tau \in Y$, which extends uniquely to $\phi\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ by Corollary 3.29(iii). The extension is tracially nuclear by Lemma 4.12. As A is $\|\cdot\|_{2,X}$-dense in $\mathcal{M}$, $\tau \circ \phi = \gamma(\tau)$ for $\tau \in Y$.

(iii). This is immediate from (ii) as amenability of $\mathcal{M}$ forces all traces in X to be uniformly amenable (by Theorem 4.9(iii), taking θ to be $\mathrm{id}_{\mathcal{M}}$). □

The classification theorem for regular amenable tracially complete C^*-algebras is obtained from a tracially complete version of the two-sided Elliott intertwining argument (the C^*-version of which can be found as [93, Corollary 2.3.4], for example; note that [38, Theorem 3] abstracts this C^*-version to an abstract intertwining result for categories with notions of inner automorphisms and metric structure on the morphism sets, and one could apply it to the category of separable tracially complete C^*-algebras). As with C^*-classification results, any isomorphism at the level of the invariants

– in this case the designated traces – lifts to an isomorphism of tracially complete C^*-algebras. The second part of Theorem D from the overiew follows from part (ii) below, with the first part of Theorem D having already been proven in Theorem 4.9.

Theorem 9.13. *Suppose that $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ are type II_1 semidiscrete factorial tracially complete C^*-algebras with property Γ such that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable and $\mathcal{N}$ is $\|\cdot\|_{2,Y}$-separable.*

(i) *$(\mathcal{M}, X) \cong (\mathcal{N}, Y)$ if and only if $X \cong Y$, and moreover,*
(ii) *For any affine homeomorphism $\gamma\colon Y \to X$, there is an isomorphism $\theta\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ such that $\tau \circ \theta = \gamma(\tau)$ for all $\tau \in Y$.*

Proof. The first part follows from the second, so fix an isomorphism γ as in (ii). Both $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ have CPoU by Theorem 1.4. So, by two applications of the existence portion of Theorem 9.12(ii), there are tracially nuclear morphisms $\phi_0\colon (\mathcal{M}, X) \to (\mathcal{N}, Y)$ and $\psi_0\colon (\mathcal{N}, Y) \to (\mathcal{M}, X)$ such that

$$\tau \circ \phi_0 = \gamma(\tau) \qquad \text{and} \qquad \sigma \circ \psi_0 = \gamma^{-1}(\sigma) \tag{9.23}$$

for all $\tau \in Y$ and $\sigma \in X$. In particular,

$$\tau \circ \phi_0 \circ \psi_0 = \tau \qquad \text{and} \qquad \sigma \circ \psi_0 \circ \phi_0 = \sigma \tag{9.24}$$

for all $\tau \in Y$ and $\sigma \in X$. Using the uniqueness portion of Theorem 9.12(ii) twice, we have $\phi_0 \circ \psi_0$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{N}}$ and $\psi_0 \circ \phi_0$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{M}}$.

Let $(\mathcal{F}'_n)_{n=1}^{\infty}$ and $(\mathcal{G}'_n)_{n=1}^{\infty}$ be increasing sequences of finite subsets of $\mathcal{M}$ and $\mathcal{N}$, respectively, whose unions are dense in the respective uniform 2-norms. We will inductively construct increasing sequences $(\mathcal{F}_n)_{n=1}^{\infty}$ and $(\mathcal{G}_n)_{n=1}^{\infty}$ of finite subsets of $\mathcal{M}$ and $\mathcal{N}$, respectively, and sequences of *-homomorphisms $(\phi_n\colon \mathcal{M} \to \mathcal{N})_{n=1}^{\infty}$ and $(\psi_n\colon \mathcal{N} \to \mathcal{M})_{n=1}^{\infty}$ such that for all $n \geq 1$,

(i) $\mathcal{F}'_n \subseteq \mathcal{F}_n$ and $\mathcal{G}'_n \subseteq \mathcal{G}_n$,
(ii) ϕ_n and ψ_n are unitarily equivalent to ϕ_0 and ψ_0, respectively,
(iii) $\phi_n(\mathcal{F}_n) \subseteq G_n$ and $\psi_n(\mathcal{G}_n) \subseteq \mathcal{F}_{n+1}$, and
(iv) for all $a \in \mathcal{F}_n$ and $b \in \mathcal{G}_n$, we have

$$\|\psi_n(\phi_n(a)) - a\|_{2,X} < 2^{-n} \tag{9.25}$$

and

$$\|\phi_n(\psi_{n-1}(b)) - b\|_{2,Y} < 2^{-n}. \tag{9.26}$$

Assuming that $\mathcal{F}_{n-1}$, $\mathcal{G}_{n-1}$, ϕ_{n-1}, and ψ_{n-1} have been constructed, let $\mathcal{F}_n \coloneqq \mathcal{F}'_n \cup \mathcal{F}_{n-1} \cup \psi_{n-1}(\mathcal{G}_{n-1})$. Since ϕ_{n-1} and ψ_{n-1} are unitarily equivalent to ϕ_0 and ψ_0, respectively, and $\phi_0 \circ \psi_0$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{M}}$, we have $\phi_{n-1} \circ \psi_{n-1}$ approximately unitarily equivalent to $\mathrm{id}_{\mathcal{M}}$. Therefore, there is a *-homomorphism $\phi_n\colon \mathcal{M} \to \mathcal{N}$ which satisfies (9.26), and is unitarily equivalent to ϕ_{n-1}, and hence also ϕ_0. The construction of $\mathcal{G}_n$ and ψ_n is similar.

If $m \geq n \geq 1$ and $a \in \mathcal{F}_n$, then

$$\begin{aligned} \|\phi_{m+1}(a) - \phi_m(a)\|_{2,Y} &\leq \|\phi_{m+1}(a - \psi_m(\phi_m(a)))\|_{2,Y} \\ &\quad + \|\phi_{m+1}(\psi_m(\phi_m(a))) - \phi_m(a)\|_{2,Y} \\ &< 2^{-m} + 2^{-m-1}. \end{aligned} \tag{9.27}$$

In particular, $(\phi_m(a))_{m=1}^{\infty}$ is norm-bounded and $\|\cdot\|_{2,Y}$-Cauchy for all $a \in \mathcal{F}_n$. So $(\phi_n(a))_{n=1}^{\infty}$ is norm-bounded and $\|\cdot\|_{2,Y}$-Cauchy for all $a \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, and hence also for all $a \in \mathcal{M}$. Similarly, $(\psi_n(b))_{n=1}^{\infty}$ is norm-bounded and $\|\cdot\|_{2,X}$-Cauchy for all $b \in \mathcal{N}$.

Define $\phi\colon \mathcal{M} \to \mathcal{N}$ and $\psi\colon \mathcal{N} \to \mathcal{M}$ by

$$\phi(a) \coloneqq \lim_{n\to\infty} \phi_n(a) \qquad \text{and} \qquad \psi(a) \coloneqq \lim_{n\to\infty} \psi_n(a). \tag{9.28}$$

By construction, ϕ and ψ are approximately unitarily equivalent to ϕ_0 and ψ_0, respectively, and in particular, $\tau\circ\phi = \tau\circ\phi_0$ for all $\tau \in Y$ and $\sigma\circ\psi = \sigma\circ\psi_0$ for all $\sigma \in X$. To complete the proof, note that ϕ and ψ are mutual inverses using (9.25) and (9.26). □

Theorem A from the overview is an immediate consequence of Theorem 9.13, provided (as set out in Footnote 1) we interpret it to mean that for unital separable nuclear $\mathcal{Z}$-stable C^*-algebras A and B, one has $\big(\overline{A}^{T(A)}, T(A)\big) \cong \big(\overline{B}^{T(B)}\big)$ if and only if $T(A)$ and $T(B)$ are affinely homeomorphic. The point is that the 'easy direction' – recovering the trace space from an isomorphism – is a tautology when the isomorphism is in the category of tracially complete C^*-algebras.[60] To see the 'hard direction', note that if A is a unital separable nuclear $\mathcal{Z}$-stable C^*-algebra, then $\big(\overline{A}^{T(A)}, T(A)\big)$ is $\|\cdot\|_{2,T(A)}$-separable, semidiscrete by Corollary 4.10, factorial by Proposition 3.23(iv), and satisfies property Γ by Propositions 5.17 and 5.22. The result then follows from Theorem 9.13(i).

The classification result for morphisms provides the following strengthened version of completely positive approximation property in which the 'upward maps' can be taken to be *-homomorphisms (and the downward maps are approximately multiplicative). Such a result holds for injective von Neumann algebras, and this was the starting point to obtaining strengthened forms of the completely positive approximation property involving *decomposable approximations* in the C^*-algebraic setting ([14, 18, 59]).[61]

Theorem 9.14. *Suppose A is a C^*-algebra, $(\mathcal{N}, Y)$ is a type II_1 factorial tracially complete C^*-algebra, and $\theta\colon A \to \mathcal{N}$ is a tracially nuclear *-homomorphism. Then there are nets*

$$A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\phi_\lambda} \mathcal{N} \tag{9.29}$$

[60] To recover $T(A)$ from $\overline{A}^{T(A)}$ as a C^*-algebra requires the forthcoming solution to the trace problem (Question 1.1) for tracially complete C^*-algebras with CPoU due to the third-named author.

[61] In [59, Section 1] a local reflexivity argument is given which shows how to use hyperfiniteness of a von Neumann algebra $\mathcal{M}$ to obtain nets of finite dimensional algebras F_i together with u.c.p. maps $\psi_i\colon \mathcal{M} \to F_i$, $\psi_i\colon F_i \to \mathcal{M}$ such that ϕ_i a *-homomorphism and $\phi_i(\psi_i(x)) \to x$ in the weak*-topology for all $x \in \mathcal{M}$. In fact all of Connes', Popa's, and Haagerup's proofs of injectivity implies hyperfiniteness ([29, 92, 57]) output such approximations in the case of factors.

of finite dimensional C^*-algebras F_λ and c.p.c. maps ϕ_λ and ψ_λ such that

(i) $\|\phi_\lambda(\psi_\lambda(a)) - \theta(a)\|_{2,Y} \to 0$ *for all* $a \in A$,
(ii) $\|\psi_\lambda(ab) - \psi_\lambda(a)\psi_\lambda(b)\|_{2,T(F_\lambda)} \to 0$ *for all* $a, b \in A$,
(iii) *each ϕ_λ is a unital *-homomorphism.*

If $\|\tau \circ \theta\|^{-1}(\tau \circ \theta)$ is quasidiagonal whenever $\tau \in Y$ with $\tau \circ \theta \neq 0$, we may further arrange for

(ii′) $\|\psi_\lambda(ab) - \psi_\lambda(a)\psi_\lambda(b)\| \to 0$ *for all* $a, b \in A$.

In either case, when A and θ are unital, we may arrange for each ψ_λ to be unital.

Proof. Adding a unit to A, we may assume that A is unital (Lemma 4.3). Further, as the conclusion of Theorem 9.14 is a local condition, it suffices to prove it when A is separable. By Theorem 4.9, $\tau \circ \theta$ is uniformly amenable for each $\tau \in T(A)$. By Theorem 9.8 (applied to the affine map $\tau \mapsto \tau \circ \theta$), there are finite dimensional C^*-algebras F_n and u.c.p. maps

$$A \xrightarrow{\psi_n} F_n \xrightarrow{\phi'_n} \mathcal{N} \tag{9.30}$$

so that for all $a, b \in A$ and $\tau \in Y$, we have

$$\|\psi_n(ab) - \psi_n(a)\psi_n(b)\|_{2,T(F_n)} \to 0, \tag{9.31}$$

and

$$|\tau(\phi'_n(\psi_n(a))) - \tau(\theta(a))| \to 0, \tag{9.32}$$

and such that each ϕ'_n is a unital *-homomorphism. Further, if $\tau \circ \theta$ is quasidiagonal for all $\tau \in T(A)$, we may replace (9.31) with

$$\|\psi_n(ab) - \psi_n(a)\psi_n(b)\| \to 0 \tag{9.33}$$

for all $a, b \in A$.

The maps $(\phi'_n \circ \psi_n)_{n=1}^\infty$ induce a *-homomorphism $\theta' \colon A \to \mathcal{N}^\infty$ with $\tau \circ \theta' = \tau \circ \theta$ for all $\tau \in Y^\infty$. By Theorem 4.9, θ' is tracially nuclear, and hence by Corollary 9.4, θ and θ' are unitarily equivalent. If $u \in \mathcal{N}^\omega$ is a unitary with $\mathrm{ad}(u) \circ \theta' = \theta$, then by Corollary 7.11, there is a sequence of unitaries $(u_n)_{n=1}^\infty \subseteq \mathcal{N}$ lifting u. The result follows by setting $\phi_n \coloneqq \mathrm{ad}(u_n) \circ \phi'_n$. □

We end by proving the structure and classification theorems from the overview (Theorem B and C). Most of the implications involved are already in place, and it remains to use the classification theorem to pass back from hyperfiniteness to the McDuff property via CPoU.

Theorem 9.15. *Let $(\mathcal{M}, X)$ be a type II_1 factorial tracially complete C^*-algebra. Then the following conditions are equivalent:*

(i) $(\mathcal{M}, X)$ *hyperfinite;*
(ii) $(\mathcal{M}, X)$ *is amenable and has CPoU;*
(iii) $(\mathcal{M}, X)$ *is amenable and satisfies property* Γ*;*
(iv) $(\mathcal{M}, X)$ *is amenable and McDuff.*

In this setting, if $\mathcal{M}$ is also assumed to be $\|\cdot\|_{2,X}$-separable, then $(\mathcal{M}, X) \cong (\mathcal{R}_X, X)$ (see Example 3.35).

Proof. The implications (iv) $\implies$ (iii), (iii) $\implies$ (ii), and (i) $\implies$ (ii) hold by Proposition 5.22, Theorem 1.4, and Theorem 8.3, respectively. Further, Theorem 9.14 applied to $\theta = \mathrm{id}_{\mathcal{M}}$ shows (ii) $\implies$ (i). Suppose (ii) holds and $(\mathcal{M}, X)$ is $\|\cdot\|_{2,X}$-separable. Both $(\mathcal{M}, X)$ and $(\mathcal{R}_X, X)$ satisfy the conditions of Theorem 9.13, and hence $(\mathcal{M}, X) \cong (\mathcal{R}_X, X)$. As $(\mathcal{R}_X, X)$ is McDuff, this also shows $(\mathcal{M}, X)$ is McDuff and finishes the proof in the separable setting.

It remains to show (ii) implies (iv) without separability. Assume $(\mathcal{M}, X)$ satisfies CPoU. By the local characterisation of McDuff's property (Proposition 5.14(ii)), it suffices to show that every finite set $\mathcal{F} \subseteq \mathcal{M}$ is contained in a unital $\|\cdot\|_{2,X}$-closed C^*-subalgebra of $\mathcal{M}$ which is factorial and McDuff as a tracially complete C^*-algebra. Using the separable inheritability of the conditions in (ii) (Theorem A.3(ii) and (vi), together with Proposition A.2), this follows from the fact that (ii) implies (iv) in the separable setting. □

Appendix A. Separabilisation

In this appendix we collect the machinery needed to reduce our main structural results – Theorems B and 1.4 – to the case of separable tracially complete C^*-algebras, and prove the non-metrisable version of Theorem 2.2.

A.1. **Separabilising tracially complete C^*-algebras.** The following definition is modelled on Blackadar's notion of separable inheritability for C^*-algebras ([7, Section II.8.5]). Recall from Section 3.1 (before Definition 3.11) that if $(\mathcal{M}, X)$ is a tracially complete C^*-algebra then a tracially complete C^*-subalgebra of $(\mathcal{M}, X)$ is $(\mathcal{M}_0, X_0)$ where $\mathcal{M}_0 \subseteq \mathcal{M}$ is a unital $\|\cdot\|_{2,X}$-closed C^*-subalgebra and $X_0 \subseteq T(\mathcal{M})$ is the set of traces arising as restrictions of traces in X to $\mathcal{M}_0$.

Definition A.1 (cf. [7, Section II.8.5])**.** We say that a property (P) of tracially complete C^*-algebras is *separably inheritable* if

(i) whenever $(\mathcal{M}, X)$ is a tracially complete C^*-algebra satisfying (P) and $S \subseteq \mathcal{M}$ is a $\|\cdot\|_{2,X}$-separable subset of $\mathcal{M}$, there is a tracially complete C^*-subalgebra $(\mathcal{M}_0, X_0)$ which is $\|\cdot\|_{2,X_0}$-separable, satisfies (P) and such that $S \subseteq \mathcal{M}_0$,

(ii) if $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of tracially complete C^*-algebras satisfying (P) and $\phi_n^{n+1} \colon (\mathcal{M}_n, X_n) \to (\mathcal{M}_{n+1}, X_{n+1})$ is an embedding for each $n \geq 1$, then $\varinjlim \big((\mathcal{M}_n, X_n), \phi_n^{n+1}\big)$ also satisfies (P).

We say that (P) is *strongly separably inheritable* if, in addition,

(iii) if $(\mathcal{M}, X)$ is a tracially complete C^*-algebra such that for every $\|\cdot\|_{2,X}$-separable subset S of $\mathcal{M}$, there exists a tracially complete C^*-subalgebra $(\mathcal{M}_0, X_0)$ satisfying (P) and containing S, then $(\mathcal{M}, X)$ satisfies (P).

Adapting the language of [99, Definition 1.4] to tracially complete C^*-algebras, condition (iii) asks that $(\mathcal{M}, X)$ satisfies (P) whenever it separably satisfies (P).

If (P) is a separably inheritable property and (Q) satisfies condition (iii), then in order to prove $(P) \implies (Q)$ for all tracially complete C^*-algebras, it is enough to prove it for $\|\cdot\|_{2,X}$-separable tracially complete C^*-algebras.

The following will allow us to construct separable subalgebras that satisfy several separably inheritable properties simultaneously. The proof is an easy modification of the corresponding result for C^*-algebras given in [7, Proposition II.8.5.3].

Proposition A.2. *The conjunction of countably many (strongly) separably inheritable properties is (strongly) separably inheritable.*

Proof. Let $\big((P_\lambda)\big)_{\lambda\in\Lambda}$ be a collection of separably inheritable properties of tracially complete C^*-algebras indexed by some countable set Λ, and let (P) be the conjunction of the (P_λ).

It is clear that Definition A.1(ii) holds for (P) as it holds for each (P_λ). To see Definition A.1(i), let $(\mathcal{M}, X)$ be a tracially complete C^*-algebras satisfying (P) and let $S \subseteq \mathcal{M}$ be a $\|\cdot\|_{2,X}$-separable subset. Fix a surjective map $f\colon \mathbb{N} \to \Lambda$ such that each $\lambda \in \Lambda$ has infinitely many preimages. Using Definition A.1(i) for the properties (P_λ), inductively construct an increasing sequence of $\|\cdot\|_{2,X}$-closed and $\|\cdot\|_{2,X}$-separable C^*-subalgebras $\mathcal{M}_n \subseteq \mathcal{M}$ such that $S \subseteq \mathcal{M}_1$ and $\mathcal{M}_n$ satisfies $(P_{f(n)})$. As each (P_λ) satisfies Definition A.1(ii), the $\|\cdot\|_{2,X}$-closed union of the $\mathcal{M}_n$ satisfies (P).

For strong separable inheritability, it is clear that Definition A.1(iii) is closed under (arbitrary) conjucntions. □

We will show that many of the properties of tracially compete C^*-algebras defined in this paper are strongly separably inheritable.

Theorem A.3. *The following properties are strongly separably inheritable:*

(i) *factoriality,*
(ii) *amenability,*
(iii) *hyperfiniteness,*
(iv) *McDuff's property*
(v) *factoriality and property* Γ*, and*
(vi) *factoriality and CPoU.*

In the case of the last two conditions, we include the factoriality condition as we have not defined property Γ or CPoU in the non-factorial setting.

The rest of this subsection is devoted to the proof. We will show each condition separately. For factoriality, most of the work is contained in Lemma 6.8.

Proof of Theorem A.3(i). A sequential direct limit of factorial tracially complete C^*-algebras is factorial by Proposition 3.34. Now suppose $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra and $S \subseteq \mathcal{M}$ is a $\|\cdot\|_{2,X}$-separable subset of $\mathcal{M}$.

By Lemma 6.8, there is a $\|\cdot\|$-separable unital C^*-algebra $A \subseteq \mathcal{M}$ containing a $\|\cdot\|_{2,X}$-dense subset of S such that

$$X_A \coloneqq \{\tau|_A : \tau \in X\} \subseteq T(A) \tag{A.1}$$

is a closed face. Let $(\mathcal{N}, Y)$ denote the tracial completion of A with respect to X_A and note that $(\mathcal{N}, Y)$ is factorial by Proposition 3.23(iv). The inclusion $A \to \mathcal{M}$ extends to a morphism $\phi\colon (\mathcal{N}, Y) \to (\mathcal{M}, X)$ by Proposition 3.25. Note that ϕ is an embedding whose range contains S. Also, the unit ball of

$\phi(\mathcal{N})$ is $\|\cdot\|_{2,X}$-closed in the unit ball of $\mathcal{M}$, and it follows from Lemma 3.27 that $\phi(\mathcal{N})$ is $\|\cdot\|_{2,X}$-closed in $\mathcal{M}$.

Finally, suppose that $(\mathcal{M}, X)$ is a tracially complete C^*-algebra such that every $\|\cdot\|_{2,X}$-separable subset S is contained in a factorial tracially complete C^*-subalgebra of $(\mathcal{M}, X)$. To show that $(\mathcal{M}, X)$ is factorial, let $\tau_1, \tau_2, \tau \in T(\mathcal{M})$ be such that $\tau \in X$ and τ is a non-trivial convex combination of τ_1 and τ_2, and we will check that $\tau_1 \in X$. For any finite subset $\mathcal{F}$ of $\mathcal{M}$, we can find a factorial tracially complete C^*-subalgebra $(\mathcal{M}_0, X_0)$ such that $\mathcal{F} \subseteq \mathcal{M}_0$. Since $\tau|_{\mathcal{M}_0}$ is a non-trivial convex combination of $\tau_1|_{\mathcal{M}_0}$ and $\tau_2|_{\mathcal{M}_0}$, and X_0 is a face, it follows that $\tau_1|_{\mathcal{M}_0} \in X_0$. By definition of X_0, this means that there exists $\sigma_{\mathcal{F}} \in X$ such that $\tau_1|_{\mathcal{M}_0} = \sigma_{\mathcal{F}}|_{\mathcal{M}_0}$, so in particular, $\tau_1|_{\mathcal{F}} = \sigma_{\mathcal{F}}|_{\mathcal{F}}$. By doing this over all finite subsets $\mathcal{F}$, we see that τ_1 is a weak*-limit of traces in X, and therefore $\tau_1 \in X$ as required. □

It is somewhat subtle to show that amenability is preserved by inductive limits. Even in the tracial von Neumann algebra setting, the only known proof that the weak closure of an increasing union of semidiscrete von Neumann algebras is semidiscrete relies on the equivalence of semidiscreteness and injectivity from Connes' work ([29]). Our argument for parts (i) and (ii) of Definition A.1 goes via the extension result for tracially nuclear morphisms (Lemma 4.12) proved using the local-to-global characterisations of amenability in Section 4.

Proof of Theorem A.3(ii). For condition (iii) of Definition A.1, fix a finite subset $\mathcal{F}$ of $\mathcal{M}$ and $\epsilon > 0$. Use the hypothesis to find an amenable tracially complete C^*-subalgebra $(\mathcal{M}_0, X_0)$ of $\mathcal{M}$ containing $\mathcal{F}$. By Arveson's extension theorem, the maps witnessing amenability of $(\mathcal{M}_0, X_0)$ can be extended to give a finite dimensional C^*-algebra F and c.p.c. maps $\mathcal{M} \xrightarrow{\psi} F \xrightarrow{\phi} \mathcal{M}$ with $\|\phi(\psi(x)) - x\| < \epsilon$ for $x \in \mathcal{F}$. Working with a net indexed over finite sets $\mathcal{F}$ and $\epsilon > 0$ gives amenability of $\mathcal{M}$.

Suppose $\big((\mathcal{M}_n, X_n)\big)_{n=1}^{\infty}$ is a sequence of amenable tracially complete C^*-algebras and $\phi_n^{n+1}\colon (\mathcal{M}_n, X_n) \to (\mathcal{M}_{n+1}, X_{n+1})$ is an embedding for each $n \geq 1$ and consider $(\mathcal{M}, X) := \varinjlim \big((\mathcal{M}_n, X_n), \phi_n^{n+1}\big)$. If $A \subseteq \mathcal{M}$ is the C^*-algebraic direct limit of the $\mathcal{M}_n$, then the inclusion $A \hookrightarrow \mathcal{M}$ is tracially nuclear (using Arveson's extension theorem as above). By Lemma 4.12, $(\mathcal{M}, X)$ is amenable, and this shows Definition A.1(ii).

Suppose now that $(\mathcal{M}, X)$ is an amenable tracially complete C^*-algebra. We'll first show that for every unital $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed C^*-subalgebra $\mathcal{M}_0 \subseteq \mathcal{M}$, there is a $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed C^*-subalgebra $\mathcal{N} \subseteq \mathcal{M}$ containing $\mathcal{M}_0$ so that the inclusion $\mathcal{M}_0 \hookrightarrow \mathcal{N}$ is tracially nuclear.

Fix a $\|\cdot\|$-separable, $\|\cdot\|_{2,X}$-dense C^*-algebra $A \subseteq \mathcal{M}_0$ and an increasing sequence $(\mathcal{F}_n)$ of finite subsets of A with $\|\cdot\|$-dense union, and for each $n \geq 1$, use Proposition 4.4 to construct a finite dimensional C^*-algebra F_n and c.p.c. maps

$$A \xrightarrow{\psi_n} F_n \xrightarrow{\phi_n} \mathcal{M} \tag{A.2}$$

so that

$$\|\phi_n(\psi_n(a)) - a\|_{2,X} < \frac{1}{n}, \qquad a \in \mathcal{F}_n. \tag{A.3}$$

Let $\mathcal{N}$ denote the tracially complete C^*-subalgebra of $\mathcal{M}$ generated by $\mathcal{M}_0 \cup \bigcup_{n=1}^{\infty} \phi_n(\mathcal{F}_n)$. By construction, $\mathcal{N}$ is $\|\cdot\|_{2,X}$-separable and the inclusion $A \hookrightarrow \mathcal{N}$ is tracially nuclear, and by Lemma 4.12, so is the inclusion $\mathcal{M}_0 \hookrightarrow \mathcal{N}$.

Applying this result inductively, given a $\|\cdot\|_{2,X}$-separable subset $S \subseteq \mathcal{M}$, we may construct an increasing sequence $(\mathcal{M}_n)_{n=1}^{\infty}$ of $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed subalgebras of $\mathcal{M}$ such that $S \subseteq \mathcal{M}_1$ and the inclusions $\mathcal{M}_n \hookrightarrow \mathcal{M}_{n+1}$ are tracially nuclear. If $\mathcal{N}$ is the $\|\cdot\|_{2,X}$-closure of the union of the $\mathcal{M}_n$, then $\mathcal{N}$ is $\|\cdot\|_{2,X}$-separable. If $A \subseteq \mathcal{N}$ is the $\|\cdot\|$-closure of the union of the $\mathcal{M}_n$, then the inclusion $A \hookrightarrow \mathcal{N}$ is tracially nuclear (by the same application of Arveson's extension theorem used in the previous parts), and hence $\mathcal{N}$ is amenable by Lemma 4.12. This shows Definition A.1(i). □

The proof that hyperfiniteness is separably inheritable is standard.

Proof of Theorem A.3(iii). Verifying Definition A.1(ii) and (iii) is a standard $\epsilon/3$ argument. To see Definition A.1(i) suppose $(\mathcal{M}, X)$ is a hyperfinite tracially complete C^*-algebra and let S be a $\|\cdot\|_{2,X}$-separable subset of $\mathcal{M}$, and let $\mathcal{M}_1$ denote the tracially complete C^*-subalgebra of $\mathcal{M}$ generated by S.

As $\mathcal{M}_1$ is $\|\cdot\|_{2,X}$-separable, fix an increasing sequence $(\mathcal{F}_k)$ of finite sets with dense union in $\mathcal{M}_1$. For each $k \geq 1$, let $F_k \subseteq \mathcal{M}$ be a finite dimensional C^*-algebra such that for every $a \in \mathcal{F}_k$, there is a $b \in F_k$ with $\|a-b\|_{2,X} < \frac{1}{k}$. Let $\mathcal{M}_2$ denote the tracially complete C^*-subalgebra generated by $\mathcal{M}_1$ and the F_k.

Iterating this construction, there is an increasing sequence of $\|\cdot\|_{2,X}$-closed C^*-algebras $\mathcal{M}_n \subseteq \mathcal{M}$ such that for every $n \geq 1$, finite set $\mathcal{F} \subseteq \mathcal{M}_n$, and $\epsilon > 0$, there is a finite dimensional C^*-algebra $F \subseteq \mathcal{M}_{n+1}$ such that for all $a \in \mathcal{F}$ there is a $b \in F$ with $\|a - b\|_{2,X} < \epsilon$. Then the $\|\cdot\|_{2,X}$-closure of the union of the $\mathcal{M}_n$ is a tracially complete hyperfinite subalgebra of $\mathcal{M}$ containing S. □

The separable inheritability of McDuff's property and property Γ are easy consequences of the local characterisation of these properties in Proposition 5.14(ii) and Proposition 5.25 respectively.

Proof of Theorem A.3(iv) *and* (v). McDuff's property is preserved by sequential inductive limits by Corollary 5.16, and the combination of factoriality and property Γ is preserved under sequential inductive limits by Propositions 3.34 and 5.25, respectively. In particular, Definition A.1(ii) holds for both conditions. Likewise, Proposition 5.14(ii) shows that Definition A.1(iii) holds for McDuff's property, while Proposition 5.23(ii) (together with the fact that Definition A.1(iii) holds for factoriality) handles this condition for factorial tracially complete C^*-algebras with property Γ.

For Definition A.1(i), let $(\mathcal{M}, X)$ be a McDuff tracially complete C^*-algebra and let $S \subseteq \mathcal{M}$ be a $\|\cdot\|_{2,X}$-separable set; write $\mathcal{M}_1$ for the $\|\cdot\|_{2,X}$-closed C^*-subalgebra of $\mathcal{M}$ generated by S which is necessarily $\|\cdot\|_{2,X}$-separable. Inductively, given a $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed C^*-subalgebra

$\mathcal{M}_n$ of $\mathcal{M}$, Proposition 5.14(ii) provides a sequence $(v_m)_{m=1}^{\infty} \subseteq \mathcal{M}$ such that for any finite subset $\mathcal{F} \subseteq \mathcal{M}_n$ and $\epsilon > 0$, there is $m \geq 1$ such that for all $a \in \mathcal{F}$,

$$\|[v_m, a]\|_{2,X} < \epsilon, \ \|v_m^* v_m + v_m v_m^* - 1_{\mathcal{M}}\|_{2,X} < \epsilon, \text{ and } \|v_m^2\|_{2,X} < \epsilon. \tag{A.4}$$

Let $\mathcal{M}_{n+1}$ be the $\|\cdot\|_{2,X}$-closed C^*-subalgebra of $\mathcal{M}$ generated by $\mathcal{M}_n \cup \{v_m : m \in \mathbb{N}\}$. Let $\mathcal{N}$ denote the $\|\cdot\|_{2,X}$-closure of the union of the $\mathcal{M}_n$. Then $\mathcal{N}$ is a $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed C^*-subalgebra of $\mathcal{M}$ containing S and satisfying the approximation property in Proposition 5.14(ii). So $\mathcal{N}$ is McDuff, and Definition A.1(i) holds.

The argument for Definition A.1(i) works very similarly for the combination of factoriality and property Γ. For the inductive step, given a $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed C^*-subalgebra of $\mathcal{M}$ which is factorial with the tracially complete structure induced from $\mathcal{M}$, we replace the sequence $(v_m)_{m=1}^{\infty}$ with a sequence $(p_m)_{m=1}^{\infty}$ of contractions such that for any finite subset $\mathcal{F} \subset \mathcal{M}_n$ and $\epsilon > 0$, there is m with

$$\|p_m - p_m^2\|_{2,X} < \epsilon, \ \|[p_m, a]\|_{2,X} < \epsilon \text{ and } \sup_{\tau \in X} \left|\tau(ap_m) - \frac{1}{2}\tau(a)\right| < \epsilon \tag{A.5}$$

for all $a \in \mathcal{F}$ which is given by the characterisation of property Γ in Proposition 5.23(ii). Now use the corresponding part of Theorem A.3(i) to obtain a $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed C^*-subalgebra $\mathcal{M}_{n+1}$ of $\mathcal{M}$ containing $\mathcal{M}_n \cup \{p_m : m \in \mathbb{N}\}$ which is factorial in the induced tracially complete structure. Just as in the previous paragraph, the $\|\cdot\|_{2,X}$-closure of the union of the $\mathcal{M}_n$ is factorial (by Theorem A.3(i)) with property Γ (via Proposition 5.23(ii)). □

Finally we establish that CPoU is strongly separably inheritable for factorial tracially complete C^*-algebras. This works in a very similar fashion to separable inheritability of the McDuff property and property Γ.

Proof of Theorem A.3(vi). The combination of factoriality and CPoU is preserved under inductive limits by Propositions 3.34 and 6.12. Similarly, Proposition 6.2(ii) (together with the fact that Definition A.1(iii) holds for factoriality) shows that factoriality and CPoU satisfies Definition A.1(iii).

Suppose now that $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra satisfying CPoU. Let $S \subseteq \mathcal{M}$ be a $\|\cdot\|_{2,X}$-separable subset of $\mathcal{M}$. We start by constructing an increasing sequence $(\mathcal{N}_n)_{n=1}^{\infty}$ of $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed C^*-subalgebras of $\mathcal{M}$ such that

- $S \subseteq \mathcal{N}_1$,
- each $\mathcal{N}_n$ is factorial as a tracially complete C^*-algebra, and
- for each $k, n \geq 1$, if $a_1, \ldots, a_k \in (\mathcal{N}_n)_+$ and $\delta > 0$ with

$$\sup_{\tau \in X} \min_{1 \leq i \leq k} \tau(a_i) < \delta, \tag{A.6}$$

then there are projections $p_1, \ldots, p_k \in (\mathcal{N}_{n+1})^{\omega} \cap \mathcal{N}_n'$ such that

$$\sum_{j=1}^{k} p_j = 1_{\mathcal{M}^{\omega}} \qquad \text{and} \qquad \tau(a_i p_i) \leq \delta \tau(p_i) \tag{A.7}$$

for all $i = 1, \ldots, k$ and $\tau \in X^{\omega}$.

We will construct the algebras $\mathcal{N}_n$ by induction starting with $\mathcal{N}_1$ being any unital factorial $\|\cdot\|_{2,X}$-closed C^*-subalgebra of $\mathcal{M}$ containing S, which exists as factoriality is separably inheritable (part (i) of Theorem A.3).

Suppose $n \geq 1$ and $\mathcal{N}_n$ has been constructed. Let S_n be a countable, $\|\cdot\|_{2,X}$-dense subset of $(\mathcal{N}_n)_+$, and let K_n be the set of all pairs $\kappa = (\mathcal{G}_\kappa, \delta_\kappa)$ where $\mathcal{G}_\kappa \subseteq S_n$ is a finite set and $\delta_\kappa > 0$ is rational with

$$\sup_{\tau \in X} \min_{a \in \mathcal{G}_\kappa} \tau(a) < \delta_\kappa. \tag{A.8}$$

Note that K_n is a countable set. Since $(\mathcal{M}, X)$ satisfies CPoU, for each $\kappa \in K_n$ and $a \in \mathcal{G}_\kappa$, there are projections $p_{\kappa,a} \in \mathcal{M}^\omega \cap \mathcal{N}_n'$ such that

$$\sum_{b \in \mathcal{G}_\kappa} p_{\kappa,b} = 1_{\mathcal{M}^\omega} \qquad \text{and} \qquad \tau(a p_{\kappa,a}) \leq \delta \tau(p_{\kappa,a}) \tag{A.9}$$

for all $a \in \mathcal{G}_\kappa$ and $\tau \in X^\omega$. For each $\kappa \in K_n$ and $a \in \mathcal{G}_\kappa$, let $(p_{\kappa,a,m})_{m=1}^\infty \subseteq \mathcal{M}$ be a sequence in $\mathcal{M}$ representing $p_{\kappa,a}$. As factoriality is separably inheritable (part (i) of the theorem), there is a unital $\|\cdot\|_{2,X}$-separable, $\|\cdot\|_{2,X}$-closed C^*-subalgebra $\mathcal{N}_{n+1} \subseteq \mathcal{M}$ which contains $\mathcal{N}_n$ and each $p_{\kappa,a,m}$ for all $m \geq 1$, $\kappa \in K_n$ and $a \in \mathcal{G}_\kappa$, and is factorial as a tracially complete C^*-algebra. Then $\mathcal{N}_{n+1}$ satisfies the required properties.

Let $\mathcal{N} \subseteq \mathcal{M}$ be the $\|\cdot\|_{2,X}$-closure of the union of the algebras $\mathcal{N}_n$. Then $\mathcal{N}$ is a $\|\cdot\|_{2,X}$-closed, $\|\cdot\|_{2,X}$-separable C^*-subalgebra of $\mathcal{M}$ which contains S. As each $\mathcal{N}_n$ is factorial, so too is $\mathcal{N}$ by Proposition 3.34. The third condition on the $\mathcal{N}_n$ ensures that $\mathcal{N}$ satisfies CPoU, and this completes the proof. □

A.2. **Proof of Theorem 2.2.** The final section of the appendix is devoted to finite dimensional approximations of non-metrisable Choquet simplices. We show how reduce Theorem 2.2 to the metrisable case ([40, Lemma 2.8], which is a corollary of the fundamental work of Lazar and Lindenstrauss in [70]). The strategy is another variation of Blackadar's separable inheritability, this time for compact convex sets; roughly, we want to know the property of being a Choquet simplex is "metrisably inheritable" among compact convex sets. We find it conceptually easier to do this in the dual picture and consider separably inheritable properties of Archimedean order unit spaces.

Choquet simplices can be characterised in terms of Archimedean order unit spaces using Kadison duality as follows. An ordered vector space V satisfies *Riesz interpolation* if for all $f_1, f_2, g_1, g_2 \in V$ with $f_i \leq g_j$ for $i, j = 1, 2$, there is an $h \in V$ with $f_i \leq h \leq g_j$ for $i, j = 1, 2$.

Theorem A.4 ([3, Corollary II.3.11])**.** *Suppose X is a compact convex set and V is an Archimedean order unit space.*

(i) *X is a Choquet simplex if and only if* $\mathrm{Aff}(X)$ *has Riesz interpolation.*
(ii) *V has Riesz interpolation if and only if $S(V)$ is a Choquet simplex.*

Proof. The first statement is [3, Corollary II.3.11], and the second statement follows from the first by Kadison duality (see Section 2.1). □

We will show Riesz interpolation is a separably inheritable property in Proposition A.6. To facilitate this, we show that Riesz interpolation can be detected on a dense set.

Lemma A.5. *Let V be an Archimedean order unit space. Then V satisfies Riesz interpolation if and only if there exists a unital dense rational subspace $V_0 \subseteq V$ with Riesz interpolation.*

Proof. The forward direction is clear by taking the subspace to be V. In the other direction, suppose $V_0 \subseteq V$ is a unital dense rational subspace of V. Since all states (i.e. positive and unital functionals) on both V_0 and V are bounded and V_0 is dense in V, the restriction map $S(V) \to S(V_0)$ is an affine homeomorphism. As V_0 has Riesz interpolation, $S(V_0)$ is a Choquet simplex by [55, Corollary 10.6], and hence $S(V)$ is a Choquet simplex. So V has Riesz interpolation. □

Finally, we arrive at the main separabilisation result needed in this subsection. The proof is similar to the proofs in the previous subsection, and also to the proofs of separabilisation results in other settings, such as those in [7, Section II.8.5]

Proposition A.6. *Riesz interpolation is a separably inheritable property of Archimedean order unit spaces in the following sense.*

(i) *If V is an Archimedean order unit space with Riesz interpolation and S is a separable subset of V, then there is a separable unital closed subspace $V_0 \subseteq V$ that contains S and has Riesz interpolation.*
(ii) *For a sequence $(V_n)_{n=1}^{\infty}$ of separable Archimedean order unit spaces with Riesz interpolation and unital order embeddings $V_n \hookrightarrow V_{n+1}$, the inductive limit of the V_n also has Riesz interpolation.*

Proof. Part (ii) follows from Lemma A.5 since the non-closed union of the V_n will have Riesz interpolation.

Assume in (i) that V and S are given, and without loss of generality, assume S is countable and $1 \in S$. Let V_1 be the rational vector space spanned by S so that V_1 is countable.

We can inductively construct an increasing sequence of countable rational subspaces $(V_n)_{n=1}^{\infty}$ of V such that for all $n \geq 1$ and $f_1, f_2, g_1, g_2 \in V_n$ with $f_i \leq g_j$ for $1 \leq i, j \leq 2$, there is an $h \in V_{n+1}$ such that $f_i \leq h \leq g_j$ for all $1 \leq i, j \leq 2$. Indeed, as V has Riesz interpolation, for any such quadruple (f_1, f_2, g_1, g_2) from V_n, we may choose a corresponding $h \in V$. As the set of all such quadruples in V_n is countable, we may define V_{n+1} as the rational span of V_n and the corresponding functions h corresponding to these quadruples.

Now, $\bigcup_{n=1}^{\infty} V_n$ has Riesz interpolation. Let V_0 be the closure of this union and note that V_0 has Riesz interpolation by Lemma A.5. □

With these results in hand, we prove Theorem 2.2 by reducing it to the metrisable setting. We restate the theorem for the convenience of the reader.

Theorem 2.2. *If X is a Choquet simplex, then there are nets of finite dimensional Choquet simplices Z_λ and continuous affine maps*

$$X \xrightarrow{\beta_\lambda} Z_\lambda \xrightarrow{\alpha_\lambda} X \tag{2.5}$$

such that $\lim_\lambda \|f \circ \alpha_\lambda \circ \beta_\lambda - f\| = 0$ for all $f \in \mathrm{Aff}(X)$.

Proof of Theorem 2.2. View $\mathbb{R}^d$ as an ordered vector space with the coordinatewise order and with order unit $(1,\dots,1)$. By Kadison duality, it suffices to show that for every finite set $\mathcal{F} \subseteq \mathrm{Aff}(X)$ and $\epsilon > 0$, there are an integer d and unital positive linear maps

$$\mathrm{Aff}(X) \xrightarrow{\psi} \mathbb{R}^d \xrightarrow{\phi} \mathrm{Aff}(X) \tag{A.10}$$

such that $\|\phi(\psi(f)) - f\| < \epsilon$ for all $f \in \mathcal{F}$.

By Proposition A.6, there is a separable unital closed subspace V_0 of $\mathrm{Aff}(X)$ that contains $\mathcal{F}$ and satisfies Riesz interpolation. By the Hahn–Banach theorem (applied in each coordinate), any unital positive map $V_0 \to \mathbb{R}^d$ extends to a unital positive map $\mathrm{Aff}(X) \to \mathbb{R}^d$ (since unital functionals are positive exactly when they have norm 1). Therefore, it suffices to find an integer d and unital positive linear maps

$$V_0 \xrightarrow{\psi} \mathbb{R}^d \xrightarrow{\phi} V_0 \tag{A.11}$$

such that $\|\phi(\psi(f)) - f\| < \epsilon$ for all $f \in \mathcal{F}$. Since V_0 has Riesz interpolation, $S(V_0)$ is a Choquet simplex by Theorem A.4(ii). Using Kadison duality to identify V_0 with $\mathrm{Aff}(S(V_0))$ the result follows from the separable case, [40, Lemma 2.8]. □

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